Internat. J. Math. & Math. Sci.
VOL. 21 NO. 4 (1998) 713-722

ON INVERSION OF H-TRANSFORM IN $\mathfrak{L}_{\nu,r}$-SPACE

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(Received December 1995 and in revised form September 25, 1997)

ABSTRACT. The paper is devoted to study the inversion of the integral transform

$$(\boldsymbol{H}f)(x) = \int_0^\infty H_{p,q}^{m,n}\left[xt \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\right] f(t)dt$$

involving the H-function as the kernel in the space $\mathfrak{L}_{\nu,r}$ of functions f such that

$$\int_0^\infty |t^\nu f(t)|^r \frac{dt}{t} < \infty \quad (1 < r < \infty,\ \nu \in \mathbb{R}).$$

KEY WORDS AND PHRASES: H-function, Integral transform.

1991 AMS SUBJECT CLASSIFICATION CODES: 33C40, 44A20

1. INTRODUCTION

This paper deals with the integral transforms of the form

$$(\boldsymbol{H}f)(x) = \int_0^\infty H_{p,q}^{m,n}\left[xt \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\right] f(t)dt, \tag{1.1}$$

where $H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\right]$ is the H-function, which is a function of general hypergeometric type being introduced by S. Pincherle in 1888 (see [2, §1.19]). For integers m,n,p,q such that $0 \leqq m \leqq q$, $0 \leqq n \leqq p$, $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+ = [0,\infty)$ $(1 \leqq i \leqq p, 1 \leqq j \leqq q)$, it can be written by

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]=H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,\alpha_1), & \cdots,(a_p,\alpha_p)\\(b_1,\beta_1), & \cdots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_L \mathcal{H}^{m,n}_{p,q}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\left|s\right.\right]z^{-s}ds, \tag{1.2}$$

where

$$\mathcal{H}^{m,n}_{p,q}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\left|s\right.\right]=\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}, \tag{1.3}$$

the contour L is specially chosen and an empty product, if it occurs, is taken to be one. The theory of this function may be found in Braaksma [1], Srivastava *et al.* [13, Chapter 1], Mathai and Saxena [8, Chapter 2] and Prudnikov *et al.* [9, §8.3]. We abbreviate the H-function (1.2) and the function (1.3) to $H(z)$ and $\mathcal{H}(s)$ when no confusion occurs. We note that the formal Mellin transform $\mathfrak{M}$ of (1.1) gives the relation

$$(\mathfrak{M}\boldsymbol{H}f)(s)=\mathcal{H}(s)(\mathfrak{M}f)(1-s). \tag{1.4}$$

Most of the known integral transforms can be put into the form (1.1), in particular, if $\alpha_1=\cdots=\alpha_p=\beta_1=\cdots=\beta_q=1$, (1.1) is the integral transform with Meijer's G-function in the kernel (Rooney [11], Samko *et al.* [12, §36]). The integral transform (1.1) with the H-function kernel or the $\boldsymbol{H}$-transform was investigated by many authors (see Bibliography in Kilbas *et al.* [5-6]). In Kilbas *et al.* [5-7] we have studied it in the space $\mathfrak{L}_{\nu,r}$ $(1\leqq r<\infty,\ \nu\in\mathbb{R})$ consisted of Lebesgue measurable complex valued functions f for which

$$\int_0^\infty |t^\nu f(t)|^r\frac{dt}{t}<\infty. \tag{1.5}$$

We have investigated the mapping properties such as the boundedness, the representation and the range of the $\boldsymbol{H}$-transform (1.1) on the space $\mathfrak{L}_{\nu,2}$ in Kilbas *et al.* [5] and on the space $\mathfrak{L}_{\nu,r}$ with any $1\leqq r<\infty$ in Kilbas *et al.* [6-7], provided that $a^*\geqq 0$, $\delta=1$ and $\Delta=0$ or $\Delta\neq 0$, respectively. In Glaeske *et al.* [3] the results were extended to any $\delta>0$. Here

$$a^*=\sum_{i=1}^{n}\alpha_i-\sum_{i=n+1}^{p}\alpha_i+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q}\beta_j; \tag{1.6}$$

