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MEAN NUMBER OF REAL ZEROS OF A RANDOM HYPERBOLIC POLYNOMIAL

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ABSTRACT. Consider the random hyperbolic polynomial, $f(x) = 1^p a_1 \cosh x + \cdots + n^p \times a_n \cosh nx$, in which n and p are integers such that $n \geq 2$, $p \geq 0$, and the coefficients $a_k (k = 1, 2, \ldots, n)$ are independent, standard normally distributed random variables. If ν_{np} is the mean number of real zeros of $f(x)$, then we prove that $\nu_{np} = \pi^{-1} \log n + O\{(\log n)^{1/2}\}$.

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1. Introduction. Let n and p be integers such that $n \geq 2$ and $p \geq 0$. We suppose that $a_k (k = 1, 2, \ldots, n)$ are independent, normally distributed random variables, each with mean 0 and variance 1, and we define the random hyperbolic polynomial $f(x)$ so that

$$f(x) = \sum_{k=1}^{n} k^p a_k \cosh kx. \tag{1.1}$$

We prove the following result.

THEOREM 1.1. *Let ν_{np} be the mean number of real zeros of f(x). Then*

$$\nu_{np} = \pi^{-1} \log n + O\left\{(\log n)^{1/2}\right\}. \tag{1.2}$$

The case when $p = 0$ was considered by Das [3], whose result was reported by Bharucha-Reid and Sambandham [1, page 110] in the form $\nu_{no} \sim \pi^{-1} \log n$. The case when $p = 1$ was discussed by Farahmand and Jahangiri [5], who found the result (1.2) in that case.

The principal term in (1.2) is independent of p. That behavior does not occur in the algebraic case [4] (replace $\cosh kx$ in (1.1) by x^k and let k range from 0 to n), for which $\nu_{np} \sim \pi^{-1}\{1 + (2p+1)^{1/2}\} \log n$ (even if p is a nonnegative real number), and also does not occur in the trigonometric case [2] (replace $\cosh kx$ in (1.1) by $\cos kx$ and count zeros on $(0, 2\pi)$), for which $\nu_{np} = \{(2p+1)/(2p+3)\}^{1/2}(2n+1) + O(n^{1/2})$ (even if p is a nonnegative real number). The error term in this last case can be replaced by $O(1)$ when $2p$ is a nonnegative integer [6, 7, 8, 9].

2. Preliminary analysis. If we apply the Kac-Rice formula to our problem, we see that

$$\nu_{np} = \pi^{-1} \int_{-\infty}^{\infty} F_{np}(x)\, dx = 2\pi^{-1} \int_{0}^{\infty} F_{np}(x)\, dx \tag{2.1}$$

in which

$$F_{np}(x) = \frac{\{A_{np}(x)C_{np}(x) - B_{np}^2(x)\}^{1/2}}{A_{np}(x)}, \tag{2.2}$$

$$A_{np}(x) = \sum_{k=1}^{n} k^{2p} \cosh^2 kx, \tag{2.3}$$

$$B_{np}(x) = \sum_{k=1}^{n} k^{2p+1} \sinh kx \cosh kx, \tag{2.4}$$

$$C_{np}(x) = \sum_{k=1}^{n} k^{2p+2} \sinh^2 kx. \tag{2.5}$$

We furnish explicit formulae for the sums in (2.3), (2.4), and (2.5) in the following lemma.

Lemma 2.1. *It is true that*

$$\begin{aligned} 2^{2p+2}A_{np}(x) &= (2n+1)^{2p} \operatorname{csch} x \sinh z \\ &\quad \times \left[\sum_{r=0}^{2p} {}_{2p}C_r (2n+1)^{-r} \varphi_r(x) + (2n+1)^{-2p} (2^{2p+2}S_{np} - \delta_{op}) \sinh x \operatorname{csch} z \right], \end{aligned} \tag{2.6}$$

$$2^{2p+3}B_{np}(x) = (2n+1)^{2p+1} \operatorname{csch} x \sinh z \sum_{r=0}^{2p+1} {}_{2p+1}C_r (2n+1)^{-r} \psi_r(x), \tag{2.7}$$

$$\begin{aligned} 2^{2p+4}C_{np}(x) &= (2n+1)^{2p+2} \operatorname{csch} x \sinh z \\ &\quad \times \left[\sum_{r=0}^{2p+2} {}_{2p+2}C_r (2n+1)^{-r} \varphi_r(x) - (2n+1)^{-2p-2} 2^{2p+4} S_{n,p+1} \sinh x \operatorname{csch} z \right], \end{aligned} \tag{2.8}$$

