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INDEFINITE EIGENVALUE PROBLEM WITH EIGENPARAMETER IN THE TWO BOUNDARY CONDITIONS

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ABSTRACT. The object of this paper is to establish an expansion theorem for a regular indefinite eigenvalue problem of second order differential equation with an eigenvalue parameter λ in the two boundary conditions We associated with this problem a J-selfadjoint operator with compact resolvent defined in a suitable Krein space and then we develop an associated eigenfunction expansion theorem

KEY WORDS AND PHRASES: An expansion theorem, a regular indefinite eigenvalue problem, eigenvalue parameter in the boundary conditions, a J-selfadjoint operator, Krein space formulation
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1. INTRODUCTION

The regular right-definite eigenvalue problem with eigenparameter in the two boundary conditions where $r(x)$ and $q(x)$ are positive functions on $[a,b]$ has been studied by Zayed and Ibrahim [1]

Daho and Langer [2] have made an extension of Everitt's paper [3] and have replaced the Hilbert space in some cases by a Pontragin space with index one. Everitt [3] has shown that for $\alpha \in [0, \frac{\pi}{2}]$ the singular Sturm-Liouville with indefinite weight function $r(x)$ can be represented by a selfadjoint operator in a suitable Hilbert space

Recently, Fleckinger and Mingarelli [4] have studied an indefinite problem with the usual homogeneous Dirichlet or Neumann boundary conditions.

The object of this paper is to study the following regular indefinite eigénvalue problem of order two consisting of the ordinary differential equation

$$\tau u = \frac{1}{r(x)}[-(pu')' + q(x)u] = \lambda u, \quad x \in [a,b] \tag{1 1}$$

together with the boundary conditions

$$M_\alpha(u) := [\alpha_1 u(a) - \alpha_2 (pu')(a)] = \lambda R_\alpha(u) := \lambda[\alpha_3 u(a) - \alpha_4 (pu')(a)], \tag{1 2}$$

$$-M_\beta(u) := -[\beta_1 u(b) - \beta_2 (pu')(b)] = \lambda R_\beta(u) := \lambda[\beta_3 u(b) - \beta_4 (pu')(b)], \tag{1 3}$$

where we assume throughout that:

(i) The functions $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are real continuous functions on $[a,b]$ with $p(x) > 0$ and continuously differentiable.

(ii) Both the weight function $r(x)$ and the potential function $q(x)$ change sign on $[a,b]$ in the sense that the problem (1 1)-(1.3) is an indefinite.

(iii) The numbers α_i, β_i; $i = 1,2,3,4$ are real such that

$$\rho_1 = \begin{vmatrix} \alpha_3 & \alpha_4 \\ \alpha_1 & \alpha_2 \end{vmatrix} > 0 \quad \text{and} \quad \rho_2 = \begin{vmatrix} \beta_3 & \beta_4 \\ \beta_1 & \beta_2 \end{vmatrix} > 0. \tag{1 4}$$

The parameter λ is a complex number.

In this paper, our approach is to give a Krein space formulation to problem (1 1)-(1.3) and a J-selfadjoint operator defined in it such that this problem can be considered as the eigenvalue problem of this operator

2. KREIN SPACE FORMULATION

DEFINITION 2.1. We define a Krein space H of three components vectors by

$$\begin{aligned} H &= L^2_{|r|}(a,b) \oplus C \oplus C \\ &= \left(L^2_{|r|^+}(a,b) \oplus C_+ \oplus C_+\right)[\dot{+}]\left(L^2_{|r|^-}(a,b) \oplus C_- \oplus C_-\right) \\ &= H^+[\dot{+}]H^- \end{aligned} \tag{2 1}$$

with indefinite inner product

$$[f,g]_H := \int_a^b r f_1 \overline{g}_1 dx + \frac{1}{\rho_1} f_2 \overline{g}_2 + \frac{1}{\rho_2} f_3 \overline{g}_3, \quad \forall f,g \in H \tag{2.2}$$

and the norm

$$\|f\|_H^2 := \int_a^b |f_1|^2 dx + \frac{1}{\rho_1}|f_2|^2 + \frac{1}{\rho_3}|f_3|^2, \quad f \in H \tag{2 3}$$

where $f,g \in H$ such that

$$f = (f_1, f_2, f_3) = (f_1, R_\alpha(f_1), R_\beta(f_1)) \tag{2 4}$$

in which

$$\begin{aligned} R_\alpha(f_1) &= \alpha_3 f_1(a) - \alpha_4 (pf_1')(a), \\ R_\beta(f_1) &= \beta_3 f_1(b) - \beta_4 (pf_1')(b), \end{aligned}$$