$$\delta=\prod_{i=1}^{p}\alpha_i^{-\alpha_i}\prod_{j=1}^{q}\beta_j^{\beta_j}; \tag{1.7}$$

$$\Delta=\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i. \tag{1.8}$$

In particular, we have proved that for certain ranges of parameters, the $\boldsymbol{H}$-transform (1.1) have the representations

$$(\boldsymbol{H}f)(x)=hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H^{m,n+1}_{p+1,q+1}\left[xt\left|\begin{matrix}(-\lambda,h),(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q},(-\lambda-1,h)\end{matrix}\right.\right]f(t)dt \tag{1.9}$$

or

$$(\boldsymbol{H}f)(x)=-hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H^{m+1,n}_{p+1,q+1}\left[xt\left|\begin{matrix}(a_i,\alpha_i)_{1,p},(-\lambda,h)\\(-\lambda-1,h),(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]f(t)dt, \tag{1.10}$$

owing to the value of $\mathrm{Re}(\lambda)$, where $\lambda \in \mathbb{C}$ and $h \in \mathbb{R} \setminus \{0\}$.

In this paper we apply the results of Kilbas *et al.* [5-7] and Glaeske *et al.* [3] to find the inverse of the integral transforms (1.1) on the space $\mathfrak{L}_{\nu,r}$ with $1 < r < \infty$ and $\nu \in \mathbb{R}$. Section 2 contains preliminary information concerning the properties of the $\boldsymbol{H}$-transform (1.1) in the space $\mathfrak{L}_{\nu,r}$ and an asymptotic behavior of the H-function (1.2) at zero and infinity. In Sections 3 and 4 we prove that the inversion of the $\boldsymbol{H}$-transform have the respective form (1.9) or (1.10):

$$f(x) = hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}$$
$$\int_0^\infty H^{q-m,p-n+1}_{p+1,q+1}\left[xt \,\middle|\, \begin{matrix} (-\lambda,h),(1-a_i-\alpha_i,\alpha_i)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n} \\ (1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m},(-\lambda-1,h) \end{matrix}\right](\boldsymbol{H}f)(t)dt \quad (1.11)$$

or

$$f(x) = -hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}$$
$$\int_0^\infty H^{q-m+1,p-n}_{p+1,q+1}\left[xt \,\middle|\, \begin{matrix} (1-a_i-\alpha_i,\alpha_i)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n},(-\lambda,h) \\ (-\lambda-1,h),(1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m} \end{matrix}\right](\boldsymbol{H}f)(t)dt, \quad (1.12)$$

provided that $a^* = 0$. Section 3 is devoted to treat on the spaces $\mathfrak{L}_{\nu,2}$ and $\mathfrak{L}_{\nu,r}$ with $\Delta = 0$, while Section 4 on the space $\mathfrak{L}_{\nu,r}$ with $\Delta \neq 0$.

The obtained results are extensions of those by Rooney [11] from $\boldsymbol{G}$-transforms to $\boldsymbol{H}$-transforms.

2. PRELIMINARIES

We give here some results from Kilbas *et al.* [5-6], Glaeske *et al.* [3] and from Kilbas and Saigo [4], Mathai and Saxena [8], Srivastava *et al.* [13] concerning the properties of $\boldsymbol{H}$-transforms (1.1) in $\mathfrak{L}_{\nu,r}$-spaces and the asymptotic behavior of the H-function at zero and infinity, respectively.

For the H-function (1.2), let a^* and Δ be defined by (1.6) and (1.8) and let

$$\alpha = \begin{cases} \max\left[-\mathrm{Re}\left(\frac{b_1}{\beta_1}\right),\cdots,-\mathrm{Re}\left(\frac{b_m}{\beta_m}\right)\right] & \text{if } m > 0, \\ -\infty & \text{if } m = 0; \end{cases} \quad (2.1)$$

$$\beta = \begin{cases} \min\left[\mathrm{Re}\left(\frac{1-a_1}{\alpha_1}\right),\cdots,\mathrm{Re}\left(\frac{1-a_n}{\alpha_n}\right)\right] & \text{if } n > 0, \\ \infty & \text{if } n = 0; \end{cases} \quad (2.2)$$