in which

$$z = (2n+1)x, \tag{2.9}$$

$$\varphi_{2r}(x) = g_{2r}(x), \qquad \varphi_{2r+1}(x) = g_{2r+1}(x) \coth z, \tag{2.10}$$

$$\psi_{2r}(x) = g_{2r}(x) \coth z, \qquad \psi_{2r+1}(x) = g_{2r+1}(x), \tag{2.11}$$

$$g_r(x) = \sinh x \left\{ \frac{d^r(\operatorname{csch} x)}{dx^r} \right\}, \tag{2.12}$$

$$2S_{np} = \sum_{k=1}^{n} k^{2p}, \tag{2.13}$$

where ${}_pC_r$ is the binomial coefficient $p!/\{r!(p-r)!\}$, and δ_{op} is the Kronecker delta, i.e., $\delta_{op} = 1$ when $p = 0$ and $\delta_{op} = 0$ when $p \neq 0$.

With the help of (2.13), the identity $2\cosh^2 kx = \cosh 2kx + 1$, it is clear that

$$\begin{aligned} 2^{2p+2} A_{np}(x) &= \frac{2d^{2p}\{\sum_{k=1}^{n}(\cosh 2kx+1)\}}{dx^{2p}} - 2n\delta_{op} + 2^{2p+2} S_{np} \\ &= d^{2p}\frac{\{4A_{no}(x)\}}{dx^{2p}} - 2n\delta_{op} + 2^{2p+2} S_{np}. \end{aligned} \tag{2.14}$$

It is known from [6, equation 2.15] that $4A_{no}(x) = 2n - 1 + \operatorname{csch} x \sinh z$, if z is defined by (2.9). Hence,

$$2^{2p+2} A_{np}(x) = \sum_{r=0}^{2p} {}_{2p}C_r \left\{\frac{d^r(\operatorname{csch} x)}{dx^r}\right\}\left\{\frac{d^{2p-r}(\sinh z)}{dx^{2p-r}}\right\} - \delta_{op} + 2^{2p+2} S_{np}. \tag{2.15}$$

If the derivatives of $\sinh z$ are calculated and the definitions (2.10) and (2.12) are used, we see that (2.6) is true. In a similar manner, it follows from (2.3), (2.4), and (2.11) that

$$\begin{aligned} 2^{2p+3} B_{np}(x) &= \frac{d\{2^{2p+2} A_{np}(x)\}}{dx} = \frac{d^{2p+1}(\operatorname{csch} x \sinh z)}{dx^{2p+1}} \\ &= (2n+1)^{2p+1}(\operatorname{csch} x \sinh z)\sum_{r=0}^{2p+1} {}_{2p+1}C_r (2n+1)^{-r}\psi_r(x), \end{aligned} \tag{2.16}$$

so that (2.7) is true. Finally, (2.8) is a consequence of (2.6) and the identity $C_{np}(x) = A_{n,p+1}(x) - 2S_{n,p+1}$.

A straightforward calculation, based on (2.6), (2.7), and (2.8), suffices to prove the following lemma.

Lemma 2.2. *It is true that*

$$\begin{aligned} &2^{4p+6}\{A_{np}(x)C_{np}(x) - B_{np}^2(x)\} = (2n+1)^{4p+2}\operatorname{csch}^2 x \sinh^2 z \\ &\quad \times \left[\sum_{r=0}^{4p+2}(2n+1)^{-r}\theta_{rp}(x) + \Theta_{np}(x)\sinh x \operatorname{csch} z - \Psi_{np}(x)\sinh^2(x)\operatorname{csch}^2 z\right] \end{aligned} \tag{2.17}$$

in which

$$\theta_{rp}(x) = \sum_{s=0}^{r}\{{}_{2p}C_s\ {}_{2p+2}C_{r-s}\varphi_s(x)\varphi_{r-s}(x) - {}_{2p+1}C_s\ {}_{2p+1}C_{r-s}\psi_S(x)\psi_{r-s}(x)\}, \tag{2.18}$$

$$\begin{aligned} \Theta_{np}(x) &= (2n+1)^{-2p}(2^{2p+2}\ S_{np} - \delta_{op})\sum_{r=0}^{2p+2} {}_{2p+2}C_r(2n+1)^{-r}\varphi_r(x) \\ &\quad - (2n+1)^{-2p-2}2^{2p+4}S_{n,p+1}\sum_{r=0}^{2p} {}_{2p}C_r(2n+1)^{-r}\varphi_r(x), \end{aligned} \tag{2.19}$$

$$\Psi_{np}(x) = (2n+1)^{-4p-2}(2^{2p+2}S_{np} - \delta_{op})2^{2p+4}S_{n,p+1}. \tag{2.20}$$

We need the more explicit formulae for $g_r(x)$ contained in the following lemma.

