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A BOUNDARY VALUE PROBLEM FOR THE WAVE EQUATION

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ABSTRACT. Traditionally, boundary value problems have been studied for elliptic differential equations. The mathematical systems described in these cases turn out to be "*well posed*". However, it is also important, both mathematically and physically, to investigate the question of boundary value problems for hyperbolic partial differential equations. In this regard, prescribing data along characteristics as formulated by Kalmenov [5] is of special interest. The most recent works in this area have resulted in a number of interesting discoveries [3, 4, 5, 7, 8]. Our aim here is to extend some of these results to a more general domain which includes the characteristics of the underlying wave equation as a part of its boundary.

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1. Introduction. The well-known two point boundary value problem for the mass-spring system has an analog in the continuum case which was first formulated in [5, 7] as follows

$$Lu = u_{tt} - u_{xx} = F(x,t), \quad (x,t) \in R, \tag{1.1}$$

$$u(x,0) = 0, \quad 0 \le x \le 2, \tag{1.2}$$

$$u(t,t) = u(1+t, 1-t), \quad 0 \le t \le 1. \tag{1.3}$$

Here, R is a region bounded by the characteristics and the line segment $t = 0, 0 \le x \le 2$, as described below.

$$R = \{(x,t) : t < x < 2-t, 0 < t < 1\}. \tag{1.4}$$

In [5], it is shown that a unique solution $u \in W_2^1(R) \cap W_2^1(\partial R) \cap C(\bar{R})$ [1] of (1.1), (1.2), and (1.3) can be constructed, and that L and L^{-1} are both self adjoint in $L_2(R)$. Furthermore, in the case where $F(x,t) = \lambda u$, a complete set of eigenfunctions and corresponding eigenvalues are explicitly computed. In [7] on the other hand, where $F(x,t) = \lambda p(x,t)u$, such explicit computations are not possible. The selfadjointness of L in $L_2^p(R)$, the set of weighted L_2 functions in R with weight p, and thereby the existence of a complete set of eigenfunctions, is established by constructing a symmetric Hilbert Schmidt kernel [9] for L^{-1}. In addition it is demonstrated that one can replace the boundary conditions (1.2) and (1.3) with

$$u_t(x,0) = 0, \quad 0 \le x \le 2, \qquad u(1,1) = 0, \tag{1.5}$$

and obtain self adjointness. In the subsequent papers [4, 8], some generalizations of these results were considered. These generalizations were focused on the dimension of the space variable while preserving both the complete continuity [9] and self adjointness of the resulting integral operator. Our goal in this paper is to study equation (1.1) along with boundary data prescribed on the characteristics and noncharacteristic C^1 curve. The effect of this change in the boundary is that the symmetry of the kernel of the integral operator is lost. However, one can preserve the compactness of the integral operator and obtain information about the spectrum of the operator L using the theory of compact operators.

2. The boundary value problem. Let $f \in C^1(0,1)$ satisfy $f(0) = 0$, $f'(t) > 1$. Denote by a, $0 < a < 1$, the solution of the implicit equation $f(t) = 2-t$. Define $\Omega \subseteq \mathbb{R}^2$ to be the region

$$\Omega = \{(s,t) : t < s < f(t), 0 < t < a\} \cup \{(s,t) : t < s < 2-t, a < t < 1\}. \tag{2.1}$$

The characteristic curves C_1, C_2 and noncharacteristic Γ are defined by

$$\begin{aligned} C_1 &= \{(s,t) : s = t,\ 0 \le t \le 1\}, \\ C_2 &= \{(s,t) : s = 2-t,\ 0 \le t \le 1\}, \\ \Gamma &= \{(s,t) : s = f(t),\ 0 \le t \le a\}. \end{aligned} \tag{2.2}$$

Let $\tau \in [0,1]$ be fixed. The line $s = -t + f(\tau) + \tau$ meets Γ and C_1 at points S_1 and S_2, respectively, and the line $s = t + f(\tau) - \tau$ meets Γ and C_2 at S_1, S_3. Denote the point of intersection of C_1 and C_2 by S_4. The coordinates of these points are as follows.

$$\begin{aligned} &S_1 = (f(\tau),\tau), \qquad S_2 = \left(\tfrac{1}{2}(f(\tau)+\tau), \tfrac{1}{2}(f(\tau)+\tau)\right), \\ &S_3 = \left(\tfrac{1}{2}(2+f(\tau)-\tau), \tfrac{1}{2}(2-f(\tau)+\tau)\right), \qquad S_4 = (1,1). \end{aligned} \tag{2.3}$$