$$a_1^* = \sum_{j=1}^m \beta_j - \sum_{i=n+1}^p \alpha_i; \quad a_2^* = \sum_{i=1}^n \alpha_i - \sum_{j=m+1}^q \beta_j; \quad a_1^* + a_2^* = a^*; \quad (2.3)$$

$$\mu = \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2}. \quad (2.4)$$

For the function $\mathcal{H}(s)$ given in (1.3), the exceptional set of $\mathcal{H}$ is meant the set of real numbers ν such that $\alpha < 1-\nu < \beta$ and $\mathcal{H}(s)$ has a zero on the line $\mathrm{Re}(s) = 1-\nu$ (see Rooney [11]). For two Banach space X and Y we denote by $[X,Y]$ the collection of bounded linear operators from X to Y.

THEOREM 2.1. [5, Theorem 3], [6, Theorem 3.3] *Suppose that $\alpha < 1-\nu < \beta$ and that either $a^* > 0$ or $a^* = 0$, $\Delta(1-\nu) + \mathrm{Re}(\mu) \leqq 0$. Then*

(a) *There is a one-to-one transform $\boldsymbol{H} \in [\mathfrak{L}_{\nu,2}, \mathfrak{L}_{1-\nu,2}]$ so that (1.4) holds for $f \in \mathfrak{L}_{\nu,2}$ and $\mathrm{Re}(s) = 1-\nu$. If $a^* = 0$, $\Delta(1-\nu) + \mathrm{Re}(\mu) = 0$ and ν is not in the exceptional set of $\mathcal{H}$, then the operator $\boldsymbol{H}$ transforms $\mathfrak{L}_{\nu,2}$ onto $\mathfrak{L}_{1-\nu,2}$.*

(b) *If* $f \in \mathfrak{L}_{\nu,2}$ *and* $\mathrm{Re}(\lambda) > (1-\nu)h-1$, $\boldsymbol{H}f$ *is given by* (1.9). *If* $f \in \mathfrak{L}_{\nu,2}$ *and* $\mathrm{Re}(\lambda) < (1-\nu)h-1$, *then* $\boldsymbol{H}f$ *is given by* (1.10).

THEOREM 2.2. [6, Theorem 4.1], [3, Theorem 1] *Let* $a^* = \Delta = 0, \mathrm{Re}(\mu) = 0$ *and* $\alpha < 1-\nu < \beta$.

(a) *The transform* $\boldsymbol{H}$ *is defined on* $\mathfrak{L}_{\nu,2}$ *and it can be extended to* $\mathfrak{L}_{\nu,r}$ *as an element of* $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,r}]$ for $1 < r < \infty$.

(b) *If* $1 < r \leqq 2$, *the transform* $\boldsymbol{H}$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$ *and there holds the equality*

$$(\mathfrak{M}\boldsymbol{H}f)(s) = \mathcal{H}(s)(\mathfrak{M}f)(1-s), \quad \mathrm{Re}(s) = 1-\nu. \tag{2.5}$$

(c) *If* $f \in \mathfrak{L}_{\nu,r}$ $(1 < r < \infty)$, *then* $\boldsymbol{H}f$ *is given by* (1.9) *for* $\mathrm{Re}(\lambda) > (1-\nu)h-1$, *while* $\boldsymbol{H}f$ *is given by* (1.10) *for* $\mathrm{Re}(\lambda) < (1-\nu)h-1$.

THEOREM 2.3. [6, Theorem 5.1], [3, Theorem 3] *Let* $a^* = 0, \Delta > 0, -\infty < \alpha < 1-\nu < \beta, 1 < r < \infty$ *and* $\Delta(1-\nu) + \mathrm{Re}(\mu) \leqq 1/2 - \gamma(r)$, *where*

$$\gamma(r) = \max\left[\frac{1}{r}, \frac{1}{r'}\right] \quad \text{with} \quad \frac{1}{r} + \frac{1}{r'} = 1. \tag{2.6}$$

(a) *The transform* $\boldsymbol{H}$ *is defined on* $\mathfrak{L}_{\nu,2}$, *and it can be extended to* $\mathfrak{L}_{\nu,r}$ *as an element of* $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,s}]$ *for all* s *with* $r \leqq s < \infty$ *such that* $s' \geqq [1/2 - \Delta(1-\nu) - \mathrm{Re}(\mu)]^{-1}$ *with* $1/s + 1/s' = 1$.