and

$$g = (g_1, g_2, g_3) = (g_1, R_\alpha(g_1), R_\beta(g_1)) \tag{2 5}$$

in which

$$\begin{aligned} R_\alpha(g_1) &= \alpha_3 g_1(a) - \alpha_4 (pg_1')(a), \\ R_\beta(g_1) &= \beta_3 g_1(b) - \beta_4 (pg_1')(b) \end{aligned}$$

while

$$\begin{aligned} H^\pm &:= P^\pm\left(L^2_{|r|}(a,b) \oplus C \oplus C\right) \\ &= L^2_{|r|^\pm}(a,b) \oplus C_\pm \oplus C_\pm, \\ C_+ &:= \{\mu \in C : Im\,\mu > 0\} \quad \text{and} \quad C_- := \{\mu \in C : Im\,\mu < 0\}. \end{aligned} \tag{2.6}$$

$P^\pm$ denote the orthogonal projectors in H such that

$$J := P^+ - P^- \quad \text{and} \quad P^+ + P^- = I. \tag{2 7}$$

Both H^+ and H^- are Hilbert spaces with respect to the scalar product $[.,.]$ and $-[.,.]$ respectively and the symbol $[\dot{+}]$ denotes the direct sum which is orthogonal with respect to the scalar product $[.,.]$, that is, we have

$$H^+ \cap H^- = \{0\} \quad \text{and} \quad [f^+, f^-] = 0 \quad \text{for} \quad f^\pm \in H^\pm;\ f = (f^+ + f^-) \in H. \tag{2.8}$$

The decomposition (2.1) gives rise to a positive definite scalar product $(.,.)$ on H.

$$(f,g) := [f^+,g^+] - [f^-,g^-] \quad \text{for} \quad f,g \in H,\ f^\pm, g^\pm \in H^\pm \tag{2 9}$$

where $f = f^+ + f^-$ and $g = g^+ + g^-$.

REMARK 2.1. (i) According to Definition 2 1, the class of Krein spaces includes Hilbert space ($H^- = 0$) as well as anti-spaces of Hilbert spaces ($H^+ = 0$); (see [5]).

(ii) The decomposable, non-degenerate inner product space H is a Krein space [5] if and only if for every fundamental symmetric operator J, the J-inner product turns H into a Hilbert space, that is, we have

$$[f,g] = (Jf,g), \quad f,g \in H. \tag{2 10}$$

DEFINITION 2.2. We define a closed linear operator $A : D(A) \to H$ by

$$Af := (\tau f_1, M_\alpha(f_1), -M_\beta(f_1)), \quad \forall f \in D(A) \tag{2.11}$$

such that

$$M_\alpha(f_1) = \alpha_1 f_1(a) - \alpha_2(pf_1')(a),$$
$$M_\beta(f_1) = \beta_1 f_1(b) - \beta_2(pf_1')(b)$$

where the domain $D(A)$ of the closed linear operator A is defined as the set of all $f = (f_1, f_2, f_3) \in H$ which satisfy the following conditions:

(i) f_1, f_1' are absolutely continuous functions on $[a,b]$ with

$$\tau f_1 \in L^2_{|r|}(a,b) \quad \text{and} \quad \int_a^b \left[p|f_1'|^2 + |q|\,|f_1|^2\right] dx < \infty,$$

(ii) $f_2 = R_\alpha(f_1)$,

(iii) $f_3 = R_\beta(f_1)$.

REMARK 2.2 (i) The domain $D(A)$ is dense in H with respect to the indefinite inner product (2.2).

(ii) λ is an eigenvalue and f_1 is a corresponding eigenfunction of problem (1.1)-(1.3) if and only if $f = (f_1, f_2, f_3) \in D(A)$ and $Af = \lambda f$ Therefore, the eigenvalues and the eigenfunctions of problem (1 1)-(1.3) are equivalent to the eigenvalues and the eigenfunctions of the operator A.

We consider the following assumptions:

$$\lim_{x \to a} \left[(pf_1')(x)\overline{g_1(x)} - f_1(x)\overline{(pg_1')(x)}\right] = 0, \tag{2 12}$$

and

$$\lim_{x \to b} \left[(pf_1')(x)\overline{g_1(x)} - f_1(x)\overline{(pg_1')(x)}\right] = 0. \tag{2.13}$$

3. THE J-SELFADJOINTNESS OF OPERATOR A

DEFINITION 3.1. In the Krein space H, a symmetric operator and a selfadjoint operator with respect to indefinite scalar product are called J-symmetric and J-selfadjoint respectively (see [5]).

LEMMA 3.1. The operator A in H is J-symmetric.