Let $\alpha \neq 1, 0$, be a constant in $\mathbb{R}$. Consider the boundary value problem

$$u_{tt} - u_{ss} = F(s,t), \quad (s,t) \in \Omega, \tag{2.4}$$

$$u(S_1) - u(S_2) = \alpha\big(u(S_3) - u(S_4)\big), \quad 0 \le \tau \le 1, \tag{2.5}$$

$$u_{|\Gamma} = \omega(t), \quad 0 \le t \le a, \tag{2.6}$$

where ω is known. Our goal is to convert problem (2.4), (2.5), and (2.6) into an integral equation and study the kernel of the resulting integral operator.

3. The solution of the boundary value problem. Make the change of variables

$$x = s - t, \qquad y = s + t. \tag{3.1}$$

Then, $u(s,t) = v(x,y)$ and equation (2.4) has the simple form

$$v_{xy} = \mathcal{F}(x,y), \tag{3.2}$$

for v, where $\mathcal{F}(x,y) = (1/4)F((y+x)/2,(y-x)/2)$. Under the last transformation the region Ω is mapped into

$$T = \{(x,y) : 0 \le x \le M, g(x) \le y \le 2\}, \tag{3.3}$$

where $g(x)$ is the solution of the equation

$$\frac{y+x}{2} = f\left(\frac{y-x}{2}\right), \tag{3.4}$$

for y, and M is the x-coordinate of the point of intersection of $y = g(x)$ and $y = 2$. We note that

$$g' = \frac{f'+1}{f'-1} > 1, \tag{3.5}$$

therefore, the two lines do intersect. Let (σ,η) be an interior point of T. For a fixed $\sigma \in [0,M]$, the boundary condition (2.5) in these coordinates will be

$$v(\sigma,g(\sigma)) - v(0,g(\sigma)) = \alpha(v(\sigma,2) - v(0,2)). \tag{3.6}$$

In order to convert (3.2) and (3.6) into an integral equation, keeping in mind that v along the curve $y = g(x)$ is given, we integrate (3.2) over the rectangle with vertices at $(0,2)$, $(\sigma,2)$, $(0,\eta)$, and (σ,η). We have

$$v(\sigma,2) - v(\sigma,\eta) - v(0,2) + v(0,\eta) = \int_\eta^2 \int_0^\sigma \mathcal{F}(x,y)\,dx\,dy. \tag{3.7}$$

Substitute $g(\sigma)$ for η in (3.7) to obtain

$$v(\sigma,g(\sigma)) - v(0,g(\sigma)) = v(\sigma,2) - v(0,2) - \int_{g(\sigma)}^2 \int_0^\sigma \mathcal{F}(x,y)\,dx\,dy. \tag{3.8}$$

The boundary condition (3.6) and equality (3.8) imply

$$v(\sigma,2) = v(0,2) + \frac{1}{1-\alpha} = \int_{g(\sigma)}^2 \int_0^\sigma \mathcal{F}(x,y)\,dx\,dy. \tag{3.9}$$

If we substitute $g^{-1}(\eta)$ for σ in (3.7) we have

$$v(g^{-1}(\eta),2) - v(g^{-1}(\eta),\eta) - v(0,2) + v(0,\eta) = \int_\eta^2 \int_0^{g^{-1}(\eta)} \mathcal{F}(x,y)\,dx\,dy. \tag{3.10}$$

The substitution of $g^{-1}(\eta)$ for σ in (3.9) yields

$$v(g^{-1}(\eta),2) = v(0,2) + \frac{1}{1-\alpha}\int_\eta^2 \int_0^{g^{-1}(\eta)} \mathcal{F}(x,y)\,dx\,dy. \tag{3.11}$$

In equations (3.10) and (3.11) eliminate $v(g^{-1}(\eta),2) - v(0,2)$ and solve for $v(0,\eta)$, then

$$v(0,\eta) = v(g^{-1}(\eta),\eta) + \frac{-\alpha}{1-\alpha}\int_\eta^2 \int_0^{g^{-1}(\eta)} \mathcal{F}(x,y)\,dx\,dy. \tag{3.12}$$

