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COMPLETE CONVERGENCE FOR SUMS OF ARRAYS OF RANDOM ELEMENTS

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ABSTRACT. Let $\{X_{ni}\}$ be an array of rowwise independent B-valued random elements and $\{a_n\}$ constants such that $0 < a_n \uparrow \infty$. Under some moment conditions for the array, it is shown that $\sum_{i=1}^{n} X_{ni}/a_n$ converges to 0 completely if and only if $\sum_{i=1}^{n} X_{ni}/a_n$ converges to 0 in probability.

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1. Introduction. Let $(B, \|\|)$ be a real separable Banach space. A separable Banach space B is said to be of type r, $1 \le r \le 2$, if there exists a constants C_r such that

$$E\left\|\sum_{i=1}^{n} X_i\right\|^r \le C_r \sum_{i=1}^{n} E\|X_i\|^r \tag{1.1}$$

for all independent B-valued random elements $X_1, \ldots, X_n$ with mean zero and finite rth moments.

A sequence $\{X_n, n \ge 1\}$ of B-valued random elements is said to converge completely to zero if for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(\|X_n\| > \epsilon) < \infty. \tag{1.2}$$

Note that complete convergence implies almost surely by the Borel-Cantelli lemma.

Now let $\{X_n, n \ge 1\}$ be a sequence of independent random variables. Let $\psi(t)$ be a positive, even and continuous function such that

$$\frac{\psi(|t|)}{|t|} \uparrow \text{ and } \frac{\psi(|t|)}{|t|^2} \downarrow \text{ as } |t| \uparrow. \tag{1.3}$$

Chung [3] strong law of large numbers (SLLN) states that if

$$EX_n = 0 \quad \text{for } n \ge 1, \qquad \sum_{n=1}^{\infty} \frac{E\psi(|X_n|)}{\psi(n)} < \infty \tag{1.4}$$

then

$$\frac{\sum_{i=1}^{n} X_i}{n} \longrightarrow 0 \text{ almost surely.} \tag{1.5}$$

Recently, Hu and Taylor [6] proved Chung type SLLN for arrays of rowwise independent random variables. More specifically, let $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of rowwise independent random variables and let $\{a_n,\ n \ge 1\}$ be a sequence of real numbers with $0 < a_n \uparrow \infty$. Let $\psi(t)$ be a positive, even and continuous function such that

$$\frac{\psi(|t|)}{|t|^p} \uparrow \text{ and } \frac{\psi(|t|)}{|t|^{p+1}} \downarrow \text{ as } |t| \uparrow \tag{1.6}$$

for some integer $p \ge 2$. Furthermore, assume that

$$EX_{ni} = 0 \quad \text{for } 1 \le i \le n,\ n \ge 1, \tag{1.7}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi(|X_{ni}|)}{\psi(a_n)} < \infty, \tag{1.8}$$

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \frac{EX_{ni}^2}{a_n^2} \right)^{2k} < \infty, \tag{1.9}$$

where k is a positive integer. Then the conditions (1.6), (1.7), (1.8), and (1.9) imply

$$\frac{1}{a_n} \sum_{i=1}^{n} X_{ni} \longrightarrow 0 \text{ almost surely.} \tag{1.10}$$

Many classical theorems hold for B-valued random elements under the assumption that the weak law of large numbers (WLLN) holds (see, Kuelbs and Zinn [8], de Acosta [4], Choi and Sung [1, 2], Wang, Rao and Yang [10], Kuczmaszewska and Szynal [7], and Sung [9]).

In this paper, we apply de Acosta [4] inequality to obtain Hu and Taylor's [6] result in a general Banach space under the assumption that WLLN holds.