PROOF. On using (2.2), (2.11) and the boundary conditions (1.2)-(1.3), we get

$$\begin{aligned}[Af,g]_H &= \int_a^b r(\tau f_1)\overline{g}_1\,dx + \frac{1}{\rho_1} M_\alpha(f_1)\overline{g}_2 - \frac{1}{\rho_2} M_\beta(f_1)\overline{g}_3 \\ &= -\int_a^b (pf_1')'\overline{g}_1\,dx + \int_a^b qf_1\overline{g}_1\,dx + \frac{1}{\rho_1}(\lambda R_\alpha(f_1))\overline{R_\alpha(g_1)} + \frac{1}{\rho_2}(\lambda R_\beta(f_1))\overline{R_\beta(g_1)}.\end{aligned} \tag{3 1}$$

Integrating the first term of (3.1) by parts twice, we get

$$[Af,g]_H = \left[(pf_1')(a)\,\overline{g_1(a)} - f_1(a)\,\overline{(pg_1')(a)}\right] + \left[-(pf_1')(b)\,\overline{g_1(b)} + f_1(b)\,\overline{(pf_1')(b)}\right]$$
$$+\int_a^b f_1\left[-\overline{(pg_1')'} + \overline{qg_1}\right]dx + \frac{1}{\rho_1}R_\alpha(f_1)\left(\lambda\,\overline{R_\alpha(g_1)}\right) + \frac{1}{\rho_2}R_\beta(f_1)\left(\lambda\,\overline{R_\beta(g_1)}\right). \quad (3.2)$$

On applying the conditions (2.12)-(2.13) on the first two terms of (3 2) and using the boundary conditions (1.2)-(1.3), we obtain

$$[Af,g]_H = \int_a^b f_1 r\,\overline{(\tau g_1)}\,dx + \frac{1}{\rho_1}R_\alpha(f_1)\,\overline{M_\alpha(g_1)} - \frac{1}{\rho_2}R_\beta(f_1)\,\overline{M_\beta(g_1)}$$
$$= [f, Ag]_H. \quad (3.3)$$

This proves that the closed linear operator A in H is J-symmetric.

Let J be a conjugation operator on H; this means that J is a conjugate-linear involution with

$$[Jf, Jf]_H = [g, f]_H \quad \forall f, g \in H. \quad (3.4)$$

DEFINITION 3.2. The closed linear operator A in H is called a J-selfadjoint in H if $D(A)$ is dense in H and

$$A = JA^*J. \quad (3.5)$$

As in Knowles [6] we can define an inner product on the domain $D(JA^*J)$ by

$$[f,g]_H^* = [Jf, Jg]_H + [A^*Jf, A^*Jg]_H, \quad \forall f, g \in H. \quad (3.6)$$

Since J is a conjugation operator on H, we find that (3.6) is equivalent to

$$[f,g]_H^* = [f,g]_H + [JA^*Jg, JA^*Jf]_H. \quad (3.7)$$

With this indefinite inner product, $D(JA^*J)$ becomes a Krein space (see Dunford and Schwartz [7, p. 1225]).

LEMMA 3.2. If A is a J-symmetric operator in H, then

$$D(JA^*J) = D(A).$$

PROOF. Let

$$g \in D(JA^*J) \ominus D(A), \quad (3.8)$$

then

$$[f,g]_H^* = 0, \quad \text{for} \quad f \in D(A). \quad (3.9)$$

Making use of (3.7), (3.9) and the fact that $JA^*Jf = Af$, for $f \in D(A)$; we get

$$[JA^*Jg, Af]_H = [-g, f]_H. \quad (3.10)$$

On using the definition of an adjoint operator, (3.10) implies

$$JA^*Jg \in D(A^*).$$

This gives

$$g \in D(A^*JA^*J). \quad (3.11)$$

From (3.8) and (3.11), we can conclude that the vector function g is the zero vector function This implies that

$$D(JA^*J) = D(A). \quad (3.12)$$

REMARK 3.1. Our closed linear operator A is J-symmetric in Krein space H, the domain $D(A)$ is dense in H and $A = JA^*J$. Therefore, the operator A is J-selfadjoint operator in H (see Race [8]).

4. THE BOUNDEDNESS OF THE OPERATOR A

In this section we shall show that the J-selfadjoint operator A in H is unbounded from above and bounded from below. To this end we need the following lemma

LEMMA 4.1. Let f, f' be continuous functions on $[a,b]$ with $\tau f_1 \in L^2_{|r|}(a,b)$ Since $p(x)$ is continuous on $[a,b]$ as well as $p(x)>0$ on $[a,b]$, then there exists a positive constant c_0 with $p(x)\geq c_0$ such that

$$\int_a^b p(x)|f_1'(x)|^2 dx \geq \frac{c_0}{(b-a)}\,|f_1(b)-f_1(a)|^2. \tag{4 1}$$

PROOF. Schwartz's inequality gives

$$\begin{aligned}\int_a^b p(x)|f_1'(x)|^2 dx &\geq c_0\int_a^b |f_1'(x)|^2 dx\\ &\geq \frac{c_0}{(b-a)}\left|\int_a^b f_1'(x)dx\right|^2\\ &= \frac{c_0}{(b-a)}\,|f_1(b)-f_1(a)|^2.\end{aligned}$$

REMARK 4.1. We assume that there is a real number γ such that

$$q(x)\geq \gamma r(x), \quad x\in[a,b] \tag{4 2}$$

for all signs of $r(x)$ and $q(x)$.