Substitute $v(0,\eta)$ from (3.12) and $v(\sigma,2)$ from (3.9) into (3.7) and solve for $v(\sigma,\eta)$ to obtain

$$\begin{aligned} v(\sigma,\eta) = v(g^{-1}(\eta),\eta) &+ \frac{1}{1-\alpha}\int_{g(\sigma)}^2 \int_0^\sigma \mathcal{F}(x,y)\,dx\,dy \\ &- \frac{\alpha}{1-\alpha}\int_\eta^2 \int_0^{g^{-1}(\eta)} \mathcal{F}(x,y)\,dx\,dy - \int_\eta^2 \int_0^\sigma \mathcal{F}(x,y)\,dx\,dy. \end{aligned} \tag{3.13}$$

If we combine the integrals in the above equation we can write (3.13) in the more compact form

$$v(\sigma,\eta) = v(g^{-1}(\eta),\eta) + \iint_T G(\sigma,\eta;x,y)\mathcal{F}(x,y)\,dx\,dy, \tag{3.14}$$

where G the Green's function in $T\times T$ is defined as follows. Let (σ,η) be a point in T. Define the sets $T_i \subseteq T$, $1 \le i \le 6$ by

$$\begin{aligned}
T_1 &= \{(x,y): 0 \le x \le \sigma,\, g(x) \le y \le g(\sigma)\},\\
T_2 &= \{(x,y): 0 \le x \le \sigma,\, g(\sigma) \le y \le \eta\},\\
T_3 &= \{(x,y): \sigma \le x \le g^{-1}(\eta),\, g(x) \le y \le \eta\},\\
T_4 &= \{(x,y): 0 \le x \le \sigma,\, \eta \le y \le 2\},\\
T_5 &= \{(x,y): \sigma \le x \le g^{-1}(\eta),\, \eta \le y \le 2\},\\
T_6 &= \{(x,y): g^{-1}(\eta) \le x \le M,\, g(x) \le y \le 2\}.
\end{aligned} \tag{3.15}$$

Then G, has values

$$\begin{aligned}
G(\sigma,\eta;x,y) &= 0, \quad (x,y) \in T_1 \cup T_3 \cup T_4 \cup T_6,\ (\sigma,\eta) \in T,\\
G(\sigma,\eta;x,y) &= \frac{1}{1-\alpha}, \quad (x,y) \in T_2,\ (\sigma,\eta) \in T,\\
G(\sigma,\eta;x,y) &= \frac{-\alpha}{1-\alpha}, \quad (x,y) \in T_5,\ (\sigma,\eta) \in T.
\end{aligned} \tag{3.16}$$

Remark. We note that for $\alpha = -1$ the function G is symmetric. Also, we had chosen the positive semi axis $t = 0$ instead of $s = f(t)$, we would have had $y = x$ as a part of the boundary of T. If, in addition, datum on $t = 0$ was chosen to be zero then the result of [7] would be applicable to v.

We have the following uniqueness result.

Theorem 3.1. *The problems (3.2) and (3.6) have a unique solution in $C^2(T)$.*

Proof. Let v_1 and v_2 be two solutions. Then their difference V satisfies the following equations:

$$V_{xy} = 0, \quad (x,y) \in T, \tag{3.17}$$

$$V(0,g(\sigma)) = -\alpha V(\sigma,2), \quad 0 \le \sigma \le M, \tag{3.18}$$

$$V(\sigma,g(\sigma)) = 0, \quad 0 \le \sigma \le M. \tag{3.19}$$

The condition (3.18) is the simplified version of condition (3.6) due to the compatibility condition

$$V(\sigma,g(\sigma)) = 0 = V(0,2) = V(0,0), \quad 0 \le \sigma \le M. \tag{3.20}$$

Choose a point (σ,η) in T, and integrate equation (3.17) over the rectangle in T with vertices at $(0,\eta)$, (σ,η), $(\sigma,2)$, and $(0,2)$. We will have,

$$V(\sigma,2) - V(\sigma,\eta) + V(0,\eta) = 0. \tag{3.21}$$

Also, if we substitute $g(\sigma)$ for η in (3.21) and use the compatibility condition we have

$$V(\sigma,2) + V(0,g(\sigma)) = 0. \tag{3.22}$$

This equation together with the condition (3.18) imply that $V(\sigma,2)$ and $V(0,g(\sigma))$ are both zero. But these are the values of V along $y=2$ and $x=0$. Substituting zero for $V(\sigma,2)$ and $V(0,\eta)$ in (3.21), we will have $V(\sigma,\eta)\equiv 0$. □

The case that the right-hand side of equation (3.2) depends on the unknown function v is treated differently. We will consider this situation in two separate settings next.