2. Main Result. To prove our main theorem, we need the following lemma which is due to de Acosta [4].

Lemma 2.1. *For each $p \ge 1$, there exists a positive constant C_p such that for separable Banach space B and any finite sequence $\{X_i,\ 1 \le i \le n\}$ of independent B-valued random elements with $E\|X_i\|^p < \infty$ $(1 \le i \le n)$, the following inequalities hold.*

(i) *For $1 \le p \le 2$,*

$$E\left|\left\|\sum_{i=1}^{n} X_i\right\| - E\left\|\sum_{i=1}^{n} X_i\right\|\right|^p \le C_p \sum_{i=1}^{n} E\|X_i\|^p. \tag{2.1}$$

(ii) *For $p > 2$,*

$$E\left|\left\|\sum_{i=1}^{n} X_i\right\| - E\left\|\sum_{i=1}^{n} X_i\right\|\right|^p \le C_p\left[\left(\sum_{i=1}^{n} E\|X_i\|^2\right)^{p/2} + \sum_{i=1}^{n} E\|X_i\|^p\right]. \tag{2.2}$$

Throughout this paper, let $\psi(t)$ be a positive, even function such that

$$\frac{\psi(|t|)}{|t|} \uparrow \text{ and } \frac{\psi(|t|)}{|t|^p} \downarrow \text{ as } |t| \uparrow \tag{2.3}$$

for some $p \ge 1$.

Theorem 2.2. *Let $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of rowwise independent B-valued random elements and $\{a_n,\ n \ge 1\}$ constants such that $0 < a_n \uparrow \infty$. Assume that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi(\|X_{ni}\|)}{\psi(a_n)} < \infty, \tag{2.4}$$

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E\|X_{ni}\|^2}{a_n^2}\right)^s < \infty \tag{2.5}$$

for some $s > 0$. Then the following statements are equivalent.

(i) $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ *in L^1.*
(ii) $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ *completely.*
(iii) $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ *almost surely.*
(iv) $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ *in probability.*

Proof. (i)$\Longrightarrow$(ii). Define $Y_{ni} = X_{ni}I(\|X_{ni}\| \le a_n)$ and $Z_{ni} = X_{ni}I(\|X_{ni}\| > a_n)$. Since $\psi(|t|)/|t|$ is an increasing function of $|t|$, we have by (2.4) that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{a_n}E\left\|\sum_{i=1}^{n}Z_{ni}\right\| &\le \sum_{n=1}^{\infty}\frac{1}{a_n}\sum_{i=1}^{n}E\|Z_{ni}\| \\ &\le \sum_{n=1}^{\infty}\frac{1}{\psi(a_n)}\sum_{i=1}^{n}E\psi(\|Z_{ni}\|) \le \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi(\|X_{ni}\|)}{\psi(a_n)} < \infty.\end{aligned} \tag{2.6}$$

It follows that

$$\frac{1}{a_n}\sum_{i=1}^{n}Z_{ni} \longrightarrow 0 \text{ completely}. \tag{2.7}$$

The proof will be completed by showing that

$$\frac{1}{a_n}\sum_{i=1}^{n}Y_{ni} \longrightarrow 0 \text{ completely}. \tag{2.8}$$

From (i) and (2.6), we have

$$\begin{aligned}\frac{1}{a_n}E\left\|\sum_{i=1}^{n}Y_{ni}\right\| &= \frac{1}{a_n}E\left\|\sum_{i=1}^{n}(X_{ni}-Z_{ni})\right\| \\ &\le \frac{1}{a_n}E\left\|\sum_{i=1}^{n}X_{ni}\right\| + \frac{1}{a_n}E\left\|\sum_{i=1}^{n}Z_{ni}\right\| \longrightarrow 0.\end{aligned} \tag{2.9}$$

Thus, to prove (2.8), it is enough to show that

$$\frac{1}{a_n}\left\|\sum_{i=1}^{n}Y_{ni}\right\| - \frac{1}{a_n}E\left\|\sum_{i=1}^{n}Y_{ni}\right\| \longrightarrow 0 \text{ completely}. \tag{2.10}$$