LEMMA 4.2. The J-selfadjoint operator A in H is bounded from below

PROOF. On using the boundary conditions (1.2)-(1.3), we get

$$\begin{aligned}[Af,f]_H &= \int_a^b r(\tau f_1)\,\overline{f_1}\,dx + \frac{1}{\rho_1}M_\alpha(f_1)\,\overline{f_2} - \frac{1}{\rho_2}M_\beta(f_1)\,\overline{f_3}\\ &= -\int_a^b (pf_1')'\,\overline{f_1}\,dx + \int_a^b q|f_1|^2 dx\\ &\quad + \frac{1}{\rho_1}M_\alpha(f_1)\,\overline{f_2} - \frac{1}{\rho_2}M_\beta(f_1)\,\overline{f_3}.\end{aligned} \tag{4 3}$$

Integrating the first term of (4.3) by parts, we find that

$$\begin{aligned}[Af,f]_H &= \int_a^b q|f_1|^2 dx + \int_a^b p|f_1'|^2 dx\\ &\quad + \left\{(pf_1')(a)\,\overline{f_1(a)} - (pf_1')(b)\,\overline{f_1(b)}\right\}\\ &\quad + \frac{1}{\rho_1}M_\alpha(f_1)\,\overline{R_\alpha(f_1)} - \frac{1}{\rho_2}M_\beta(f_1)\,\overline{R_\beta(f_1)}.\end{aligned} \tag{4 4}$$

Substituting (4.1), (4.2) into (4.4) and using the boundary conditions (1.2)-(1.3), we obtain

$$\begin{aligned}[Af,f]_H &\geq \gamma\int_a^b r|f_1|^2 dx + \frac{c_0}{(b-a)}\,|f_1(b)-f_1(a)|^2\\ &\quad + \left\{(pf_1')(a)\,\overline{f_1(a)} - (pf_1')(b)\,\overline{f_1(b)}\right\}\\ &\quad + \frac{1}{\rho_1}\{\alpha_1 f_1(a) - \alpha_2(pf_1')(a)\}\{\alpha_3\,\overline{f_1(a)} - \alpha_4\,\overline{(pf_1')(a)}\}\\ &\quad + \frac{1}{\rho_2}\{-\beta_1 f_1(b) + \beta_2(pf_1')(b)\}\{\beta_3\,\overline{f_1(b)} - \beta_4\,\overline{(pf_1')(b)}\}.\end{aligned} \tag{4 5}$$

The formula (4.5) can be simplified to take the form

$$[Af,f]_H \geq \gamma \int_a^b r|f_1|^2 dx + \frac{1}{\rho_1}\left\{ f_1(a)\overline{f_1(a)}\left[\alpha_1\alpha_3 + \frac{c_0\rho_1}{(b-a)}\left(1-\frac{f_1(b)}{f_1(a)}\right)\right] \right.$$
$$\left. - \alpha_1\alpha_4\left[\overline{f_1(a)}\,(pf_1')(a) + f_1(a)\,\overline{(pf_1')(a)}\right] + \alpha_2\alpha_4 (pf_1')(a)\,\overline{(pf_1')(a)} \right\}$$
$$+ \frac{1}{\rho_2}\left\{ f_1(b)\overline{f_1(b)}\left[-\beta_1\beta_3 + \frac{c_0\rho_2}{(b-a)}\left(1-\frac{f_1(a)}{f_1(b)}\right)\right] \right.$$
$$\left. + \beta_1\beta_4\left[\overline{f_1(b)}\,(pf_1')(b) + f_1(b)\,\overline{(pf_1')(b)}\right] + \beta_2\beta_4 (pf_1')(b)\,\overline{(pf_1')(b)} \right\}. \qquad (4\ 6)$$