(b) *If* $1 < r \leqq 2$, *the transform* $\boldsymbol{H}$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$ *and there holds the equality* (2.5).

(c) *If* $f \in \mathfrak{L}_{\nu,r}$ *and* $g \in \mathfrak{L}_{\nu,s}$ *with* $1 < r < \infty, 1 < s < \infty, 1/r + 1/s \geqq 1$ *and* $\Delta(1-\nu) + \mathrm{Re}(\mu) \leqq 1/2 - \max[\gamma(r), \gamma(s)]$, *then the relation*

$$\int_0^\infty f(x)(\boldsymbol{H}g)(x)dx = \int_0^\infty g(x)(\boldsymbol{H}f)(x)dx \tag{2.7}$$

holds.

The following two assertions give the asymptotic behavior of the the H-function (1.2) at zero and infinity provided that the poles of Gamma functions in the numerator of $\mathcal{H}(s)$ do not coincide, i.e.

$$\beta_j(a_i - 1 - k) \neq \alpha_i(b_j + l) \quad (i = 1, \cdots, n; j = 1, \cdots, m; k, l = 0, 1, 2, \cdots). \tag{2.8}$$

THEOREM 2.4. [8, §1.1.6], [13, §2.2] *Let the condition* (2.8) *be satisfied and poles of Gamma functions* $\Gamma(b_j + \beta_j s)$ $(j = 1, \cdots, m)$ *be simple, i.e.*

$$\beta_i(b_j + k) \neq \beta_j(b_i + l) \quad (i \neq j; i, j = 1, \cdots, m; k, l = 0, 1, 2, \cdots). \tag{2.9}$$

If $\Delta \geqq 0$, *then*

$$H^{m,n}_{p,q}(z) = O(z^\rho) \quad (|z| \to 0) \quad \text{with} \quad \rho = \min_{1 \leqq j \leqq m}\left[\frac{\mathrm{Re}(b_j)}{\beta_j}\right]. \tag{2.10}$$

THEOREM 2.5. [4, Corollary 3] *Let* a^*, Δ *and* μ *be given by* (1.6), (1.8) *and* (2.4), *respectively. Let the conditions in* (2.8) *be satisfied and poles of Gamma functions* $\Gamma(1 - a_i - \alpha_i s)$ $(i = 1, \cdots, n)$ *be simple, i.e.*

$$\alpha_j(1 - a_i + k) \neq \alpha_i(1 - a_j + l) \quad (i \neq j;\ i, j = 1, \cdots, n;\ k, l = 0, 1, 2, \cdots). \tag{2.11}$$

If $a^* = 0$ and $\Delta > 0$, then

$$H^{m,n}_{p,q}(z) = O(z^{\varrho}) \quad (|z| \to \infty) \quad \text{with} \quad \varrho = \max\left[\max_{1\leqq i\leqq n}\left[\frac{\mathrm{Re}(a_i)-1}{\alpha_i}\right], \frac{\mathrm{Re}(\mu)+1/2}{\Delta}\right]. \tag{2.12}$$

REMARK 2.1. It was proved in Kilbas and Saigo [4, §6] that if poles of Gamma functions $\Gamma(1-a_i-\alpha_i s)$ $(i = 1,\cdots,n)$ are not simple (i.e. conditions in (2.11) are not satisfied), then the H-function (1.1) have power-logarithmic asymptotics at infinity. In this case the logaritmic multiplier $[\log(z)]^N$ with N being the maximal number of orders of the poles may be added to the power multiplier z^{ϱ} and hence the asymptotic estimate $O(z^{\varrho})$ in (2.12) may be replaced by $O\left(z^{\varrho}[\log(z)]^N\right)$. The same result is valid in the case of the asymptotics of the H-function (1.1) at zero, and the estimate $O(z^{\rho})$ in (2.10) may be replaced by $O\left(z^{\rho}[\log(z)]^M\right)$, where M is the maximal number of orders of the points at which the poles of $\Gamma(b_j+\beta_j s)$ $(j = 1,\cdots,m)$ coincide.

