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SOME APPLICATIONS OF A DIFFERENTIAL SUBORDINATION

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ABSTRACT. A number of interesting criteria were given by earlier workers for a normalized analytic function to be in the familiar class $\mathcal{S}^*$ of starlike functions. The main object of the present paper is to extend and improve each of these earlier results. An application associated with an integral operator $\mathcal{F}_c\,(c>-1)$ is also considered.

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1. Introduction. Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z)=z+\sum_{k=n+1}^{\infty}a_k z^k,\quad (n\in\mathbb{N}:=\{1,2,3,\dots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U}=\{z:z\in\mathbb{C}\text{ and }|z|<1\}$. Also, let $\mathcal{S}^*$ be the class of starlike functions in $\mathcal{U}$, defined by (cf., e.g., [2, 11])

$$\mathcal{S}^*:=\left\{f(z)\in\mathcal{A}(1):\Re\left(\frac{zf'(z)}{f(z)}\right)>0,\,(z\in\mathcal{U})\right\}. \tag{1.2}$$

For analytic functions $g(z)$ and $h(z)$ with $g(0)=h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $w(z)$ such that $w(0)=0$, $|w(z)|<1$, $(z\in\mathcal{U})$, and $g(z)=h(w(z))$. We denote this subordination by $g(z)\prec h(z)$.

For a function $f(z)$ belonging to the class $\mathcal{A}(1)$, Bernardi [1] defined the integral operator $\mathcal{F}_c$ as follows:

$$(\mathcal{F}_c f)(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt,\quad (c>-1;\ z\in\mathcal{U}). \tag{1.3}$$

We note that $\mathcal{F}_c f\in\mathcal{A}(n)$ if $f\in\mathcal{A}(n)$. In particular, the operator $\mathcal{F}_1$ was studied earlier by Libera [3]. (Also, see Owa and Srivastava [8, p. 126 et seq.]).

R. Singh and S. Singh [10] proved that if $f(z)\in\mathcal{A}(1)$ and

$$\Re\{f'(z)+zf''(z)\}>-\tfrac{1}{4},\quad (z\in\mathcal{U}), \tag{1.4}$$

then $f(z)\in\mathcal{S}^*$.

Recently, Yi and Ding [12] improved the above-mentioned result of R. Singh and S. Singh [10] by showing that if $f(z)\in\mathcal{A}(1)$ and

$$\Re\{f'(z)+zf''(z)\}>1-\frac{3}{4(1-\log 2)^2+2}\approx -0.263,\quad (z\in\mathcal{U}), \tag{1.5}$$

then $f(z) \in \mathcal{S}^*$.

Furthermore, Nunokawa and Thomas [6] proved that if $f(z) \in \mathcal{A}(1)$ and

$$\Re\{f'(z)\} > -0.0175\ldots, \quad (z \in \mathcal{U}), \tag{1.6}$$

then $\mathcal{F}_1 f \in \mathcal{S}^*$

In this paper, we extend and improve each of these earlier results in [6, 12] and also consider an interesting application associated with the integral operator $\mathcal{F}_c$.

2. Preliminary results.

The following results are required in our investigation.

Lemma 1 (Yi and Ding [12, Lem. 1]). *Suppose that the function $\phi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$ satisfies the condition $\Re\{\phi(ix, y; z)\} \le \delta$ for all real x and $y \le -(1/2)(1+x^2)$ and all $z \in \mathcal{U}$. If $p(z) = 1 + p_1 z + p_2 z + \cdots$ is analytic in $\mathcal{U}$ and*

$$\Re\{\phi(p(z), zp'(z); z)\} > \delta, \quad (z \in \mathcal{U}), \tag{2.1}$$

then $\Re\{p(z)\} > 0$ in $\mathcal{U}$.

Lemma 2 (Owa and Nunokawa [7, Thm. 1]). *Let $p(z)$ be analytic in $\mathcal{U}$ with*

$$p(0) = 1, \qquad p'(0) = \cdots = p^{(n-1)}(0) = 0. \tag{2.2}$$

If $p(z)$ satisfies the inequality

$$\Re\{p(z) + \alpha z p'(z)\} > \beta, \quad (z \in \mathcal{U}), \tag{2.3}$$

then

$$\Re\{p(z)\} > \beta + (1-\beta)\left\{2\int_0^1 \frac{d\rho}{1+\rho^{n\Re(\alpha)}} - 1\right\}, \quad (z \in \mathcal{U}), \tag{2.4}$$

where $\alpha \ne 0$, $\Re(\alpha) \ge 0$, and $\beta < 1$.