4. The eigenvalue problem

4.1. The datum along Γ is identically zero. Let λ' be a parameter, and p a nonnegative measurable function in $L_\infty(T)$. Denote by B the Banach space of weighted square integrable functions, i.e.,

$$B = L_2^p(T) = \left\{\phi : \iint_T |\phi|^2 p\,dx\,dy < \infty\right\}, \tag{4.1}$$

with norm in B defined by,

$$\|\phi\|_2^p = \left(\iint_T |\phi|^2 p\,dx\,dy\right)^{1/2}. \tag{4.2}$$

In what follows we consider the equation

$$v_{xy} = \lambda' p v, \quad (x,y)\in T \tag{4.3}$$

along with the conditions (3.18) and (3.19). The integration of equation (4.3), over a rectangle inside the region T as before yields the following integral equation for v,

$$v(\sigma,\eta) = \lambda' \iint_T G(\sigma,\eta;x,y)pv\,dx\,dy, \tag{4.4}$$

where G is the same Green's function used as a kernel of the integral in equation (3.14). Equation (4.4) can be written in the operator notation form

$$\lambda v = Kv, \tag{4.5}$$

where, for $\phi\in B$,

$$(K\phi)(\sigma,\eta) = \iint_T G(\sigma,\eta;x,y)\phi(x,y)p(x,y)\,dx\,dy, \tag{4.6}$$

and $\lambda = 1/\lambda'$. For $v\in B$, (4.5) is an eigenvalue problem of functional analysis. The theory of the spectrum of a compact operator [2, 9] allows us to investigate the solutions of this operator equation. Our task is to show that K is a compact (or completely continuous, e.g., [9]) operator in B. Here, for the sake of convenience, we include a few definitions that we will use in the sequel.

Definition 4.1. A set of functions $\{\phi\}$ is said to be uniformly bounded if there exists a constant c such that $|\phi|\le c$, for all functions ϕ in the set.

Definition 4.2. A set of continuous functions $\{\phi\}$ is said to be equicontinuous in a region Ω, if given ϵ, there exists $\delta(\epsilon)$ such that for any two points P_1, P_2 in Ω,

$$|P_1 - P_2| < \epsilon, \tag{4.7}$$

implies

$$|\phi(P_1) - \phi(P_2)| < \delta(\epsilon), \tag{4.8}$$

for all functions ϕ in the set [2]. According to Arzela's theorem [9], any uniformly bounded equicontinuous set $\{\phi\}$ of functions has a uniformly convergent subset.

Theorem 4.1. *The operator K is compact in B.*

Proof. Consider the set $\{\phi\} \subset B$ of functions uniformly bounded in the norm $\|\cdot\|_2^p$ so that,

$$\left(\|\phi\|_2^p\right)^2 = \iint_T |\phi|^2 p\,dx\,dy \le L'. \tag{4.9}$$

We have,

$$|K\phi|^2 \le \iint_T |G(\sigma,\eta;x,y)|^2 p\,dx\,dy \iint_T |\phi|^2 p\,dx\,dy, \tag{4.10}$$

by Hölder inequality. The first integral on the right-hand side of inequality (4.10) is bounded because G is bounded and p is integrable in T. The second integral is bounded because of the assumption (4.9). Therefore, the bound on the right-hand side of the inequality (4.10) is independent of ϕ, i.e., the set $\{K\phi\}$ is uniformly bounded. Now, let $\epsilon > 0$ and choose any two points P_1, P_2 in T, so that $|P_1 - P_2| < \epsilon$. To show that $|(K\phi)(P_1) - K(\phi)(P_2)| < \delta(\epsilon)$, independent of P_1, P_2, we have by Hölder inequality,

$$\begin{aligned} &|K\phi(P_1) - k\phi(P_2)|^2 \\ &\quad \le \iint_T |G(P_1;x,y) - G(P_2;x,y)|^2 p\,dx\,dy \iint_T |\phi|^2 p\,dx\,dy. \end{aligned} \tag{4.11}$$