3. INVERSION OF H-TRANSFORM IN $\mathfrak{L}_{\nu,2}$ AND $\mathfrak{L}_{\nu,r}$ WHEN $\Delta = 0$

In this and next sections we investigate that $\boldsymbol{H}$-transform will have the inverse of the form (1.11) or (1.12). If $f \in \mathfrak{L}_{\nu,2}$, and $\boldsymbol{H}$ is defined on $\mathfrak{L}_{\nu,r}$, then according to Theorem 2.2, the equality (2.5) holds under the assumption there. This fact implies the relation

$$(\mathfrak{M}f)(s) = \frac{(\mathfrak{M}\boldsymbol{H}f)(1-s)}{\mathcal{H}(1-s)} \tag{3.1}$$

for $\mathrm{Re}(s) = \nu$. By (1.3) we have

$$\frac{1}{\mathcal{H}(1-s)} = \mathcal{H}^{q-m,p-n}_{p,q}\left[\begin{array}{c} (1-a_i-\alpha_i,\alpha_i)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n} \\ (1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m} \end{array}\middle| s\right] \equiv \mathcal{H}_0(s), \tag{3.2}$$

and hence (3.1) takes the form

$$(\mathfrak{M}f)(s) = (\mathfrak{M}\boldsymbol{H}f)(1-s)\mathcal{H}_0(s) \quad (\mathrm{Re}(s) = \nu). \tag{3.3}$$

We denote by $\alpha_0, \beta_0, a_0^*, a_{01}^*, a_{02}^*, \delta_0, \Delta_0$ and μ_0 for $\mathcal{H}_0$ instead of those for $\mathcal{H}$. Then we find

$$\alpha_0 = \begin{cases} \max\left[\dfrac{\mathrm{Re}(b_{m+1})-1}{\beta_{m+1}}+1,\cdots,\dfrac{\mathrm{Re}(b_q)-1}{\beta_q}+1\right] & \text{if } q > m, \\ -\infty & \text{if } q = m; \end{cases} \tag{3.4}$$

$$\beta_0 = \begin{cases} \min\left[\dfrac{\mathrm{Re}\,(a_{n+1})}{\alpha_{n+1}}+1,\cdots,\dfrac{\mathrm{Re}\,(a_p)}{\alpha_p}+1\right] & \text{if } p > n, \\ \infty & \text{if } p = n; \end{cases} \tag{3.5}$$

$$a_0^* = -a^*;\quad a_{01}^* = -a_2^*;\quad a_{02}^* = -a_1^*;\quad \delta_0 = \delta;\quad \Delta_0 = \Delta;\quad \mu_0 = -\mu - \Delta. \tag{3.6}$$

We also note that if $\alpha_0 < \nu < \beta_0$, ν is not in the exceptional set of $\mathcal{H}_0$.

First we consider the case $r = 2$.

THEOREM 3.1. *Let $\alpha < 1-\nu < \beta, \alpha_0 < \nu < \beta_0, a^* = 0$ and $\Delta(1-\nu)+\mathrm{Re}(\mu) = 0$. If $f \in \mathfrak{L}_{\nu,2}$, the relation (1.11) holds for $\mathrm{Re}(\lambda) > \nu h - 1$ and the relation (1.12) holds for $\mathrm{Re}(\lambda) < \nu h - 1$.*