First consider the case of $1 \le p \le 2$. From Markov's inequality and Lemma 2.1(i), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|\frac{1}{a_n}\left\|\sum_{i=1}^{n} Y_{ni}\right\|-\frac{1}{a_n}E\left\|\sum_{i=1}^{n} Y_{ni}\right\|\right|>\epsilon\right)\\
&\qquad\le \frac{1}{\epsilon^p}\sum_{n=1}^{\infty}\frac{1}{a_n^p}E\left|\left\|\sum_{i=1}^{n} Y_{ni}\right\|-E\left\|\sum_{i=1}^{n} Y_{ni}\right\|\right|^p\\
&\qquad\le \frac{C_p}{\epsilon^p}\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\|Y_{ni}\|^p}{a_n^p}\le \frac{C_p}{\epsilon^p}\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi(\|Y_{ni}\|)}{\psi(a_n)}\\
&\qquad\le \frac{C_p}{\epsilon^p}\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi(\|X_{ni}\|)}{\psi(a_n)}<\infty,
\end{aligned}\tag{2.11}$$

since $\psi(|t|)/|t|^p\downarrow$ and (2.4). Thus (2.10) holds.

Now consider the case of $p>2$. Note that $\psi(|t|)/|t|^p\downarrow$ implies $\psi(|t|)/|t|^q\downarrow$ for each $q\ge p$. Let $q=\max\{p,2s\}$. Then we have by Markov's inequality and Lemma 2.1(ii) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|\frac{1}{a_n}\left\|\sum_{i=1}^{n} Y_{ni}\right\|-\frac{1}{a_n}E\left\|\sum_{i=1}^{n} Y_{ni}\right\|\right|>\epsilon\right)\\
&\qquad\le \frac{1}{\epsilon^q}\sum_{n=1}^{\infty}\frac{1}{a_n^q}E\left|\left\|\sum_{i=1}^{n} Y_{ni}\right\|-E\left\|\sum_{i=1}^{n} Y_{ni}\right\|\right|^q\\
&\qquad\le \frac{C_q}{\epsilon^q}\sum_{n=1}^{\infty}\frac{1}{a_n^q}\left[\left(\sum_{i=1}^{n}E\|Y_{ni}\|^2\right)^{q/2}+\sum_{i=1}^{n}E\|Y_{ni}\|^q\right]\\
&\qquad= \frac{C_q}{\epsilon^q}\sum_{n=1}^{\infty}\left(\frac{\sum_{i=1}^{n}E\|Y_{ni}\|^2}{a_n^2}\right)^{q/2}+\frac{C_q}{\epsilon^q}\sum_{n=1}^{\infty}\frac{1}{a_n^q}\sum_{i=1}^{n}E\|Y_{ni}\|^q.
\end{aligned}\tag{2.12}$$

Since $q\ge p$, $\psi(|t|)/|t|^p\downarrow$ implies $\psi(|t|)/|t|^q\downarrow$, and so

$$\sum_{n=1}^{\infty}\frac{1}{a_n^q}\sum_{i=1}^{n}E\|Y_{ni}\|^q\le\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi(\|Y_{ni}\|)}{\psi(a_n)}\le\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi(\|X_{ni}\|)}{\psi(a_n)}<\infty.\tag{2.13}$$

Also,

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{\sum_{i=1}^{n}E\|Y_{ni}\|^2}{a_n^2}\right)^{q/2}&\le\left[\sum_{n=1}^{\infty}\left(\frac{\sum_{i=1}^{n}E\|Y_{ni}\|^2}{a_n^2}\right)^{s}\right]^{q/2s}\\
&\le\left[\sum_{n=1}^{\infty}\left(\frac{\sum_{i=1}^{n}E\|X_{ni}\|^2}{a_n^2}\right)^{s}\right]^{q/2s}<\infty,
\end{aligned}\tag{2.14}$$

since $q\ge 2s$ and (2.5). Combining (2.12), (2.13), and (2.14) yields (2.10). Thus (i)$\Longrightarrow$(ii) is proved. Since the implications (ii)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(iv) are obvious, it remains to show that (iv)$\Longrightarrow$(i).