The first integral on the right-hand side can be made small as follows. Let P_1 and P_2 have coordinates (σ,η) and (σ',η'), respectively. Assume without loss of generality that $\sigma' < \sigma$ and $\eta' < \eta$. Define the rectangles R_1–R_6 in T as follows (Figure 1)

$$\begin{aligned} R_1 &= \{(x,y) : 0 \le x \le \sigma',\ \eta' \le y \le \eta\}, \\ R_2 &= \{(x,y) : 0 \le x \le \sigma',\ g(\sigma') \le y \le g(\sigma)\}, \\ R_3 &= \{(x,y) : \sigma' \le x \le \sigma,\ g(\sigma) \le y \le \eta'\}, \\ R_4 &= \{(x,y) : \sigma' \le x \le \sigma,\ \eta \le y \le 2\}, \\ R_5 &= \{(x,y) : \sigma \le x \le g^{-1}(\eta'),\ \eta' \le y \le \eta\}, \\ R_6 &= \{(x,y) : g^{-1}(\eta') \le x \le g^{-1}(\eta),\ \eta \le y \le 2\}. \end{aligned} \tag{4.12}$$

We note that, since g is a continuous function, the area ΔR_i of each rectangle is a function of ϵ. Let $A(\epsilon)$ represent,

$$A(\epsilon) = \max\{\Delta R_i,\ i = 1,\dots,6\}. \tag{4.13}$$

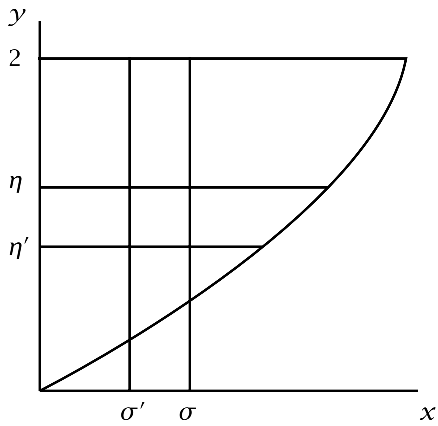

FIGURE 1. The regions R_i.

The difference $|G(P_1;x,y)-G(P_2;x,y)|$ takes on the following values in T.

$$\begin{aligned}
|G(P_1;x,y)-G(P_2;x,y)| &= \left|\frac{1}{1-\alpha}\right|, \quad (x,y)\in R_1\cup R_2\cup R_3,\\
|G(P_1;x,y)-G(P_2;x,y)| &= \left|\frac{\alpha}{1-\alpha}\right|, \quad (x,y)\in R_4\cup R_5\cup R_6,\\
|G(P_1;x,y)-G(P_2;x,y)| &= 0, \quad (x,y)\in T-\bigcup_{i=1}^{6}R_i.
\end{aligned}\tag{4.14}$$

Let

$$M' = \max_T |G(P_1;x,y)-G(P_2;x,y)| = \max\left\{\left|\frac{1}{1-\alpha}\right|,\left|\frac{\alpha}{1-\alpha}\right|\right\}\tag{4.15}$$

which is independent of P_1 and P_2. Then we have,

$$\iint_T |G(P_1;x,y)-G(P_2;x,y)|^2 p\,dx\,dy \le M'^2\sum_i\iint_{R_i} p\,dx\,dy \le 6M'^2\|p\|_\infty A(\epsilon).\tag{4.16}$$

The inequalities (4.10), (4.11), and (4.16) imply that

$$|K\phi(P_1)-K\phi(P_2)| \le \delta(\epsilon),\tag{4.17}$$

where $\delta(\epsilon) = 6L'M'^2\|p\|_\infty A(\epsilon)$. This shows that the set $\{K\phi\}$ is equicontinuous. Therefore, by Arzela's theorem $\{K\phi\}$ has a uniformly convergent subset. But uniform convergence implies convergence in $\|\cdot\|_2^p$ norm. Hence, K maps a bounded subset of B to a compact set, i.e., K is a compact operator. □

