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# NORM ATTAINING BILINEAR FORMS ON $L_1(\mu)$

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ABSTRACT. Given a finite measure $\mu$, we show that the set of norm attaining bilinear forms is dense in the space of all continuous bilinear forms on $L_1(\mu)$ if and only if $\mu$ is purely atomic.

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**1. Introduction.** A classical result of Bishop and Phelps [3] asserts that the set of norm attaining linear functionals on a Banach space is dense in the dual space. Very recently some attention has been paid to the question if the Bishop-Phelps theorem still holds for multilinear forms. To pose the problem more precisely, given a real or complex Banach space $X$ and a natural number $N$, let us denote by $\mathcal{L}^N(X)$ the space of all continuous $N$-linear forms on $X$ and let us say that $\varphi \in \mathcal{L}^N(X)$ attains its norm if there are $x_1,\ldots,x_N \in B_X$ (the closed unit ball of $X$) such that

$$|\varphi(x_1,\ldots,x_N)| = \|\varphi\| := \sup\{|\varphi(y_1,\ldots,y_N)| : y_1,\ldots,y_N \in B_X\}. \tag{1.1}$$

We denote by $\mathcal{AL}^N(X)$ the set of norm attaining continuous $N$-linear forms on $X$. The question is whether $\mathcal{AL}^N(X)$ is dense in $\mathcal{L}^N(X)$ or not. Unlike the linear case, the answer to this question is negative, an example of a Banach space $X$ such that $\mathcal{AL}^2(X)$ is not dense in $\mathcal{L}^2(X)$ was recently exhibited by Acosta, Aguirre, and Payá [1]. In [4], Choi gave a more striking counterexample by showing that $\mathcal{AL}^2(L_1[0,1])$ is not dense in $\mathcal{L}^2(L_1[0,1])$. In the positive direction, Aron, Finet, and Werner [2] showed that $\mathcal{AL}^N(X)$ is dense in $\mathcal{L}^N(X)$ whenever $X$ satisfies either the Radon-Nikodým property or the so-called property $(\alpha)$. Choi and Kim [5] obtained the same result for a Banach space $X$ with a monotone shrinking basis and the Dunford-Pettis property (e.g., $c_0$). Jiménez and Payá [7] gave, for each $N$, an example of a Banach space $X$ such that $\mathcal{AL}^N(X)$ is dense in $\mathcal{L}^N(X)$ but $\mathcal{AL}^{N+1}(X)$ is not dense in $\mathcal{L}^{N+1}(X)$, actually $X$ is the canonical predual of a suitable Lorentz sequence space.

In this paper, we discuss the denseness of norm attaining multilinear forms on the space $L_1(\mu)$, where $\mu$ is an arbitrary finite measure. We show that $\mathcal{AL}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ (for all $N$, or just for $N = 2$) if and only if $\mu$ is purely atomic. Half of this characterization follows from the main result in [2], since $L_1(\mu)$ satisfies the Radon-Nikodým property if $\mu$ is purely atomic. For the converse, we first extend Choi's example [4] to show that $\mathcal{AL}^2(L_1(\mu))$ is not dense in $\mathcal{L}^2(L_1(\mu))$ where $\mu$ is the product measure on an arbitrary product of copies of the unit interval. Then the result follows from the isometric classification of $L_1$-spaces (cf. [8]) through an elementary lemma which deals with the denseness of $\mathcal{AL}^2(X)$ when $X = Y \oplus_1 Z$ is the $l_1$-sum of two Banach spaces.

**2. Results.** In what follows $(\Omega, \mathcal{A}, \mu)$ will be a finite measure space. Let us start by recalling that the Banach space $\mathcal{L}^2(L_1(\mu))$ of all continuous bilinear forms on $L_1(\mu)$ is isometrically isomorphic to $L_\infty(\mu \otimes \mu)$, where $\mu \otimes \mu$ denotes the product measure on $\Omega \times \Omega$. More precisely, the bilinear form $\varphi$ which corresponds to a function $h \in L_\infty(\mu \otimes \mu)$ is given by

$$\varphi(f,g) = \int_{\Omega \times \Omega} h(u,v) f(u) g(v) \, d\mu(u) \, d\mu(v), \tag{2.1}$$

for every $f, g \in L_1(\mu)$ (see [6]). Choi [4], has shown that $\mathcal{AL}^2(L_1[0,1])$ is not dense in $\mathcal{L}^2(L_1[0,1])$. This result can be extended in the following way.

