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# TOTALLY REAL SUBMANIFOLDS IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. In this paper, we establish the following result: Let $M$ be an $n$-dimensional complete totally real minimal submanifold immersed in $CP^n$ with Ricci curvature bounded from below. Then either $M$ is totally geodesic or $\inf r \le (3n+1)(n-2)/3$, where $r$ is the scalar curvature of $M$.

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**1. Introduction.** Let $CP^n$ be the $n$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $c = 4$ and let $M$ be an $n$-dimensional totally real submanifold of $CP^n$. Let $r$ be the scalar curvature of $M$. If $M$ is compact, then many authors studied them and obtained many beautiful results (for example [2, 4, 5]).

In this paper, we make use of Yau's maximum principle to study the complete totally real minimal submanifold with Ricci curvature bounded from below and obtain the following result.

**THEOREM 1.** *Let $M$ be an $n$-dimensional complete totally real minimal manifold immersed in $CP^n$ with Ricci curvature bounded from below. Then either $M$ is totally geodesic or* $\inf r \le (3n+1)(n-2)/3$.

**2. Preliminaries.** Let $M$ be an $n$-dimensional totally real minimal submanifold of $CP^n$. We choose a local field of orthonormal frames $e_1,\dots,e_n,e_{1^*} = Je_1,\dots,e_{n^*} = Je_n$ ($J$ is the complex structure of $CP^n$), such that, restricted to $M$, the vectors $e_1,\dots,e_n$ are tangent to $M$. We make use of the following convention on the range of indices

$$A,B,C,\dots = 1,\dots,n,1^*,\dots,n^*; \quad i,j,k,\dots = 1,\dots,n. \tag{2.1}$$

With respect to the frame field of $CP^n$, let $w^A$ be the field of dual frames. Then the structure equations of $CP^n$ are given by

$$dw^A = -\sum w_B^A \wedge w^B, \quad w_A^B + w_B^A = 0, \tag{2.2}$$

$$dw_B^A = -\sum w_C^A \wedge w_B^C + \frac{1}{2}\sum \bar{R}_{BCD}^A w^C \wedge w^D, \tag{2.3}$$

$$\bar{R}_{BCD}^A = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}, \tag{2.4}$$

where $J = J_{AB} e_A \otimes e_B$, so that

$$(J_{AB}) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{2.5}$$

where $I_n$ is the identity matrix of order $n$. We restrict these forms to $M$. Then from [2], we have

$$w^{i^*} = 0, \quad w_i^j = w_{i^*}^{j^*}, \quad w_j^{i^*} = w_i^{j^*}, \tag{2.6}$$

$$w_i^{k^*} = \sum h_{ij}^{k^*} w^j, \quad h_{ij}^{k^*} = h_{ji}^{k^*} = h_{jk}^{i^*} = h_{ik}^{j^*}, \tag{2.7}$$

$$dw^i = -\sum w_j^i \wedge w^j, \quad w_i^j + w_j^i = 0, \tag{2.8}$$

$$dw_i^j = -\sum w_i^k \wedge w_k^j + \frac{1}{2} \sum R_{ikl}^j w^k \wedge w^l, \tag{2.9}$$

$$R_{jkl}^i = \bar{R}_{jkl}^i w^k + \sum \left( h_{ik}^{m^*} h_{jl}^{m^*} - h_{il}^{m^*} h_{jk}^{m^*} \right), \tag{2.10}$$

$$dw_{j^*}^{i^*} = -\sum w_{k^*}^{i^*} \wedge w_{j^*}^{k^*} + \frac{1}{2} \sum R_{j^* kl}^{i^*} w^k \wedge w^l, \tag{2.11}$$

$$R_{j^* kl}^{i^*} = \bar{R}_{j^* kl}^{i^*} + \sum \left( h_{km}^{i^*} h_{ml}^{j^*} - h_{ml}^{i^*} h_{km}^{j^*} \right). \tag{2.12}$$

The second fundamental form $h$ of $M$ in $CP^n$ is defined as $h = \sum h_{ij}^{k^*} w^i \otimes e_{k^*}$, whose squared length is $\|h\|^2 = \sum (h_{ij}^{k^*})^2$.

If $M$ is minimal in $CP^n$, i.e., trace $h = 0$, then from (2.4) and (2.10), we have

$$r = n(n-1) - \|h\|^2, \tag{2.13}$$

where $r$ is the scalar curvature of $M$.

Define $h_{ijk}^{m^*}$ and $h_{ijkl}^{m^*}$ by

$$\sum h_{ijk}^{m^*} w^k = dh_{ij}^{m^*} - \sum h_{kj}^{m^*} w_i^k - \sum h_{ik}^{m^*} w_j^k + \sum h_{ij}^{l^*} w_{l^*}^{m^*}, \tag{2.14}$$

$$\sum h_{ijkl}^{m^*} w^l = dh_{ijk}^{m^*} - \sum h_{ljk}^{m^*} w_i^l - \sum h_{ilk}^{m^*} w_j^l - \sum h_{ijl}^{m^*} w_k^l + \sum h_{ijk}^{l^*} w_{l^*}^{m^*}, \tag{2.15}$$

respectively.