Now, we are in a position to discuss the spectrum of K. We note first that, the conjugate operator K^* is generated by the kernel $G(x,y;\sigma,\eta)$, and therefore is bounded. This suffices to imply that K^*K is compact. Furthermore, if we equip the space B with the inner product

$$(\phi,\psi)=\iint_T \phi\overline{\psi}p\,dx\,dy, \tag{4.18}$$

the operator K^*K is also selfadjoint and positive over B. The eigenvalues of K^*K are positive and countable. We let J be the index set of the eigenvalues of K^*K and call μ_j,

$$\mu_j=\sqrt{\lambda_j},\quad j\in J \tag{4.19}$$

the singular values of K. Also, $\mathcal{N}(K)$ will represent the space of functions in $L_2^p(T)$ that are mapped to zero by K [6],

$$\mathcal{N}(K)=\{\phi : K\phi=0\}. \tag{4.20}$$

We have the following theorem [6, 9]

Theorem 4.2 (Singular Value Decomposition). *Let $\mu_1\geq\mu_2\geq\mu_3\cdots>0$ be the ordered sequence of positive singular values of K, counted relative to its multiplicity. Then there exists orthonormal systems (ϕ_j) and (ψ_j) both subsets of $L_2^p(T)$ with the following properties:*

$$K\phi_j=\mu_j\psi_j\quad\text{and}\quad K^*\psi_j=\mu_j\phi_j\quad\text{for all } j\in J. \tag{4.21}$$

The system (μ_j,ϕ_j,ψ_j) is called the singular system of K. Every ϕ in $L_2^p(T)$ possesses the singular value decomposition

$$\phi=\phi_0+\sum_{j\in J}(\phi,\phi_j)\phi_j \tag{4.22}$$

for some $\phi_0\in\mathcal{N}(K)$ and

$$K\phi=\sum_{j\in J}\mu_j(\phi,\phi_j)\psi_j. \tag{4.23}$$

A necessary and sufficient condition for the existence of a solution of equation (4.5) is given by a theorem of Picard [6].

Theorem 4.3. *Let (μ_j,v_j,u_j) be a singular system for the compact operator K. A necessary and sufficient condition for the equation (4.5) to be solvable is that*

$$v\in\mathcal{N}(K^*)^{\perp}\qquad\text{and}\qquad\sum_{j\in J}\frac{1}{\mu_j^2}\left|(v,u_j)\right|^2<\infty. \tag{4.24}$$

In this case

$$v=\sum_{j\in J}\frac{1}{\mu_j}(v,u_j)v_j \tag{4.25}$$

is a solution of equation (4.5).

4.2. The datum along Γ is not identically zero. We now turn to the more general case where the datum along the curve $y=g(x)$ is not identically zero and the right-hand side of (3.2) depends on v. The problem we are considering is

$$v_{xy} = \lambda' p v, \quad (x,y) \in T, \tag{4.26}$$

$$v(\sigma,g(\sigma)) - v(0,g(\sigma)) = \alpha(v(\sigma,2) - v(0,2)), \quad 0 \le \sigma \le M, \tag{4.27}$$

$$v(\sigma,g(\sigma)) = \omega(\sigma), \quad 0 \le \sigma \le M. \tag{4.28}$$

The method of Section 3 shows that the solution of (4.26), (4.27), and (4.28) is given by the nonhomogeneous integral equation

$$v(\sigma,\eta) = \omega(g^{-1}(\eta)) + \lambda' \iint_T G(\sigma,\eta;x,y) p v \, dx \, dy, \quad (\sigma,\eta) \in T. \tag{4.29}$$

Employing the operator notation of (4.5), we can rewrite equation (4.29) in the form

$$v = \omega + \lambda' K v. \tag{4.30}$$

For the discussion on the existence of a solution to (4.30), we use *Fredholm Alternative* theorem [2, 9] for compact operators. We first note that the problem conjugate to (4.26), (4.27), and (4.28) is of the form

$$v'_{xy} = \lambda' p v', \quad (x,y) \in T, \tag{4.31}$$

$$v'(\sigma,g(\sigma)) - v'(0,g(\sigma)) = \frac{1}{\alpha}(v'(\sigma,2) - v'(0,2)), \quad 0 \le \sigma \le M, \tag{4.32}$$

$$v'(\sigma,g(\sigma)) = \omega'(\sigma), \quad 0 \le \sigma \le M. \tag{4.33}$$

The solution of (4.31), (4.32), and (4.33) is

$$\begin{aligned} v'(\sigma,\eta) = {} & \omega'(g^{-1}(\eta)) \\ & + \lambda' \iint_T G^*(\sigma,\eta;x,y) p(x,y) v'(x,y) \, dx \, dy, \quad (\sigma,\eta) \in T, \end{aligned} \tag{4.34}$$