Lemma 2.3. *There are constants* β_{rs} $(s = 0, 1, \ldots, [r/2])$ *such that*

$$g_{2r}(x) = \sum_{s=0}^{r} \beta_{2r,s} \operatorname{csch}^{2s} x, \tag{2.21}$$

$$g_{2r+1}(x) = \sum_{s=0}^{r} \beta_{2r+1,s} \operatorname{csch}^{2s} x \coth x. \tag{2.22}$$

It follows from (2.12) that (2.21) is true when $r = 0$ if $\beta_{00} = 1$. A differentiation of (2.12) shows that

$$g_{r+1}(x) = \frac{dg_r}{dx} - g_r(x) \coth x. \tag{2.23}$$

If (2.21) is true for r, we infer from (2.23) that (2.22) is true for r, provided that

$$\beta_{2r+1,s} = -(2s+1)\beta_{2r,s}. \tag{2.24}$$

Similarly, the truth of (2.21) with r replaced by $r+1$ is assured when

$$\beta_{2r+2,s} = -(2s+1)\beta_{2r+1,s} - 2s\beta_{2r+1,s-1}. \tag{2.25}$$

We record for future reference the cases when $r = 0$, 1, and 2:

$$g_0(x) = 1, \qquad g_1(x) = -\coth x, \qquad g_2(x) = 1 + 2\operatorname{csch}^2 x. \tag{2.26}$$

3. Estimates of the terms in (2.6) and (2.17) when x is not too small. We suppose that $x \geq \varepsilon$, in which

$$\varepsilon = \frac{w}{(2n+1)}, \qquad w = (\log n)^{1/2}. \tag{3.1}$$

Lemma 3.1. *If* $n_o = 8104$ *and* $n \geq n_0$*, the functions* $\sinh^3 x \operatorname{csch} z$*,* $\sinh x \operatorname{csch} z$*, and* $\sinh^4 x \operatorname{csch}^2 z$ *are decreasing functions of* x *when* $x \geq \varepsilon$*.*

We observe that

$$\begin{aligned}\frac{\operatorname{csch}^2 x \operatorname{sech} x \sinh^2 z \operatorname{sech} z \, d(\sinh^3 x \operatorname{csch} z)}{dx} &= 3\tanh z - (2n+1)\tanh x \\ &< 3 - (2n+1)\tanh \varepsilon.\end{aligned} \tag{3.2}$$

Also,

$$\frac{\cosh^2 \varepsilon \, d\{(2n+1)\tanh \varepsilon\}}{dn} = \sinh 2\varepsilon - 2\varepsilon + (2nw)^{-1} > 0. \tag{3.3}$$

Therefore, $(2n+1)\tanh\varepsilon > 3$ when $n \geq n_0$ because $(2n+1)\tanh\varepsilon > 3$ when $n = 8104$. It follows that $\sinh^3 x \operatorname{csch} z$ is decreasing when $x \geq \varepsilon$ and $n \geq n_0$. The other functions in the lemma are decreasing because $(\sinh^3 x \operatorname{csch} z)^{1/3} \operatorname{csch}^{2/3} z$ and $(\sinh^3 x \operatorname{csch} z)^{4/3}$ $\operatorname{csch}^{2/3} z$ are. The third term on the right hand side of (2.17) is estimated in the following lemma.

Lemma 3.2. *When $n \geq n_0$ and $x \geq \varepsilon$, it is true that*

$$\Psi_{np}(x)\sinh^2 x \operatorname{csch}^2 z = O(w^4 e^{-2w})(2n+1)^{-2}\operatorname{csch}^2 x. \tag{3.4}$$

It follows from an explicit formula [6, equation (2.12)] for S_{np} that $S_{np} = O\{(2n+1)^{2p+1}\}$. Then (2.20) and Lemma 3.1 imply that

$$\begin{aligned}\Psi_{np}(x)\sinh^4 x \operatorname{csch}^2 z &= O\{(2n+1)^2\sinh^4\varepsilon \operatorname{csch}^2 w\}\\ &= O\{(2n+1)^2\varepsilon^4 e^{-2w}\}.\end{aligned} \tag{3.5}$$

Lemma 3.2 is an immediate consequence of this result and (3.1).