Choose the real numbers α_i, β_i; $i = 1,2,3,4$ provided $\rho_1, \rho_2 > 0$ and the constant $c_0 > 0$ and choose the values $p(a) > 0$, $p(b) > 0$, $f_1(a)$, $f_1(b)$, $f_1'(a)$, $f_1'(b)$ such that the following inequality is valid

$$[Af,f]_H \geq \gamma \int_a^b r|f_1|^2 dx$$
$$+ \frac{1}{\rho_1}\left\{ \alpha_3^2 f_1(a)\overline{f_1(a)} - \alpha_3\alpha_4\left[\overline{f_1(a)}\,(pf_1')(a) + f_1(a)\,\overline{(pf_1')(a)}\right] \right.$$
$$\left. + \alpha_4^2 (pf_1')(a)\,\overline{(pf_1')(b)} \right\}$$
$$+ \frac{1}{\rho_2}\left\{ \beta_3^2 f_1(b)\overline{f_1(b)} - \beta_3\beta_4\left[\overline{f_1(b)}\,(pf_1')(b) + f_1(b)\,\overline{(pf_1')(b)}\right] \right.$$
$$\left. + \beta_4^2 (pf_1')(b)\,\overline{(pf_1')(b)} \right\}. \qquad (4\ 7)$$

The inequality (4 7) can be rewritten in the form

$$\begin{aligned}[Af,f]_H &\geq \gamma \int_a^b r|f_1|^2 dx + \frac{1}{\rho_1} R_\alpha(f_1)\,\overline{R_\alpha(f_1)} + \frac{1}{\rho_2} R_\beta(f_1)\,\overline{R_\beta(f_1)} \\ &= \gamma \int_a^b r|f_1|^2 dx + \frac{1}{\rho_1}|f_2|^2 + \frac{1}{\rho_2}|f_3|^2 \\ &\geq c\,\|f\|_H^2, \end{aligned} \qquad (4\ 8)$$

where the constant $c = \min(\gamma, 1)$.

LEMMA 4.3. The J-selfadjoint operator A in H is unbounded from above.

PROOF. Let $\chi(x)$ be a test function in the Krein space H with compact support on $[a,b]$ and define a sequence of this test function by

$$\chi_m(x) := \chi(mx) \quad \text{for} \quad x \in [a,b], \quad m = 1,2,3,..., \qquad (4\ 9)$$

On using the same arguments of Lemma 4.2, we can show that

$$[A\chi_m, \chi_m]_H \geq c_1 m^2 \|\chi_m\|_H^2, \qquad (4\ 10)$$

where c_1 is a constant. Letting $m \to \infty$, we get

$$\lim_{m\to\infty} [A\chi_m, \chi_m]_H = \infty. \qquad (4\ 11)$$

This proves that A in H is unbounded from above.

5. THE EIGENVALUES OF OPERATOR A

The problem (1 1)-(1.3) in the indefinite case gives us positive and negative eigenvalues Thus we consider the infinite sequence of the eigenvalues of A:

$$-\infty < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_n < \lambda_{n+1} \leq \lambda_{n+2} \leq ... \qquad (5\ 1)$$

where $\lambda_n < 0 < \lambda_{n+1}$. For brevity, the eigenvalues and eigenfunctions are together called eigenpairs

DEFINITION 5.1. The J-selfadjoint operator A is called J-non-negative if $[Af,f]_H \geq 0$, $f \in D(A)$.

THEOREM 5.1. Consider the problem (1.1)-(1.3). There exists at least a finite number of distinct positive eigenvalues $\lambda_{n+1}, \lambda_{n+2}, ..., \lambda_{n+s}$ whose associated eigenfunctions Φ_j, $j = n+1, n+2, ..., n+s$, satisfy $[A\Phi_j, \Phi_j]_H \geq 0$

PROOF. Suppose there exists $h < n+s$ and the eigenpairs of operator A are λ_j, Φ_j; $j = n+1, n+2, ...$ such that

$$[A\Phi_j, \Phi_j]_H \geq 0. \tag{5.2}$$

We have $0 \leq \{[A\Phi_j, \Phi_j]_H / [\Phi_j, \Phi_j]_H\} = \lambda_j$, $j = n+1, n+2, n+3, ..., h$. Since $h < n+s$, then $0 \leq \{[A\Phi_h, \Phi_h]_H / [\Phi_h, \Phi_h]_H\} = \lambda_h < \lambda_{n+s}$

Therefore, there exists at least a finite number h; $n+1 < h < n+s$. Thus there exists at least a finite number of distinct positive eigenvalues $\lambda_{n+1}, \lambda_{n+2}, ..., \lambda_{n+s}$; $s = 1, 2, 3, ...$. This admits the existence of infinite number positive eigenvalues $\lambda_{n+1}, \lambda_{n+2}, ..., \lambda_{n+s}, ...$ as $s \to \infty$.

THEOREM 5.2. Consider the problem (1.1)-(1.3). There exists at most finite number of distinct negative eigenvalues $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ whose associated eigenfunctions Φ_i; $i = 1, 2, 3, ..., n$, satisfy $[A\Phi_i, \Phi_i]_H < 0$.

From above we deduce that the set of real eigenvalues of operator A (negative and positive eigenvalues), is bounded from below and unbounded from above.