PROOF. We apply Theorem 2.1 with $\mathcal{H}$ being replaced by $\mathcal{H}_0$ and ν by $1-\nu$. By the assumption and (3.6) we have

$$a_0^* = -a^* = 0, \tag{3.7}$$

$$\Delta_0[1-(1-\nu)] + \mathrm{Re}(\mu_0) = \Delta\nu - \mathrm{Re}(\mu) - \Delta = -[\Delta(1-\nu) + \mathrm{Re}(\mu)] = 0 \tag{3.8}$$

and $\alpha_0 < 1-(1-\nu) < \beta_0$, and thus Theorem 2.1(a) applies. Then there is a one-to-one transform $\boldsymbol{H}_0 \in [\mathfrak{L}_{1-\nu,2}, \mathfrak{L}_{\nu,2}]$ so that the relation

$$(\mathfrak{M}\boldsymbol{H}_0 f)(s) = \mathcal{H}_0(s)(\mathfrak{M}f)(1-s) \tag{3.9}$$

holds for $f \in \mathfrak{L}_{1-\nu,2}$ and $\mathrm{Re}(s) = \nu$. Further if $f \in \mathfrak{L}_{\nu,2}$, $\boldsymbol{H}f \in \mathfrak{L}_{1-\nu,2}$ and it follows from (3.9), (1.4) and (3.2) that

$$(\mathfrak{M}\boldsymbol{H}_0\boldsymbol{H}f)(s) = \mathcal{H}_0(s)(\mathfrak{M}\boldsymbol{H}f)(1-s) = \mathcal{H}_0(s)\mathcal{H}(1-s)(\mathfrak{M}f)(s) = (\mathfrak{M}f)(s),$$

if $\mathrm{Re}(s) = \nu$. Hence $\mathfrak{M}\boldsymbol{H}_0\boldsymbol{H}f = \mathfrak{M}f$ and

$$\boldsymbol{H}_0\boldsymbol{H}f = f \quad \text{for} \quad f \in \mathfrak{L}_{\nu,2}. \tag{3.10}$$

Applying Theorem 2.1(b) with $\mathcal{H}$ being replaced by $\mathcal{H}_0$ and ν by $1-\nu$, we obtain for $f \in \mathfrak{L}_{1-\nu,2}$ that

$$(\boldsymbol{H}_0 f)(x) = hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h} \cdot \int_0^\infty H^{q-m,p-n+1}_{p+1,q+1}\left[xt \left| \begin{matrix} (-\lambda,h),(1-a_i-\alpha_i,\alpha_i)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n} \\ (1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m},(-\lambda-1,h) \end{matrix} \right.\right] f(t)dt, \tag{3.11}$$

if $\mathrm{Re}(\lambda) > [1-(1-\nu)]h - 1$ and

$$(\boldsymbol{H}_0 f)(x) = -hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h} \cdot \int_0^\infty H^{q-m+1,p-n}_{p+1,q+1}\left[xt \left| \begin{matrix} (1-a_i-\alpha_i,\alpha_i)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n},(-\lambda,h) \\ (-\lambda-1,h),(1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m} \end{matrix} \right.\right] f(t)dt, \tag{3.12}$$

if $\mathrm{Re}(\lambda) < [1-(1-\nu)]h - 1$. Replacing f by $\boldsymbol{H}f$ and using (3.10) we have the relations (1.11) and (1.12) for $f \in \mathfrak{L}_{\nu,2}$, if $\mathrm{Re}(\lambda) > \nu h - 1$ and $\mathrm{Re}(\lambda) < \nu h - 1$, respectively, which completes the proof of theorem.

Next results is the extension of Theorem 3.1 to $\mathfrak{L}_{\nu,r}$-spaces for any $1 < r < \infty$, provided that $\Delta = 0$ and $\mathrm{Re}(\mu) = 0$.

THEOREM 3.2. *Let $\alpha < 1-\nu < \beta, \alpha_0 < \nu < \beta_0, a^* = 0, \Delta = 0$ and $\mathrm{Re}(\mu) = 0$. If $f \in \mathfrak{L}_{\nu,r}$ $(1 < r < \infty)$, the relation* (1.11) *holds for* $\mathrm{Re}(\lambda) > \nu h - 1$ *and the relation* (1.12) *holds for* $\mathrm{Re}(\lambda) < \nu h - 1$.

