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RELATED FIXED POINTS FOR SET VALUED MAPPINGS ON TWO METRIC SPACES

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Abstract. Some related fixed points theorems for set valued mappings on two complete and compact metric spaces are proved.

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We let (X,d) be a complete metric space and let $B(X)$ be the set of all nonempty subsets of X. As in [1, 2], we define the function $\delta(A,B)$ with A and B in $B(X)$ by $\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}$. If A consists of a single point a we write $\delta(A,B) = \delta(a,B)$. If B also consists of single point b, we write $\delta(A,B) = \delta(a,B) = \delta(a,b) = d(a,b)$. It follows immediately that $\delta(A,B) = \delta(B,A) \geq 0$, and $\delta(A,B) \leq \delta(A,C) + \delta(C,B)$ for all A, B in $B(X)$.

Now if $\{A_n : n = 1,2,\dots\}$ is a sequence of sets in $B(X)$, we say that it converges to the set A in $B(X)$ if

(i) each point $a \in A$ is the limit of some convergent sequence $\{a_n \in A_n : n = 1,2,\dots\}$,

(ii) for arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subset A_\epsilon$ for $n > N$, where A_ϵ is the union of all open spheres with centers in A and radius ϵ.

The set A is then said to be the limit of the sequence $\{A_n\}$.

The following lemma was proved [1].

Lemma. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X,d) which converge to the bounded subsets A and B, respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.*

Now, let F be a mapping of X into $B(X)$. We say that the mapping F is continuous at a point X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$. We say that F is continuous mapping of X into $B(X)$ if F is continuous at each point x in X. We say that a point z in X is a fixed point of F if z is in Fz. If A is in $B(X)$, we define the set $FA = \bigcup_{a\in A} Fa$.

Theorem 1. *Let (X,d_1) and (Y,d_2) be complete metrics spaces, let F be mapping of X into $B(Y)$ and let G be mapping of Y into $B(X)$ satisfying the inequalities*

$$\delta_1(GFx,GFx') \leq c\max\left\{d_1(x,x'),\delta_1(x,GFx),\delta_1(x',GFx'),\delta_2(Fx,Fx')\right\}, \quad (1)$$

$$\delta_2(FGy,FGy') \le c\max\left\{d_2(y,y'),\delta_2(y,FGy),\delta_1(y',FGy'),\delta_1(Gy,Gy')\right\} \quad (2)$$

for all x,x' in X and y,y', where $0 \le c < 1$. If F is continuous, then GF has a unique fixed point z in X and FG has a unique fixed point w in Y.

Proof. Let x_1 be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y, respectively, as follows. Choose a point y_1 in Fx_1 and then a point x_2 in Gy_1. In general, having chosen x_n in X and y_n in Y choose x_{n+1} in Gy_n and then y_{n+1} in Fx_{n+1} for $n = 1,2,\dots$. Then,

$$\begin{aligned} d_1(x_{n+1},x_{n+2}) &\le \delta_1(GFx_n,GFx_{n+1}) \\ &\le c\max\left\{d_1(x_n,x_{n+1}),\delta_1(x_n,GFx_n),\ \delta_1(x_{n+1},GFx_n),\right. \\ &\qquad \left.\delta_1(x_{n+1},GFx_{n+1}),\delta_2(Fx_n,Fx_{n+1})\right\} \\ &\le c\max\left\{\delta_1(GFx_{n-1},GFx_n),\delta_1(GFx_n,GFx_{n+1}),\delta_2(Fx_n,Fx_{n+1})\right\} \\ &= c\max\left\{\delta_1(GFx_{n-1},GFx_n),\delta_2(Fx_n,Fx_{n+1})\right\} \end{aligned} \quad (3)$$

and, similarly,

$$\begin{aligned} d_2(y_{n+1},y_{n+2}) &\le \delta_2(FGy_n,GFy_{n+1}) \\ &\le c\max\left\{\delta_2(FGy_{n-1},FGy_n),\delta_1(Gy_n,Gy_{n+1})\right\}. \end{aligned} \quad (4)$$

It follow that, for $r = 1,2,\dots,$

$$\begin{aligned} d_1(x_{n+1},x_{n+r+1}) &\le \delta_1(GFx_n,GFx_{n+r}) \\ &\le \delta_1(GFx_n,GFx_{n+1}) + \cdots + \delta_1(GFx_{n+r-1},GFx_{n+r}) \\ &\le (c^n + c^{n+1} + \cdots + c^{n+r-1})\delta_1(x_1,GFx_1) < \epsilon \end{aligned} \quad (5)$$

for n greater than some N, since $c < 1$. The sequence $\{x_n\}$ is, therefore, a Cauchy sequence in the complete metric space X and so has a limit z in X. Similarly, the sequence $\{y_n\}$ is a sequence in complete metric space Y and so has a limit w in Y.