6. THE RESOLVENT OPERATOR AND THE EXPANSION THEOREM

Suppose that $\psi_1(x;\lambda)$, $\psi_2(x;\lambda)$, are the fundamental solutions of (1.1) on $[a,b]$ which satisfy the initial conditions:

$$\left.\begin{aligned} \psi_1(a;\lambda) = \alpha_2 - \lambda\alpha_4, \quad p(a)\psi_1'(a;\lambda) = \alpha_1 - \lambda\alpha_3 \\ \psi_2(b;\lambda) = \beta_2 + \lambda\beta_4, \quad p(b)\psi_2'(b;\lambda) = \beta_1 + \lambda\beta_3 \end{aligned}\right\} \tag{6.1}$$

where $\lambda \in C$ is not an eigenvalue of the operator A, and put

$$\begin{aligned} \omega_x(\lambda) &:= W(\psi_1, \psi_2)(x;\lambda) \\ &= \psi_1(x;\lambda)p(x)\psi_2'(x;\lambda) - p(x)\psi_1'(x;\lambda)\psi_2(x;\lambda) \end{aligned} \tag{6.2}$$

which is independent of $x \in [a,b]$. Putting $x = a$, we therefore have:

$$\begin{aligned} \omega_a(\lambda) &= -[\alpha_1\psi_2(a;\lambda) - \alpha_2 p(a)\psi_2'(a;\lambda)] + \lambda[\alpha_3\psi_2(a;\lambda) - \alpha_4 p(a)\psi_2'(a;\lambda)] \\ &= -M_\alpha(\psi_2(a;\lambda)) + \lambda R_\alpha(\psi_2(a;\lambda)). \end{aligned} \tag{6.3}$$

Similarly, putting $x = b$, we therefore have:

$$\begin{aligned} \omega_b(\lambda) &= [\beta_1\psi_1(b;\lambda) - \beta_2 p(b)\psi_1'(b;\lambda)] + \lambda[\beta_3\psi_1(b;\lambda) - \beta_4 p(b)\psi_1'(b;\lambda)] \\ &= M_\beta(\psi_1(b;\lambda)) + \lambda R_\beta(\psi_1(b;\lambda)). \end{aligned} \tag{6.4}$$

It is also from (6.1) that

$$R_\alpha(\psi_1(a;\lambda)) = \rho_1 \quad \text{and} \quad R_\beta(\psi_2(b;\lambda)) = \rho_2 \tag{6.5}$$

where ρ_1, ρ_2 are given by (1.4).

From (6.1) it is clear that for all $\lambda \in C$, $M_\alpha(\psi_1(a;\lambda)) = \lambda R_\alpha(\psi_1(a;\lambda))$, which gives that $\psi_1(x;\lambda)$ satisfies the first boundary condition (1.2) at $x = a$ and $M_\beta(\psi_2(b;\lambda)) = -\lambda R_\beta(\psi_2(b;\lambda))$, which gives that $\psi_2(x;\lambda)$ satisfies the second boundary condition (1.3) at $x = b$. Employing the same type of argument as in the regular Sturm-Liouville problem [9, Sec. 1.8] it follows that the zeros of $\omega_x(\lambda)$ are real and that if λ_n, $n = 1, 2, 3, ...$ denotes these zeros, the three-component vectors

$$\Psi_n := (\psi_1(x;\lambda_n), R_\alpha(\psi_1(x;\lambda_n)), R_\beta(\psi_1(x;\lambda_n))) \tag{6.6}$$

are eigenfunctions of operator A satisfies the orthogonality relation

$$[\Psi_n, \Psi_m]_H = 0 \quad \text{for} \quad n \neq m, \tag{6.7}$$

where the indefinite inner product is given by (2.2). The initial conditions (6.1) also serve to guarantee that $\psi_1(x;\lambda)$ and $\psi_2(x;\lambda)$ are entire in λ for fixed x, and so it follows that $\omega_x(\lambda)$ is an entire function of λ.

We let

$$\Phi_n := \frac{\Psi_n}{\|\psi_n\|} := (\Phi_n(x), R_\alpha(\Phi_n(x)), R_\beta(\Phi_n(x))) \tag{6.8}$$

denote the normalized eigenfunctions and let $k_n \neq 0$ denote the real constant for which

$$\psi_2(x;\lambda_n) = k_n\psi_1(x;\lambda_n) \quad \forall x \in [a,b] \tag{6.9}$$

for each zero of $\omega_x(\lambda)$. By Green's formula with $r(x)$ which changes sign on $[a,b]$, we have

$$\int_a^b r(x)\psi_1(x;\lambda_n)\psi_1(x;\lambda)dx = \pm \frac{W(\psi_1(b;\lambda_n),\psi_1(b;\lambda)) - W(\psi_1(a;\lambda_n),\psi_1(a;\lambda))}{\lambda_n - \lambda}. \tag{6.10}$$