Lemma 3 (Owa and Nunokawa [7, Ex. 1]). *Let $\alpha > 0$ and $\beta < 1$. If $f(z) \in \mathcal{A}(n)$ satisfies the inequality*

$$\Re\{f'(z) + \alpha z f''(z)\} > \beta, \quad (z \in \mathcal{U}), \tag{2.5}$$

then

$$\Re\{f'(z)\} > \beta + (1-\beta)\{2\delta(n,\alpha) - 1\}, \quad (z \in \mathcal{U}), \tag{2.6}$$

where

$$\delta(n,\alpha) = \int_0^1 \frac{d\rho}{1+\rho^{n\alpha}}. \tag{2.7}$$

Incidentally, the value of $\delta(n,\alpha)$ in (2.7) can be expressed as the Gauss hypergeometric function

$${}_2F_1\left(1, \frac{1}{n\alpha}; 1 + \frac{1}{n\alpha}; -1\right) \tag{2.8}$$

which may also be rewritten in terms of the difference of two Digamma (or ψ-) functions

$$\frac{1}{2n\alpha}\left[\psi\left(\frac{1+n\alpha}{2n\alpha}\right) - \psi\left(\frac{1}{2n\alpha}\right)\right] \quad \left(\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}\right). \tag{2.9}$$

We also note that the inequality (2.5) is equivalent to the subordination given by

$$f'(z)+\alpha z f''(z) \prec \frac{1+(1-2\beta)z}{1-z}. \tag{2.10}$$

3. Main results. The following theorem is a generalization of the main result of Yi and Ding [12].

Theorem. *Let $\delta(n,\alpha)$ be as defined in Lemma 3 and let $\theta = 0.911621907$, $\alpha \geq 0.17418$, and*

$$\alpha - \frac{(1-\alpha)^2}{3\alpha}\tan^2\theta < \frac{2\delta(n,\alpha)-1}{\{1-\delta(n,\alpha)\}\{2\delta(n,1)-1\}}. \tag{3.1}$$

If $f \in \mathcal{A}(n)$ satisfies the inequality

$$\Re\{f'(z)+\alpha z f''(z)\} > 1-\frac{\dfrac{2}{\alpha}+\left(1-\dfrac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}{\dfrac{2}{\alpha}+4\{1-\delta(n,1)\}\{1-\delta(n,\alpha)\}\left(1-\dfrac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}, \quad (z\in\mathcal{U}), \tag{3.2}$$

then $f(z) \in \mathcal{S}^*$.

Proof. Making use of Lemma 3 and the inequality (3.2), we obtain

$$\begin{aligned}\Re\{f'(z)\} &> \beta+(1-\beta)\{2\delta(n,\alpha)-1\}\\ &= 2\{\delta(n,\alpha)-1\}\left[\frac{\dfrac{2}{\alpha}+\left(1-\dfrac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}{\dfrac{2}{\alpha}+4\{1-\delta(n,1)\}\{1-\delta(n,\alpha)\}\left(1-\dfrac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}\right]+1\\ &=: \gamma, \quad (z\in\mathcal{U}),\end{aligned} \tag{3.3}$$

where

$$\beta = 1-\frac{\dfrac{2}{\alpha}+\left(1-\dfrac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}{\dfrac{2}{\alpha}+4\{1-\delta(n,1)\}\{1-\delta(n,\alpha)\}\left(1-\dfrac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}. \tag{3.4}$$

Since $\alpha \geq 0.17418$ and

$$\tfrac{1}{2} < \delta(n,\alpha) < 1, \quad (\alpha > 0;\ n\in\mathbb{N}), \tag{3.5}$$

we have

$$\frac{2}{\alpha}+4\{1-\delta(n,1)\}\{1-\delta(n,\alpha)\}\left(1-\frac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right) > 0. \tag{3.6}$$

Hence, by (3.1), we find from (3.3) that

$$0 < \gamma < 1. \tag{3.7}$$

If we put $p(z) = z^{-1}f(z)$, then

$$\Re\{f'(z)\} = \Re\{p(z)+zp'(z)\} > \gamma, \quad (z\in\mathcal{U}), \tag{3.8}$$

which, in view of Lemma 2, implies that

$$\Re\left\{\frac{f(z)}{z}\right\} > \gamma + (1-\gamma)\{2\delta(n,1)-1\}, \quad (z \in \mathcal{U}). \tag{3.9}$$

By using (3.5) and (3.7), we get

$$\Re\left\{\frac{f(z)}{z}\right\} > 0, \quad (z \in \mathcal{U}). \tag{3.10}$$

Next, we let

$$q(z) = \frac{zf'(z)}{f(z)} \qquad \text{and} \qquad \lambda(z) = \frac{f(z)}{z}. \tag{3.11}$$

Then

$$\Re\{\lambda(z)\} > \gamma + (1-\gamma)\{2\delta(n,1)-1\}, \quad (z \in \mathcal{U}) \tag{3.12}$$

and

$$\begin{aligned} f'(z) + \alpha z f''(z) &= \lambda(z)\left[\alpha z q'(z) + (1-\alpha)q(z) + \alpha\{q(z)\}^2\right] \\ &= \phi(q(z), zq'(z); z), \end{aligned} \tag{3.13}$$

where $\phi(u,v;z) = \lambda(z)[\alpha u^2 + (1-\alpha)u + \alpha v]$.