**Lemma 2.1.** *Let $\nu$ be an arbitrary nonzero finite measure and $\mu = \nu \otimes m$, where $m$ denotes Lebesgue measure on $I = [0,1]$. Then $\mathcal{AL}^2(L_1(\mu))$ is not dense in $\mathcal{L}^2(L_1(\mu))$.*

**Proof.** Let $U$ be the set where $\nu$ is defined, so that $\mu$ works on $\Omega = I \times U$ and $\mu \otimes \mu$ lives on the set $\Omega \times \Omega = I \times U \times I \times U$. We want a function $h \in L_\infty(\mu \otimes \mu)$ such that the corresponding bilinear form cannot be approximated by norm attaining bilinear forms. Actually $h$ will be the characteristic function $\chi_T$ of a suitable measurable set $T \subseteq \Omega \times \Omega$ with positive measure. The same argument used by Choi [4, Theorem 3] shows that the bilinear form corresponding to $\chi_T$ belongs to the closure of $\mathcal{AL}^2(L_1(\mu))$ only if there are measurable sets $E, F \subseteq \Omega$ with $\mu(E) > 0$, $\mu(F) > 0$, such that $[\mu \otimes \mu]$ $(E \times F \setminus T) = 0$. Therefore, we are left with finding a measurable set $T \subseteq \Omega \times \Omega$ with $[\mu \otimes \mu](T) > 0$ such that $[\mu \otimes \mu]$ $(E \times F \setminus T) > 0$ for any pair $E, F$ of measurable subsets of $\Omega$ with $\mu(E) > 0$, $\mu(F) > 0$.

By [4, Lemma 2] there exists a set $S \subseteq I \times I$ with the analogous property. More concretely, $S$ is a measurable set in $I \times I$, with $[m \otimes m](S) > 0$, such that $[m \otimes m](A \times B \setminus S)$ for any pair $A, B$ of measurable subsets of $I$ with $m(A) > 0$ and $m(B) > 0$. To get our set $T$, we modify $S$ in the obvious way, namely we define

$$T = \{(s,u,t,v) \in I \times U \times I \times U : (s,t) \in S\}. \tag{2.2}$$

Clearly, $T$ is a measurable set in $\Omega \times \Omega$, with positive measure. Let $E, F \subseteq \Omega$ be measurable sets in $\Omega$ with $\mu(E) > 0$, $\mu(F) > 0$, write $H = E \times F \setminus T$ and assume that $[\mu \otimes \mu](H) = 0$ to get a contradiction. For $u, v \in U$, let us consider the section

$$H^{(u,v)} = \{(s,t) \in I \times I : (s,u,t,v) \in H\}, \tag{2.3}$$

and note that $H^{(u,v)} = E^u \times F^v \setminus S$, where

$$E^u = \{s \in I : (s,u) \in E\}, \qquad F^v = \{t \in I : (t,v) \in F\}. \tag{2.4}$$

By Fubini's theorem (or the definition of the product measure) we have

$$0 = [m\otimes\nu\otimes m\otimes\nu](H) = \int_{U\times U}[m\otimes m](H^{(u,v)})\,d\nu(u)\,d\nu(v), \tag{2.5}$$

so

$$0 = [m\otimes m](H^{(u,v)}) = [m\otimes m](E^u\times F^v\setminus S), \tag{2.6}$$

for $[\nu\otimes\nu]$—almost every $(u,v)\in U\times U$. The property satisfied by $S$ then implies that

$$m(E^u)m(F^v) = [m\otimes m](E^u\times F^v) = 0, \tag{2.7}$$

for $[\nu\otimes\nu]$—almost every $(u,v)\in U\times U$ and by applying to $E\times F$ the same argument used with $H$, we get