We find that

$$\begin{aligned} W(\psi_1(b;\lambda_n),\psi_1(b;\lambda)) &= k_n^{-1}[(\beta_2 + \lambda_n\beta_4)p(b)\psi_1'(b;\lambda) - (\beta_1 + \lambda_n\beta_3)\psi_1(b;\lambda)] \\ &= -k_n^{-1}[\omega_b(\lambda) + (\lambda_n - \lambda)R_\beta(\psi_1(b;\lambda))] \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} W(\psi_1(a;\lambda_n),\psi_1(a;\lambda)) &= k_n^{-1}[(\alpha_1 - \lambda\alpha_3)\psi_2(a;\lambda_n) - (\alpha_2 - \lambda\alpha_4)p(a)\psi_2'(a;\lambda_n)] \\ &= -k_n^{-1}[\omega_a(\lambda_n) + (\lambda - \lambda_n)R_\alpha(\psi_2(a;\lambda_n))] \\ &= -k_n^{-1}(\lambda - \lambda_n)R_\alpha(\psi_2(a;\lambda_n)) \end{aligned} \tag{6.12}$$

where $\omega_a(\lambda_n) = 0$.

Substituting (6.11) and (6.12) into (6.10) we obtain

$$\int_a^b r(x)\psi_1(x;\lambda_n)\psi_1(x;\lambda)dx = \pm k_n^{-1}\left\{\frac{\omega_b(\lambda)}{\lambda - \lambda_n} - R_\alpha(\psi_2(a;\lambda_n)) - R_\beta(\psi_1(b;\lambda))\right\}. \tag{6.13}$$

Letting $\lambda \to \lambda_n$, we get

$$\int_a^b r(x)\{\psi_1(x;\lambda_n)\}^2 dx = \pm k_n^{-1}\{\omega_b'(\lambda_n) - R_\alpha(\psi_2(a;\lambda_n)) - R_\beta(\psi_1(b;\lambda_n))\}. \tag{6.14}$$

From (6.5) and (6.9) with $\lambda_n = \lambda$, we get

$$\left.\begin{aligned} &R_\alpha(\psi_2(a;\lambda_n)) = k_n R_\alpha(\psi_1(a;\lambda_n)) = k_n\rho_1 \\ \text{and}\quad & \\ &R_\beta(\psi_1(b;\lambda_n)) = k_n^{-1}R_\beta(\psi_2(b;\lambda_n)) = k_n^{-1}\rho_2 \end{aligned}\right\} \tag{6.15}$$

On using (6.15) to eliminate k_n from (6.14), we can find that

$$\begin{aligned} \|\Psi_n\|^2 &= \int_a^b r(x)|\psi_1(x;\lambda_n)|^2 dx + \frac{1}{\rho_1}|R_\alpha(\psi_1(a;\lambda_n))|^2 + \frac{1}{\rho_2}|R_\beta(\psi_1(b;\lambda_n))|^2 \\ &= \pm\rho_2^{-1}R_\beta(\psi_1(b;\lambda_n))\omega_b'(\lambda_n). \end{aligned} \tag{6.16}$$

Now, for $f = (f_1(x), f_2, f_3) \in H$, we define $\Psi = (\Psi_1(x), \Psi_2, \Psi_3) \in D(A)$ as the unique solution of inhomogeneous operator equation

$$(\lambda I - A)\Psi = f. \tag{6.17}$$

Therefore,

$$\left.\begin{array}{l}(\lambda I-\tau)\Psi_1(x)=f_1(x)\\ (\lambda\alpha_3-\alpha_1)\Psi_1(a)-(\lambda\alpha_4-\alpha_2)(p\Psi_1')(a)=f_2\\ (\lambda\beta_3+\beta_1)\Psi_1(b)-(\lambda\beta_4+\beta_2)(p\Psi_1')(b)=f_3\end{array}\right\} \quad (6\ 18)$$

On applying the method of variation of parameters, we get

$$\begin{aligned}\Psi_1(x) = & -\psi_1(x;\lambda)\int_a^x \frac{\psi_2(t;\lambda)(rf_1)(t)}{\omega_t(\lambda)}dt+\psi_2(x;\lambda)\int_a^x \frac{\psi_1(t;\lambda)(rf_1)(t)}{\omega_t(\lambda)}dt\\ & + d_1\psi_1(x;\lambda)+d_2\psi_2(x;\lambda),\end{aligned} \quad (6\ 19)$$

where d_1, d_2 are constants and $r(x)$ changes sign on $[a,b]$. From (6.18) and (6 19) together with the initial conditions (6.1), we can get the constants d_1, d_2 in the form

$$d_1=\frac{1}{\omega_x(\lambda)}\left[f_3+\int_a^b \psi_2(t;\lambda)(rf_1)(t)dt\right] \quad (6.20a)$$

and

$$d_2=\frac{f_2}{\omega_x(\lambda)} \quad (6.20b)$$

provided that $r(x)$ changes its sign on $[a,b]$ according to our indefinite problem (1 1)-(1.3)