By setting $\lambda(z) = a + bi$, we get

$$\begin{aligned} \Re\{\phi(ix,y;z)\} &\leq -\frac{1}{2}\{3\alpha a x^2 + 2b(1-\alpha)x + \alpha a\} \\ &\leq -\frac{a}{2}\left\{\alpha - \frac{1}{3\alpha}(1-\alpha)^2\left(\frac{b}{a}\right)^2\right\} \end{aligned} \tag{3.14}$$

for all real x and $y \leq -(1/2)(1+x^2)$. Since $\Re\{f'(z)\} > 0\,(z \in \mathcal{U})$ implies that $\lambda(z) \prec L(z) := -1 - (2/z)\log(1-z)$, we have $\lambda(\mathcal{U}) \subset L(\mathcal{U})$, where (see [9])

$$L(\mathcal{U}) \subset \{\omega : \Re(\omega) > 2\log 2 - 1\} \cap \{\omega : |\Im(\omega)| < \pi\} \cap \{\omega : |\arg(\omega)| < \theta = 0.911621907\}. \tag{3.15}$$

By using (3.9) and (3.14), we obtain

$$\begin{aligned} \Re\{\phi(ix,y;z)\} &\leq -\frac{a}{2}\left\{\alpha - \frac{(1-\alpha)^2}{3\alpha}\tan^2\theta\right\} \\ &\leq \beta, \quad (z \in \mathcal{U}). \end{aligned} \tag{3.16}$$

Hence, by Lemma 1, we get

$$\Re\{q(z)\} = \Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathcal{U}). \tag{3.17}$$

This evidently completes the proof of the theorem. □

Corollary 1. *Let* $\theta = 0.911621907$, $\alpha \geq 0.17418$, *and*

$$\alpha - \frac{(1-\alpha)^2}{3\alpha}\tan^2\theta < \frac{2\delta(1,\alpha)-1}{\{1-\delta(1,\alpha)\}(2\log 2 - 1)}. \tag{3.18}$$

If $f \in \mathcal{A}(1)$ *satisfies the inequality*

$$\Re\{f'(z)+\alpha z f''(z)\} > 1 - \frac{\dfrac{2}{\alpha} + \left(1 - \dfrac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}{\dfrac{2}{\alpha} + 4(1-\log 2)\{1-\delta(1,\alpha)\}\left(1 - \dfrac{(1-\alpha)^2}{3\alpha^2}\tan^2\theta\right)}, \quad (z \in \mathcal{U}), \quad (3.19)$$

then $f(z) \in \mathcal{S}^*$.

Remark 1. For $\alpha = 1$, Corollary 1 immediately yields the main result of Yi and Ding [12, Thm., p. 614].

Remark 2. A result of Ponnusamy [9, Thm. 4] can be obtained by taking $\beta = 0$ in the proof of our theorem.

It is not difficult to apply the definition (1.3) in order to show that

$$f'(z) = (\mathcal{F}_c f)'(z) + \frac{1}{c+1} z (\mathcal{F}_c f)''(z). \quad (3.20)$$

Thus, by the theorem, we arrive at the following application:

Corollary 2. *Let* $\theta = 0.911621907$, $-1 < c \le 4.741187$, *and*

$$\frac{1}{c+1} - \frac{c^2}{3(c+1)}\tan^2\theta < \frac{2\delta\left(n, \dfrac{1}{c+1}\right) - 1}{\left\{1 - \delta\left(1, \dfrac{1}{c+1}\right)\right\}\{2\delta(n,1)-1\}}. \quad (3.21)$$

If $f \in \mathcal{A}(n)$ *satisfies the inequality*

$$\Re\{f'(z)\} > 1 - \frac{2(c+1) + \left(1 - \dfrac{1}{3}c^2\tan^2\theta\right)}{2(c+1) + 4\{1-\delta(n,1)\}\left\{1 - \delta\left(n, \dfrac{1}{c+1}\right)\right\}\left(1 - \dfrac{1}{3}c^2\tan^2\theta\right)}, \quad (z \in \mathcal{U}) \quad (3.22)$$

then $\mathcal{F}_c f \in \mathcal{S}^*$, *where* $\mathcal{F}_c$ *is defined by (1.3).*

By setting $c = n = 1$ in Corollary 2, we obtain Corollary 3 below, which shows that the constant -0.0175 in the inequality (1.6) of Nunokawa and Thomas [6] can be reduced further.

Corollary 3. *Let* $\theta = 0.911621907$. *If* $f \in \mathcal{A}(1)$ *satisfies the inequality*

$$\Re\{f'(z)\} > 1 - \frac{5 - (1/3)\tan^2\theta}{4 + 8(1-\log 2)^2\left(1 - (1/3)\tan^2\theta\right)} \approx -0.025311\ldots, \quad (z \in \mathcal{U}), \quad (3.23)$$

then $\mathcal{F}_1 f \in \mathcal{S}^*$.

Proof. Since

$$\frac{1}{2} - \frac{1}{6}\tan^2\theta = 0.222356 \ (\theta = 0.911621907) \quad \text{and} \quad \frac{3 - 4\log 2}{(2\log 2 - 1)^2} = 1.523967\ldots, \quad (3.24)$$

the proof of Corollary 3 is completed by setting $c = n = 1$ in Corollary 2. □

Remark 3. Several nonsharp results, obtained by various other authors (cf., e.g., [9]), correspond to the further special cases of Corollary 2 when $c = 0$ and $c = 1$.

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