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A NOTE ON (gDF)-SPACES

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ABSTRACT. Certain locally convex spaces of scalar-valued mappings are shown to be finite-dimensional.

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1. Introduction. Radenovič [6], generalizing a result of Iyahen [2], has shown that if E is a Banach space and $(E,\sigma(E,E'))$ (or $(E',\sigma(E',E))$) is a (DF)-space [1], then E is finite-dimensional. His result has been extended to arbitrary locally convex spaces by Krassowska and Śliwa [3].

In [4, 5], (DF)-spaces have been generalized as follows: a locally convex space (E,τ) is a (gDF)-space if

(a) (E,τ) has a fundamental sequence $(B_n)_{n\in\mathbb{N}}$ of bounded sets, and

(b) τ is the finest locally convex topology on E that agrees with τ on each B_n.

In this note, we prove that if an arbitrary vector space of scalar-valued mappings is a (gDF)-space under the locally convex topology of pointwise convergence, then it is finite-dimensional. As a consequence, the above-mentioned theorem of Krassowska and Śliwa readily follows.

2. The result. Throughout this note, all vector spaces under consideration are vector spaces over a field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. In our result, E denotes an arbitrary set and H denotes a subspace of the vector space of all mappings from E into $\mathbb{K}$. We consider on H the separated locally convex topology of pointwise convergence and represent by H' the topological dual of H.

THEOREM 2.1. *The following conditions are equivalent:*

(a) *H is a finite-dimensional vector space;*

(b) *H is a (DF)-space;*

(c) *H is a (gDF)-space.*

PROOF. It is clear that (a) implies (b) and (b) implies (c) (every (DF)-space is a (gDF)-space).

Suppose that condition (c) holds. If H is infinite-dimensional, there exists a countable linearly independent subset $\{\varphi_n; n\in\mathbb{N}\}$ of H'. Let $(B_n)_{n\in\mathbb{N}}$ be an increasing fundamental sequence of bounded subsets of H. Then, $(B_n^0)_{n\in\mathbb{N}}$ is a decreasing sequence of neighborhoods of zero in $(H',\beta(H',H))$ forming a fundamental system of neighborhoods of zero in $(H', \beta(H', H))$. For each $n \in \mathbb{N}$, fix an $\alpha_n > 0$ such that $\alpha_n \varphi_n \in B_n^0$; then $(\alpha_n \varphi_n)_{n\in\mathbb{N}}$ converges to zero in $(H', \beta(H', H))$. By [5, Theorem 1.1.7], the set $\Gamma = \{\alpha_n \varphi_n; n \in \mathbb{N}\}$ is equicontinuous. Hence, there exist $x_1, \dots, x_m \in E$ and there exists an $\alpha > 0$ such that the relations

$$f \in H, \qquad |f(x_1)| \le \alpha, \dots, |f(x_m)| \le \alpha, \qquad \varphi \in \Gamma \tag{2.1}$$

imply

$$|\varphi(f)| \le 1. \tag{2.2}$$

For each $i = 1, \dots, m$, let $\delta_i \in H'$ be given by $\delta_i(f) = f(x_i)$ for $f \in H$, and put $F = \{\delta_1, \dots, \delta_m\}$. We claim that $\Gamma \subset [F]$, where $[F]$ is the finite-dimensional vector space generated by F. Indeed, let $\varphi \in \Gamma$ and take an $f \in H$ such that $\delta_1(f) = \cdots = \delta_m(f) = 0$. Then, for all $\lambda \in \mathbb{K}$,

$$|(\lambda f)(x_1)| = |\delta_1(\lambda f)| = 0 \le \alpha, \dots, |(\lambda f)(x_m)| = |\delta_m(\lambda f)| = 0 \le \alpha. \tag{2.3}$$

Consequently, $|\varphi(\lambda f)| = |\lambda||\varphi(f)| \le 1$. By the arbitrariness of $\lambda, \varphi(f) = 0$. By [7, Lemma 5, Chapter II], $\varphi \in [F]$. Therefore the vector space generated by the set $\{\varphi_n; n \in \mathbb{N}\}$ is finite-dimensional, which contradicts the choice of $(\varphi_n)_{n\in\mathbb{N}}$. This completes the proof of the theorem. □

Remark 2.2. The theorem of Krassowska and Šliwa mentioned at the beginning of this note follows from Theorem 2.1. In fact, let E be a separated locally convex space. If $(E', \sigma(E', E))$ is a (DF)-space, then E' is finite-dimensional by Theorem 2.1, and so E is finite-dimensional. Hence, E is finite-dimensional if $(E, \sigma(E, E'))$ is a (DF)-space.

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