Lemma 3.3. *When $x \geq \varepsilon$, it is true that $g_r(x) = O(\varepsilon^{-r})$, $\varphi_r(x) = O(\varepsilon^{-r})$, and $\psi_r(x) = O(\varepsilon^{-r})$.*

The lemma follows immediately from (2.10), (2.11), (2.21), and (2.22), and the facts that

$$\begin{aligned}&\operatorname{csch} x \leq \operatorname{csch}\varepsilon < \varepsilon^{-1},\\ &\coth x \leq \coth\varepsilon < \varepsilon^{-1}\cosh\varepsilon_o,\\ &\coth z \leq \coth w \leq \coth w_0,\end{aligned} \tag{3.6}$$

in which $\varepsilon_o = w_0/(2n_o+1)$ and $w_0 = (\log n_0)^{1/2}$. Now, we can estimate the second term on the right-hand side of (2.17).

Lemma 3.4. *When $n \geq n_0$ and $x \geq \varepsilon$, it is true that*

$$\Theta_{np}(x)\sinh x \operatorname{csch} z = O(w^3 e^{-w})(2n+1)^{-2}\operatorname{csch}^2 x. \tag{3.7}$$

We deduce from (2.19), Lemmas 3.1 and 3.3, and the earlier observation that $S_{np} = O\{(2n+1)^{2p+1}\}$ that

$$\begin{aligned}\Theta_{np}(x)\sinh^3 x \operatorname{csch} z &= O\left\{(2n+1)\sum_{r=0}^{2p+2} O(w^{-r})\sinh^3\varepsilon \operatorname{csch} w\right\}\\ &= O\{(2n+1)\varepsilon^3 e^{-w}\} = O(w^3 e^{-w})(2n+1)^{-2}.\end{aligned} \tag{3.8}$$

This equation suffices to prove Lemma 3.4.

The analysis to obtain an estimate for θ_{rp} is more recondite. We use (2.10), (2.11), (2.18), and the identity $\coth^2 z = 1 + \operatorname{csch}^2 z$, to see that

$$\theta_{2r,p} = \sum_{s=0}^{2r} L_{2r,sp}\, g_s(x)\, g_{2r-s}(x) + M_{rp}(x)\operatorname{csch}^2 z, \tag{3.9}$$

in which

$$L_{rsp} = {}_{2p}C_s\ {}_{2p+2}C_{r-s} - {}_{2p+1}C_s\ {}_{2p+1}C_{r-s}, \tag{3.10}$$

$$\begin{aligned}M_{rp}(x) = &\sum_{s=0}^{r-1} {}_{2p}C_{2s+1}\ {}_{2p+2}C_{2r-2s-1} g_{2s+1}(x)\, g_{2r-2s-1}(x)\\ &-\sum_{s=0}^{r} {}_{2p+1}C_{2s}\ {}_{2p+1}C_{2r-2s}\ g_{2s}(x) g_{2r-2s}(x).\end{aligned} \tag{3.11}$$

In a similar manner, we also see that

$$\theta_{2r+1,p}(x) = \sum_{s=0}^{2r+1} L_{2r+1,sp}\; g_s(x)\; g_{2r+1-s}(x) \coth z. \tag{3.12}$$

Because we infer from (3.9) and Lemma 3.3 that $M_{rp}(x) = O(\varepsilon^{-2r})$, it follows, from (3.9) and (3.12), that

$$\theta_{rp}(x) = \sum_{s=0}^{r} L_{rsp}\; g_s(x)\; g_{r-s}(x)(\coth z)^{u_r} + O(\varepsilon^{-2r})\operatorname{csch}^2 z \tag{3.13}$$

in which $u_r = \{1-(-1)^r\}/2$. Moreover, Lemma 2.3 implies that

$$g_r(x) = \sum_{h=0}^{[r/2]} \beta_{rh} \operatorname{csch}^{2h} x (\coth z)^{u_r}, \tag{3.14}$$

so that there are constants γ_{rsh} such that

$$g_s(x) g_{r-s}(x) = \sum_{h=0}^{[r/2]} \gamma_{rsh} \operatorname{csch}^{2h} x (\coth x)^{u_r}. \tag{3.15}$$

In the derivation of (3.15), it is helpful to consider separately the cases when r is even and r is odd. When r is even and s is odd, we also need the identity $\coth^2 x = 1 + \operatorname{csch}^2 x$. An easy induction using (2.24) and (2.25) when $s = 0$ shows that $\beta_{ro} = (-1)^r$; hence $\gamma_{rso} = (-1)^r$.