$$0 = [\mu\otimes\mu](E\times F) = \mu(E)\mu(F), \tag{2.8}$$

which is the required contradiction. □

Let us point out the special case of the above lemma that will be needed in the proof of our main result. Given an arbitrary nonempty set $\Lambda$, consider the product $[0,1]^\Lambda$ of so many copies of $[0,1]$ as indicated by $\Lambda$, with product measure $\mu$. More concretely, $\mu$ is the unique positive normalized regular Borel measure on $[0,1]^\Lambda$ (provided with the product topology) such that, given a family $\{A_\lambda : \lambda\in\Lambda\}$ of Borel sets in $[0,1]$,

$$\mu\left(\prod_{\lambda\in\Lambda}A_\lambda\right) = \prod_{\lambda\in\Lambda}m(A_\lambda) = \inf\left\{\prod_{\lambda\in J}m(A_\lambda) : J\subset\Lambda,\ J \text{ finite}\right\}, \tag{2.9}$$

if all but countably many $A_\lambda$'s are equal to $[0,1]$ and $\mu(\prod_{\lambda\in\Lambda}A_\lambda) = 0$ otherwise. The space $L_1(\mu)$ is usually denoted by $L_1([0,1]^\Lambda)$ (cf. [8, page 120]). By fixing $\lambda_0\in\Lambda$ and denoting by $\nu$ the product measure on $[0,1]^{\Lambda\setminus\{\lambda_0\}}$ we have clearly $\mu = m\otimes\nu$ and Lemma 2.1 tells us that $\mathcal{AL}^2(L_1[0,1]^\Lambda)$ is not dense in $\mathcal{AL}^2(L_1[0,1]^\Lambda)$.

We need another elementary lemma. By $Y\oplus_1 Z$ we denote the $\ell_1$-sum of two Banach spaces $Y$ and $Z$, i.e., $\|y+z\| = \|y\|+\|z\|$ for arbitrary $y\in Y$, $z\in Z$.

**Lemma 2.2.** *Let $Y$, $Z$ be Banach spaces and $X = Y\oplus_1 Z$. If $\mathcal{AL}^2(X)$ is dense in $\mathcal{L}^2(X)$, then $\mathcal{AL}^2(Y)$ is dense in $\mathcal{L}^2(Y)$.*

**Proof.** Let $\varphi\in\mathcal{AL}^2(Y)$ with $\|\varphi\| = 1$ and $0<\epsilon<(1/2)$ be given. Define $\bar\varphi(x_1,x_2) = \varphi(Px_1,Px_2)$, $\forall x_1,x_2\in X$, where $P : X\to Y$ is the natural projection, and note that $\bar\varphi\in\mathcal{L}^2(X)$, $\|\bar\varphi\| = 1$. Since $\mathcal{AL}^2(X)$ is dense in $\mathcal{L}^2(X)$, there exists $\bar\psi\in\mathcal{AL}^2(X)$ such that $\|\bar\psi-\bar\varphi\|<\epsilon$ and $\|\bar\psi\| = 1$. Let $x_1 = y_1+z_1$, $x_2 = y_2+z_2$ with $y_1,y_2\in Y$, $z_1,z_2\in Z$ be such that $\|y_1\|+\|z_1\| = \|x_1\|\le 1$, $\|y_2\|+\|z_2\| = \|x_2\|\le 1$ and $|\bar\psi(x_1,x_2)| = 1$. Then we have

$$\begin{aligned}|\bar\psi(x_1,x_2)-\bar\psi(y_1,y_2)| &\le |\bar\psi(x_1,x_2)-\bar\varphi(x_1,x_2)| + |\bar\varphi(x_1,x_2)-\bar\psi(y_1,y_2)|\\ &= |\bar\psi(x_1,x_2)-\bar\varphi(x_1,x_2)| + |\bar\varphi(y_1,y_2)-\bar\psi(y_1,y_2)|\\ &\le \|\bar\psi-\bar\varphi\|(\|x_1\|\|x_2\|+\|y_1\|\|y_2\|) < 2\epsilon < 1\end{aligned} \tag{2.10}$$