Let $H_{l^*}$ and $\Delta$ denote the $(n \times n)$-matrix $(h_{ij}^{l^*})$ and the Laplacian on $M$, respectively. By a simple calculation, we have (cf. [2])

$$\begin{aligned} \frac{1}{2} \Delta \|h\|^2 = {} & \sum (h_{ijk}^{l^*})^2 + (n+1)\|h\|^2 + \sum \operatorname{tr}(H_{i^*} H_{j^*} - H_{j^*} H_{i^*})^2 \\ & - \sum (\operatorname{tr} H_{i^*} \operatorname{tr} H_{j^*})^2. \end{aligned} \tag{2.16}$$

The following lemma is important in this paper.

**Lemma 1** [6]. *Let $M^n$ be a complete Riemannian manifold with Ricci curvature bounded from below and let $f$ be a $C^2$-function bounded from above on $M^n$, then for all $\epsilon > 0$, there exists a point $x \in M^n$ at which*

(i) $\sup f - \epsilon < f(x)$;

(ii) $\|\nabla f(x)\| < \epsilon$;

(iii) $\Delta f(x) < \epsilon$.

**Proof of the main theorem.** By [3], we have $\sum(\operatorname{tr} H_{i^*}H_{j^*})^2 = \sum\big(\operatorname{tr} H_{i^*}^2\big)^2$. From [1], we know that $\sum \operatorname{tr}(H_{i^*}H_{j^*} - H_{j^*}H_{i^*})^2 - \sum\big(\operatorname{tr} H_{i^*}^2\big)^2 \geq -3/2\|h\|^4$. So, from (2.16), we obtain

$$\frac{1}{2}\Delta\|h\|^2 \geq \|h\|^2\big((n+1) - 3/2\|h\|^2\big). \tag{2.17}$$

We know that $\|h\|^2 = n(n-1) - r$. By the condition of the theorem, we conclude that $\|h\|^2$ is bounded. We define $f = \|h\|^2$ and $F = (f+a)^{1/2}$ (where $a > 0$ is any positive constant number). $F$ is bounded. We have

$$dF = \frac{1}{2}(f+a)^{-1/2}df, \tag{2.18}$$

$$\begin{aligned}\Delta F &= \frac{1}{2}\Big(-\frac{1}{2}(f+a)^{-3/2}\|df\|^2 + (f+a)^{-1/2}\Delta f\Big)\\ &= \frac{1}{2}(-2\|dF\|^2 + \Delta f)(f+a)^{-1/2},\end{aligned} \tag{2.19}$$

i.e.,

$$\Delta F = \frac{1}{2F}(-2\|dF\|^2 + \Delta f). \tag{2.20}$$

Hence, $F\Delta F = -\|dF\|^2 + 1/2\Delta f$ or $1/2\Delta f = F\Delta F + \|dF\|^2$.

Applying Lemma 1 to $F$, we have for all $\epsilon > 0$, there exists a point $x \in M$ such that at $x$

$$||dF(x) < \epsilon||; \tag{2.21}$$

$$\Delta F(x) < \epsilon; \tag{2.22}$$

$$F(x) > \sup F - \epsilon. \tag{2.23}$$

From (2.21), (2.22), and (2.23), we have

$$\frac{1}{2}\Delta f < \epsilon^2 + F\epsilon = \epsilon(\epsilon + F). \tag{2.24}$$

We take a sequence $\{\epsilon_m\}$ such that $\epsilon_m \longrightarrow 0 (m \longrightarrow \infty)$ and for all $m$, there exists a point $x_m \in M$ such that (2.21), (2.22), and (2.23) hold. Therefore, $\epsilon_m(\epsilon_m + F(x_m)) \longrightarrow 0 (m \longrightarrow \infty)$ (because $F$ is bounded).

From (2.23), we have $F(x_m) > \sup F - \epsilon_m$. Because $\{F(x_m)\}$ is a bounded sequence. So we get $F(x_m) \longrightarrow F_0$ (if necessary, we can choose a subsequence). Hence, $F_0 \geq \sup F$. So we have

$$F_0 = \sup F. \tag{2.25}$$

From the definition of $F$, we get

$$f(x_m) \longrightarrow f = \sup f. \tag{2.26}$$

(2.17) and (2.24) imply that

$$f\Big((n+1) - \frac{3}{2}f\Big) \leq \frac{1}{2}\Delta f \leq \epsilon(\epsilon + F), \tag{2.27}$$

and

$$f(x_m)\Big((n+1) - \frac{3}{2}f(x_m)\Big) < \epsilon_m^2 + \epsilon_m F(x_m) \leq \epsilon_m^2 + \epsilon_m F_0 \tag{2.28}$$

let $m \longrightarrow \infty$, then $\epsilon_m \longrightarrow 0$ and $f(x_m) \longrightarrow f_0$. Hence,

$$f_0\Big((n+1) - \frac{3}{2} f_0\Big) \le 0. \tag{2.29}$$

(i) if $f_0 = 0$, we have $f = \|h\|^2 \equiv 0$. Hence, $M$ is totally geodesic.

(ii) if $f_0 > 0$, we have $(n+1) - 3/2f_0 \le 0$ and $f_0 \ge 2/3(n+1)$, that is, $\sup \|h\|^2 \ge 2/3(n+1)$. Therefore, $\inf r \le (3n+1)(n-2)/3$. This completes the proof. □

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