The combinatorial identity

$$\sum_{s=0}^{r} L_{rsp} = 0 \tag{3.16}$$

is well known (and is easy to prove). We now deduce, from (3.13), (3.15), and (3.16), that

$$\begin{aligned} \theta_{rp}(x) \sinh^2 x = {} & \sum_{s=0}^{r} L_{rsp} \sum_{h=1}^{[r/2]} \gamma_{rsh} \operatorname{csch}^{2h-2} x (\coth x \coth z)^{u_r} \\ & + O(\varepsilon^{-r}) \sinh^2 x \operatorname{csch}^2 z. \end{aligned} \tag{3.17}$$

We showed in the proof of Lemma 3.3 that

$$\operatorname{csch} x = O(\varepsilon^{-1}), \qquad \coth x = O(\varepsilon^{-1}), \qquad \coth z = O(1). \tag{3.18}$$

Because it follows from Lemma 2.3 that

$$\sinh^2 x \operatorname{csch}^2 z \le \sinh^2 \varepsilon \operatorname{csch}^2 w = O(\varepsilon^2 \mathrm{e}^{-2w}), \tag{3.19}$$

we conclude that the following lemma is true.

Lemma 3.5. *When $n \ge n_o$ and $x \ge \varepsilon$, it is true that*

$$\theta_{rp}(x) = O(\varepsilon^{2-r})\{1 + O(\mathrm{e}^{-2w})\} \operatorname{csch}^2 x. \tag{3.20}$$

We also need the more precise estimates of $\theta_{rp}(x)$ when $r = 0$, 1, and 2, deducible from (2.10), (2.11), (2.18), and (2.26), that are recorded below:

$$\begin{aligned}\theta_{0p}(x) &= -\operatorname{csch}^2 z = O(w^2 \mathrm{e}^{-2w})(2n+1)^{-2}\operatorname{csch}^2 x,\\ \theta_{1p}(x) &= 0,\\ \theta_{2p}(x) &= (1-4p^2\operatorname{csch}^2 z + 2p\sinh x \operatorname{csch} z)\operatorname{csch}^2 x\\ &= \{1+O(\mathrm{e}^{-2w}) + (2n+1)^{-1}O(w\,\mathrm{e}^{-w})\}\operatorname{csch}^2 x\\ &= \{1+O(\mathrm{e}^{-2w})\}\operatorname{csch}^2 x.\end{aligned} \tag{3.21}$$

Finally, the methods used above can be applied to (2.6) to yield an easy proof of the following lemma.

Lemma 3.6. *When $n \ge n_o$ and $x \ge \varepsilon$, it is true that*

$$2^{2p+2}A_{np} = (2n+1)^{2p}\operatorname{csch} x \sinh z[1+O(w^{-1})]. \tag{3.22}$$

Proof of Theorem 1.1. If we use Lemmas 3.2, 3.4, and 3.5, we infer from (2.17), and (3.21) that, when $n \ge n_o$ and $x \ge \varepsilon$,

$$2^{4p+6}\{A_{np}(x)C_{np}(x) - B_{np}^2(x)\} = (2n+1)^{4p}\operatorname{csch}^4 x \sinh^2 z[1+O(w^{-1})]. \tag{3.23}$$

It now follows from (2.2) and Lemma 3.6 that, when $n \ge n_0$ and $x \ge \varepsilon$,

$$2F_{np}(x)\,dx = \{1+O(w^{-1})\}\operatorname{csch} x, \tag{3.24}$$

$$\begin{aligned}2\int_\varepsilon^\infty F_{np}(x)\,dx &= \{1+O(w^{-1})\}\log\left\{\coth\left(\frac{\varepsilon}{2}\right)\right\}\\ &= \{1+O(w^{-1})\}\{1+O(w^{-2}\log w)\}\log n,\end{aligned} \tag{3.25}$$

$$2\pi^{-1}\int_\varepsilon^\infty F_{np}(x)\,dx = \pi^{-1}\log n + O\left\{(\log n)^{1/2}\right\}. \tag{3.26}$$

Next, we observe that (2.2), (2.3), and (2.5) imply that

$$0 \le C_{np}(x) \le n^2\sum_{k=1}^{n} k^{2p}\sinh^2 kx < n^2 A_{np}(x), \tag{3.27}$$

$$0 \le F_{np}(x) \le \left\{\frac{C_{np}(x)}{A_{np}(x)}\right\}^{1/2} < n, \tag{3.28}$$

$$2\pi^{-1}\int_0^\varepsilon F_{np}(x)\,dx < 2\pi^{-1}n\varepsilon < \pi^{-1}w = O\{(\log n)^{1/2}\}. \tag{3.29}$$

If we add (3.26) and (3.29) and use (2.1), we see that the theorem is true. □

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