hence $\bar\psi(y_1,y_2)\ne 0$, so $y_1\ne 0, y_2\ne 0$. Moreover we have

$$\begin{aligned} 1 = |\bar{\psi}(x_1,x_2)| &\le |\bar{\psi}(y_1,y_2)| + |\bar{\psi}(y_1,z_2)| + \bar{\psi}(z_1,y_2)| + |\bar{\psi}(z_1,z_2)| \\ &\le \|y_1\|\|y_2\| + \|y_1\|\|z_2\| + \|z_1\|\|y_2\| + \|z_1\|\|z_2\| \\ &= (\|y_1\| + \|z_1\|)(\|y_2\| + \|z_2\|) = \|x_1\|\|x_2\| \le 1, \end{aligned} \tag{2.11}$$

and it follows that

$$\left|\bar{\psi}\left(\frac{y_1}{\|y_1\|}, \frac{y_2}{\|y_2\|}\right)\right| = 1. \tag{2.12}$$

Now let $\psi$ be the restriction of $\bar{\psi}$ to $Y \times Y$. Clearly $\|\psi\| \le 1$ and the above equality shows that $\psi \in \mathcal{AL}^2(Y)$. Note finally that $\|\psi - \varphi\| \le \|\bar{\psi} - \bar{\varphi}\| < \epsilon$. □

We are now ready for the main result.

**Theorem 2.3.** *Given a finite measure $\mu$, the following statements are equivalent*
(1) *$\mu$ is purely atomic.*
(2) *$\mathcal{AL}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ for any natural number $N$.*
(3) *$\mathcal{AL}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ for some $N \ge 2$.*
(4) *$\mathcal{AL}^2(L_1(\mu))$ is dense in $\mathcal{L}^2(L_1(\mu))$.*

**Proof.** (1)⟹(2). If $\mu$ is purely atomic, then $L_1(\mu)$ has the Radon-Nikodým property, and (2) follows from [2, Theorem 1].

(2)⟹(3). This is trivial.

(3)⟹(4). This is follows from [7, Proposition 2.1].

(4)⟹(1). We use the isometric classification of $L_1$-spaces (see [8, Theorem 14.9]). Arguing by contradiction, we assume that (4) holds and that $\mu$ is not purely atomic. By the above-mentioned theorem, we can write $L_1(\mu)$ in the form

$$L_1(\mu) \cong L_1([0,1]^\Lambda) \oplus_1 Z, \tag{2.13}$$

for some nonempty set $\Lambda$ and some Banach space $Z$. By Lemma 2.2 we get that $\mathcal{AL}^2(L_1([0,1]^\Lambda))$ is dense in $\mathcal{L}^2(L_1([0,1]^\Lambda))$, which contradicts Lemma 2.1. □

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## References

[1] M. D. Acosta, F. J. Aguirre, and R. Payá, *There is no bilinear Bishop-Phelps theorem*, Israel J. Math. **93** (1996), 221–227. MR 97d:46018. Zbl 852.46010.

[2] R. M. Aron, C. Finet, and E. Werner, *Some remarks on norm-attaining n-linear forms*, Function Spaces (Edwardsville, IL, 1994) (New York), Lecture Notes in Pure and Appl. Math., 172, Dekker, 1995, pp. 19–28. MR 96h:46061. Zbl 851.46008.

[3] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. **67** (1961), 97–98. MR 23#A503. Zbl 098.07905.

[4] Y. S. Choi, *Norm attaining bilinear forms on $L^1[0,1]$*, J. Math. Anal. Appl. **211** (1997), no. 1, 295–300. MR 98e:46015. Zbl 888.46007.

[5] Y. S. Choi and S. G. Kim, *Norm or numerical radius attaining multilinear mappings and polynomials*, J. London Math. Soc. (2) **54** (1996), no. 1, 135–147. MR 97f:46029. Zbl 858.47005.

[6] C. Finet and R. Payá, *Norm attaining operators from $L_1$ into $L_\infty$*, Israel J. Math. **108** (1998), 139–143. MR 2000c:47061. Zbl 929.47037.
[7] M. Jiménez-Sevilla and R. Payá, *Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces*, Studia Math. **127** (1998), no. 2, 99–112. MR 98k:46031. Zbl 909.46015.
[8] H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag, New York, 1974. MR 58#12308. Zbl 285.46024.

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