Further

$$\begin{aligned} \delta_1(z,GFx_n) &\le d_1(z,x_{m+1}) + \delta_1(x_{m+1},GFx_n) \\ &\le d_1(z,x_{m+1}) + \delta_1(GFx_m,GFx_n), \end{aligned} \quad (6)$$

since $x_{m+1} \in GFx_m$. Thus, on using inequality (5), we have

$$\delta_1(z,GFx_n) \le d_1(z,x_{m+1}) + \epsilon \quad (7)$$

for m, $n \ge N$. Letting m tends to infinity it follows that

$$\delta_1(z,GFx_n) < \epsilon \quad (8)$$

for $n > N$ and so

$$\lim_{n\to\infty} GFx_n = \{z\} \quad (9)$$

since ϵ is arbitrary. Similarly,

$$\lim_{n\to\infty} FGy_n = \{w\} = \lim_{n\to\infty} Fx_n \quad (10)$$

since x_{n+1} is in Gy_n. Using the continuity of F, we see that

$$\lim_{n\to\infty} Fx_n = Fz = \{w\}. \tag{11}$$

Using inequality (1), we now have

$$\delta_1(GFx_n, GFz) \le c\max\left\{d_1(x_n,z), \delta_1(x_n, GFx_n), \delta_1(z, GFz), \delta_2(Fz, Fx_n)\right\}. \tag{12}$$

Letting n tends to infinity and using (9) and (11), we have

$$\delta_1(z, GFz) \le c\delta_1(z, GFz). \tag{13}$$

Since $c < 1$, $\delta_1(z, GFz) = 0$ and, so, we must have $GFz = \{z\}$, proving that z is a fixed point of GF.

Further, using (11), we have

$$FGw = FGFz = Fz = w, \tag{14}$$

proving that w is a fixed point of FG.

Now suppose that GF has a second fixed point z'. Then using inequalities (1) and (2), we have

$$\begin{aligned}
\delta_1(z', GFz') &\le \delta_1(GFz', GFz') \\
&\le c\max\left\{d_1(z',z'), \delta_1(z', GFz'), \delta_2(Fz', Fz')\right\} \\
&= c\delta_2(Fz', Fz') \le c\delta_2(Fz', FGFz') \le c\delta_2(FGFz', FGFz') \\
&\le c^2\max\left\{\delta_2(Fz', Fz'), \delta_2(Fz', FGFz'), \delta_1(GFz', FGz')\right\} \\
&= c^2\delta_2(GFz', GFz')
\end{aligned} \tag{15}$$

and so Fz' is a singleton and $GFz' = \{z'\}$, since $c < 1$. Thus,

$$\begin{aligned}
d_1(z,z') &= \delta_1(GFz, GFz') \\
&\le c\max\left\{d_1(z,z'), \delta_1(z, GFz), \delta_1(z', GFz'), \delta_2(Fz, Fz')\right\} \\
&= cd_2(Fz, Fz').
\end{aligned} \tag{16}$$

But

$$\begin{aligned}
d_2(Fz,Fz') &\le \delta_2(FGFz, FGFz') \\
&\le c\max\left\{\delta_2(Fz,Fz'), \delta_2(Fz, FGFz), \delta_2(Fz', FGFz'), \delta_1(GFz, GFz')\right\} \\
&= c\max\left\{d_2(Fz,Fz'), d_2(Fz,Fz), d_2(Fz',Fz'), d_1(z,z')\right\} \\
&= cd_1(z,z')
\end{aligned} \tag{17}$$

and so

$$d_1(z,z') \le c^2 d_1(z,z'). \tag{18}$$

Since $c < 1$, the uniqueness of z follows.

Similarly, w is the unique fixed point of FG. This completes the proof of the theorem. □

If we let F be a single valued mapping T of X into Y and G be a single valued mapping of Y into X, we obtain the following result given in [3].

Corollary 1. *Let (X,d_1) and (Y,d_2) be complete metric spaces. If T is a continuous mapping of X into Y, and S is a mapping of Y into X satisfying the inequalities*

$$\begin{aligned} d_1(STx,STx') &\le c\max\left\{d_1(x,x'),d_1(x,STx),d_1(x',STx'),d_2(Tx,Tx')\right\},\\ d_2(STy,STy') &\le c\max\left\{d_2(y,y'),d_2(y,TSy),d_2(y',TSy'),d_1(Sy,Sy')\right\}\end{aligned} \tag{19}$$

for all x, x' in X and y,y' in Y, where $0 \le c < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further $Tz = w$ and $Sw = z$.