as it can be verified by direct substitution. We write (4.34) in the abbreviated form

$$v' = \omega' + \lambda' K^* v', \tag{4.35}$$

and state the following result

Theorem 4.4. *The nonhomogeneous equations*

$$v = \omega + \lambda' K v, \tag{4.36}$$

$$v' = \omega' + \lambda' K^* v', \tag{4.37}$$

have unique solutions for any ω, and ω' in B, if and only if the homogeneous equations

$$v = \lambda' K v, \tag{4.38}$$

$$v' = \lambda' K^* v', \tag{4.39}$$

have only the zero solutions. Furthermore, if one of the homogeneous equations has a nonzero solution, then they both have the same finite number of linearly independent solutions. In this case the equations (4.36) and (4.37) have solutions if and only if ω and ω' are orthogonal to all solutions of (4.39) and (4.38), respectively. Moreover, the general solution for (4.36) is found by adding a particular solution of (4.36) to the general solution of (4.38).

5. Higher dimensions. The extension of the boundary value problem (1.1), (1.2), and (1.3) to analogous selfadjoint problems in higher dimensions is also of much interest. In [8], the equation (1.1) in n-dimensional space

$$L = u_{tt} - \Delta u = \lambda u, \tag{5.1}$$

is considered. It is shown there that, in a characteristic cone, it is possible to construct symmetric Green functions which will convert the operator L of (5.1) to a selfadjoint integral operator. However, the boundary values that give rise to such selfadjoint problems are not known. In [4], a two space dimensional extension of (1.1) with boundary conditions analogous to (1.2) and (1.3) is studied. Here, we give an example of an n-dimensional boundary value problem that can be reduced to (1.1), (1.2), and (1.3), due to its special form and radial symmetry. Let $x \in \mathbb{R}^n$ have the Euclidean norm $|x| = r$. Let $f \in C^1(0,1)$ be the function of Section 2. Define $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$ by

$$\Omega = \{(x,t) : t \le |x| \le f(t), 0 \le t \le a\} \cup \{(x,t) : t \le |x| \le 2-t, a \le t \le 1\}. \tag{5.2}$$

The characteristic surfaces C_1, C_2 and noncharacteristic Γ are defined by

$$C_1 = \{(x,t) : r = t,\ 0 \le t \le 1\}, \tag{5.3}$$

$$C_2 = \{(x,t) : r = 2-t,\ 0 \le t \le 1\}, \tag{5.4}$$

$$\Gamma = \{(x,t) : r = f(t),\ 0 \le t \le a\}. \tag{5.5}$$

The points S_1–S_4 of Section 2 correspond to $(n-1)$-spheres that lie on the boundary of Ω. Let $\alpha \ne 1$, 0, be a constant in $\mathbb{R}$. Consider the boundary value problem

$$u_{tt} - \Delta u - \frac{(n-1)(n-3)}{4|x|^2} u = F(|x|,t), \quad (x,t) \in \Omega, \tag{5.6}$$

$$u_{|S_1} - u_{|S_2} = \alpha\left(u_{|S_3} - u_{|S_4}\right), \quad 0 \le \tau \le 1, \tag{5.7}$$

$$u_{|\Gamma} = \omega(t), \quad 0 \le t \le a. \tag{5.8}$$

Assume that $u(x,t) = u(r,t)$, and make the change of variables to polar coordinates. Equation (5.6) becomes

$$u_{tt} - u_{rr} - \left(\frac{n-1}{r}\right) u_r - \frac{(n-1)(n-3)}{4r^2} u = F(r,t). \tag{5.9}$$

Upon further change of variable $r^{(1-n)/2} u = U$ we have

$$U_{tt} - U_{rr} = r^{(n-1)/2} F(r,t). \tag{5.10}$$

Equation (5.10) along with conditions (5.7) and (5.8) can be treated by the procedures of Sections 2 and 3. The only difference now is that when dealing with the case $F(r,t) = \lambda p(r,t) u$, the factor r^{n-1} appears on the right-hand side of (5.10), i.e., we will have

$$U_{tt} - U_{rr} = \lambda r^{n-1} p(r,t) U. \tag{5.11}$$

However, since this factor is continuous over Ω, it does not pose any new difficulty. We can modify the weight function to $P(r,t) = r^{n-1} p(r,t)$, proceed as before and obtain the results of Section 4.

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