Consequently, we deduce that

$$\Psi_1(x)=\frac{1}{\omega_x(\lambda)}\left[f_3\psi_1(x;\lambda)+f_2\psi_2(x;\lambda)\right]+\int_a^b G(x,t;\lambda)(rf_1)(t)dt, \quad (6.21)$$

$$\Psi_2=R_\alpha(\Psi_1), \quad (6.22)$$

$$\Psi_3=R_\beta(\Psi_1), \quad (6\ 23)$$

where $G(x,t;\lambda)$ is the Green's function of our indefinite problem (1.1)-(1.3) defined by

$$G(x,t;\lambda)=\begin{cases}\frac{\psi_2(x;\lambda)\psi_1(t;\lambda)}{\omega_x(\lambda)} & \text{for } a\le t\le x<b\\ \\ \frac{\psi_1(x,\lambda)\psi_2(t,\lambda)}{\omega_x(\lambda)} & \text{for } a<x\le t\le b\end{cases} \quad (6\ 24)$$

The form of the equations (6.21)-(6.24) shows that the resolvent operator $R(\lambda;A)=(\lambda I-A)^{-1}$ is actually compact; for details of arguments of Theorem 5 in Hellwig [10, p. 120] can be used

REMARK 6.1. (i) $\lambda=0$ is not an eigenvalue of A in Krein space H.

(ii) Since A is a J-selfadjoint operator, then it has real eigenvalues and the corresponding real eigenfunctions are orthonormal.

(iii) On using Theorem 3 in Hellwig [10, p. 30], we deduce that the density of $D(A)$ in Krein space H gives the completeness of the orthonormal system of the real eigenfunctions of A.

The results of our investigations are summarized in the following expansion theorem

THEOREM 6.1. The closed linear operator A in Krein space H has an unbounded set of real eigenvalues of finite multiplicity without accumulation points in $(-\infty,\infty)$, and they can be ordered according to size

$$\begin{gathered}-\infty<\lambda_1\le\lambda_2\le\ldots\le\lambda_n<\lambda_{n+1}\le\lambda_{n+2}\le\ldots\le\lambda_{n+s}\le\ldots;\\ \lambda_{n+s}\to\infty \quad \text{as} \quad s\to\infty \quad \text{with} \quad \lambda_n<0<\lambda_{n+1}.\end{gathered}$$

If the corresponding real eigenfunctions $\Phi_1,\Phi_2,\ldots,\Phi_n,\Phi_{n+1},\ldots$ form a complete orthonormal system, then for any function $f(x)\in H$, we have the expansion

$$f = \sum_{n=1}^{\infty} [f, \Phi_n]_H \Phi_n \tag{6 25}$$

in the sense of strong convergence in H

REFERENCES

[1] ZAYED, E.M.E. and IBRAHIM, S.F.M., Regular eigenvalue problem with eigenparameter in the boundary conditions, *Bull. Calcu. Math. Soc.* **84** (1992), 379-393.

[2] DAHO, K. and LANGER, H., Some remarks on a paper by W N. Everitt, *Proc. Royal Soc. Edin.* **78A** (1977), 71-79.

[3] EVERITT, W.N., Some remarks on a differential expression with an indefinite weight function, *Math. Stud.* **13** (Amsterdam North Holland) (1974), 13-28.

[4] FLECKINGER, J. and MINGARELLI, A.B., On the eigenvalues of non-definite elliptic operators, *Diff. Equ.*, Editors: Knowles and Lewis, Elsevier Science Pub. B.V. (North-Holland) (1984), 219-227.

[5] BOGNAR, J., *Indefinite Inner Product Spaces*, Springer-Verlag, Berlin-N.Y. (1974)

[6] KNOWLES, I.W., On the boundary conditions characterizing J-selfadjoint extensions of J-symmetric operators, *J. Diff. Equ.* **40** (1981), 193-216.

[7] DUNFORD, N. and SCHWARTZ, J.T., *Linear Operators Part II*, Wiley, N.Y. (1963).

[8] RACE, D., The theory of J-selfadjoint extensions of J-symmetric operators, *J. Diff. Equ.* **57**, 2 (1985), 258-274.

[9] TITCHMARSH, E.C., *Eigenfunction Expansions Associated With Second Order Differential Equations I*, London, Oxford University Press (1962)

[10] HELLWIG, G., *Differential Operators in Mathematical Physics*, Addison-Weslley Pub Com, U.S.A. (1967).