Theorem 2. *Let (X,d_1) and (Y,d_2) be compact metric spaces. If F is a continuous mapping of X into $B(Y)$, and G is a continuous mapping of Y into $B(X)$ satisfying the inequalities*

$$\begin{aligned} \delta_1(GFx,GFx') &< \max\left\{d_1(x,x'),\delta_1(x,GFx),\delta_1(x',GFx'),\delta_2(Fx,Fx')\right\},\\ \delta_2(FGy,FGy') &< \max\left\{d_2(y,y'),\delta_2(y',FGy),\delta_2(y',FGy'),\delta_1(Gy,Gy')\right\}\end{aligned} \tag{20}$$

for all x,x' in X and y,y' in Y for which the right-hand sides of the inequalities are positive. Then FG has a unique fixed point z in X and GF has a unique fixed point w in Y. Further $FGz = \{z\}$ and $GFw = \{w\}$.

Proof. Let us denote the right-hand side of inequalities (20) by $h(x,x')$ and $k(y,y')$, respectively. First of all suppose that $h(x,x') \ne 0$ for all $x,x' \in X$ and $k(y,y') \ne 0$ for all $y,y' \in Y$. Define the real-valued function $f(x,x')$ on X^2 by

$$f(x,x') = \frac{\delta_1(GFx,GFx')}{h(x,x')}. \tag{21}$$

Then if $\{(x_n,x'_n)\}$ is an arbitrary sequence in X^2 converning to (x,x'), it follows from the lemma and the continuity of F and G the sequence $\{f(x_n,x'_n)\}$ converges to $f(x,x')$. The function f is therefore a continuous function defined on the compact metric space X^2 and so achieves its maximum value c_1. Because of inequality (9), $c_1 < 1$ and so

$$\delta_1(GFx,GFx') \le c_1\max\left\{d_1(x,x'),\delta(x,GFx),\delta_1(x',GFx'),\delta_2(Fx,Fx')\right\} \tag{22}$$

for all x,x' in X.

Similarly, there exists $c_2 < 1$ such that

$$\delta_2(FGy,FGy') \le c_2\max\left\{d_2(y,y'),\delta_2(y,FGy),\delta_2(y',FGy'),\delta_1(Gy,Gy')\right\} \tag{23}$$

for all y,y' in Y. It follows that the conditions of Theorem 2 are satisfied with $c = \max\{c_1,c_2\}$ and, so, once again there exist z in X and w in Y such that $GFz = \{z\}$ and $FGw = \{w\}$.

Now, suppose that $h(x,x') = 0$ for some x,x' in X. Then $GFx = GFx' = \{x\} = \{x'\}$ is a singleton $\{z\}$ and then Fz is a singleton $\{w\}$. It follows that z is a fixed point of

GF and $GFz = \{z\}$. Further,

$$FGw = FGFz = Fz = \{w\} \tag{24}$$

and so w is a fixed point of FG.

It follows similarly that if $k(y,y') = 0$ for some y, y' in Y, then again GF has a fixed point z and FG has a fixed point w.

Now let us suppose that GF has a second fixed point z' in X so that z' is in GFz'. Then on using inequalities (20), we have, on assuming that $\delta_2(Fz',Fz') \neq 0$,

$$\begin{aligned}\delta_1(z',GFz') &\le \delta_1(GFz',GFz')\\ &< \max\left\{d_1(z',z'),\delta_1(z',GFz'),\delta_2(Fz',Fz')\right\}\\ &= \delta_2(Fz',Fz') \le \delta_2(Fz',FGFz') \le \delta_2(FGFz',FGFz')\\ &< \max\left\{\delta_2(Fz',Fz'),\delta_2(Fz',FGFz'),\delta_1(GFz',FGz')\right\}\\ &= c^2\delta_2(GFz',GFz')\end{aligned} \tag{25}$$

a contradiction and so Fz' is a singleton and $GFz' = \{z'\}$. Thus, if $z \neq z'$

$$\begin{aligned}d_1(z,z') &= \delta_1(GFz,GFz')\\ &< \max\left\{d_1(z,z'),\delta_1(z,GFz),\delta_1(z',GFz'),\delta_2(Fz,Fz')\right\}\\ &= d_2(Fz,Fz').\end{aligned} \tag{26}$$

But if $Fz \neq Fz'$, we have

$$\begin{aligned}d_2(Fz,Fz') &\le \delta_2(FGFz,FGFz')\\ &< \max\left\{\delta_2(Fz,Fz'),\delta_2(Fz,FGFz),\delta_2(Fz',FGFz'),\delta_1(GFz,GFz')\right\}\\ &= \max\left\{\delta_2(Fz,Fz'),d_2(Fz,Fz),d_2(Fz',Fz'),d_1(z,z')\right\}\\ &= d_1(z,z')\end{aligned} \tag{27}$$

and so

$$d_1(z,z') < d_1(z,z'), \tag{28}$$

a contradiction. The uniqueness of z follows.

Similarly, w is the unique fixed point of FG. This completes the proof of the theorem. □

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for all x, x' in X and y, y' in Y for which the right-hand sides of the inequalities are positive, then ST has a fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

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