Assume that (iv) holds. From Lemma 2.1(i) and (2.5)

$$E\left|\frac{1}{a_n}\left\|\sum_{i=1}^{n} X_{ni}\right\|-\frac{1}{a_n}E\left\|\sum_{i=1}^{n} X_{ni}\right\|\right|^2\le\frac{C_2}{a_n^2}\sum_{i=1}^{n}E\|X_{ni}\|^2\longrightarrow 0,\tag{2.15}$$

which entails

$$\frac{1}{a_n}\left\|\sum_{i=1}^{n} X_{ni}\right\|-\frac{1}{a_n}E\left\|\sum_{i=1}^{n} X_{ni}\right\|\longrightarrow 0 \text{ in probability.}\tag{2.16}$$

It follows by (iv) that $E\|\sum_{i=1}^{n} X_{ni}\|/a_n \longrightarrow 0$, and so (i) holds. Thus the proof of Theorem 2.2 is completed. □

The following theorem states that Theorem 2.2 holds even if the condition (2.5) is replaced by

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E\|X_{ni}\|^{r}}{a_n^{r}}\right)^{s}<\infty, \tag{2.17}$$

for some $1 \le r \le 2$ and $s > 0$.

Theorem 2.3. *Let $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of rowwise independent B-valued random elements and $\{a_n,\ n \ge 1\}$ constants such that $0 < a_n \uparrow \infty$. Assume that (2.4) and (2.17) hold. Then the following statements are equivalent.*

(i) $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ *in* L^1.
(ii) $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ *completely.*
(iii) $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ *almost surely.*
(iv) $(1/a_n)\sum_{i=1}^{n} X_{ni} \to 0$ *in probability.*

Proof. Let $\{Y_{ni}\}$ and $\{Z_{ni}\}$ be as in the proof of Theorem 2.2. From the proof of (i)⟹(ii) in Theorem 2.2, we have

$$\sum_{n=1}^{\infty}\frac{1}{a_n}E\left\|\sum_{i=1}^{n} Z_{ni}\right\|<\infty, \tag{2.18}$$

which implies $\sum_{i=1}^{n} Z_{ni}/a_n \to 0$ in L^1, completely, almost surely, and in probability. Hence, it is enough to show that

$$\begin{aligned}\frac{1}{a_n}\sum_{i=1}^{n} Y_{ni} \longrightarrow 0 \text{ in } L^1 &\iff \frac{1}{a_n}\sum_{i=1}^{n} Y_{ni} \longrightarrow 0 \text{ completely}\\ &\iff \frac{1}{a_n}\sum_{i=1}^{n} Y_{ni} \longrightarrow 0 \text{ almost surely}\\ &\iff \frac{1}{a_n}\sum_{i=1}^{n} Y_{ni} \longrightarrow 0 \text{ in probability.}\end{aligned} \tag{2.19}$$

Since $Y_{ni} = X_{ni} I(\|X_{ni}\| \le a_n)$, it follows that $E\psi(\|Y_{ni}\|) \le E\psi(\|X_{ni}\|)$ and

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E\|Y_{ni}\|^{2}}{a_n^{2}}\right)^{s} \le \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E\|Y_{ni}\|^{r}}{a_n^{r}}\right)^{s} \le \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E\|X_{ni}\|^{r}}{a_n^{r}}\right)^{s}. \tag{2.20}$$

Thus $\{Y_{ni}\}$ satisfies the conditions of Theorem 2.2, and so (2.19) holds by Theorem 2.2. □

Corollary 2.4. *Let $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of rowwise independent B-valued random elements and $\{a_n,\ n \ge 1\}$ constants such that $0 < a_n \uparrow \infty$. Assume that $EX_{ni} = 0$ and B is of type r $(1 \le r \le 2)$. Then (2.4) and (2.17) imply that*

$$\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \longrightarrow 0 \text{ almost surely.} \tag{2.21}$$

PROOF. By Theorem 2.3, it is enough to show that

$$\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \longrightarrow 0 \text{ in } L^1. \tag{2.22}$$

Since B is of type r and $EX_{ni} = 0$, it follows by (2.17) that

$$E\left\|\frac{1}{a_n}\sum_{i=1}^{n} X_{ni}\right\|^r \le \frac{C_r}{a_n^r}\sum_{i=1}^{n} E\|X_{ni}\|^r \longrightarrow 0, \tag{2.23}$$

and so (2.22) holds. □

REMARK 2.5. The condition (2.3) is weaker than (1.6). Hu and Chung [5] proved Corollary 2.4 under the stronger condition (1.6).

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