PROOF. We apply Theorem 2.2 with $\mathcal{H}$ being replaced by $\mathcal{H}_0$ and ν by $\nu - 1$. By the assumption and (3.6), we have $a_0^* = \Delta_0 = 0, \mathrm{Re}(\mu_0) = 0$ and $\alpha_0 < 1-(1-\nu) < \beta_0$, and thus Theorem 2.2(a) can be applied. In accordance with this theorem, $\boldsymbol{H}_0$ can be extended to $\mathfrak{L}_{1-\nu,r}$ as an element of $\boldsymbol{H}_0 \in [\mathfrak{L}_{1-\nu,r}, \mathfrak{L}_{\nu,r}]$. By virtue of (3.10) $\boldsymbol{H}_0\boldsymbol{H}$ is identical operator in $\mathfrak{L}_{\nu,2}$. By Rooney [11, Lemma 2.2] $\mathfrak{L}_{\nu,2}$ is dense in $\mathfrak{L}_{\nu,r}$ and since $\boldsymbol{H} \in [\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,r}]$ and $\boldsymbol{H}_0 \in [\mathfrak{L}_{1-\nu,r}, \mathfrak{L}_{\nu,r}]$, the operator $\boldsymbol{H}_0\boldsymbol{H}$ is identical in $\mathfrak{L}_{\nu,r}$ and hence

$$\boldsymbol{H}_0\boldsymbol{H}f = f \quad \text{for} \quad f \in \mathfrak{L}_{\nu,r}. \tag{3.13}$$

Applying Theorem 2.2(c) with $\mathcal{H}$ being replaced by $\mathcal{H}_0$ and ν by $1-\nu$, we obtain that the relations (3.11) and (3.12) hold for $f \in \mathfrak{L}_{1-\nu,r}$, when $\mathrm{Re}(\lambda) > [1-(1-\nu)]h - 1$ and $\mathrm{Re}(\lambda) < [1-(1-\nu)]h-1$, respectively. Replacing f by $\boldsymbol{H}f$ and using (3.13), we arrive at (1.11) and (1.12) for $f \in \mathfrak{L}_{1-\nu,r}$, if $\mathrm{Re}(\lambda) > \nu h - 1$ and $\mathrm{Re}(\lambda) < \nu h - 1$, respectively, which completes the proof of theorem.

REMARK 3.1. If $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$ which means that the H-function (1.2) is Meijer's G-function, then $\Delta = q - p$ and Theorems 8.1 and 8.2 in Rooney [11] follow from Theorems 3.1 and 3.2.

4. INVERSION OF H-TRANSFORM IN $\mathfrak{L}_{\nu,r}$ WHEN $\Delta \neq 0$

We now investigate under what condition the $\boldsymbol{H}$-transform with $\Delta \neq 0$ will have the inverse of the form (1.11) or (1.12). First, we consider the case $\Delta > 0$. To obtain the inversion of the $\boldsymbol{H}$-transform on $\mathfrak{L}_{\nu,r}$ we use the relation (2.7).

THEOREM 4.1. *Let* $1 < r < \infty, -\infty < \alpha < 1-\nu < \beta, \alpha_0 < \nu < \min\{\beta_0, [\mathrm{Re}(\mu+1/2)/\Delta] + 1\}, a^* = 0, \Delta > 0$ *and* $\Delta(1-\nu) + \mathrm{Re}(\mu) \leqq 1/2 - \gamma(r)$, *where* $\gamma(r)$ *is given in* (2.6). *If* $f \in \mathfrak{L}_{\nu,r}$, *then the relations* (1.11) *and* (1.12) *hold for* $\mathrm{Re}(\lambda) > \nu h - 1$ *and for* $\mathrm{Re}(\lambda) < \nu h - 1$, *respectively.*

PROOF. According to Theorem 2.3(a), the $\boldsymbol{H}$-transform is defined on $\mathfrak{L}_{\nu,r}$. First we consider the case $\mathrm{Re}(\lambda) > \nu h - 1$. Let $H_1(t)$ be the function

$$H_1(t) = H^{q-m,p-n+1}_{p+1,q+1}\left[t \,\middle|\, \begin{matrix} (-\lambda,h),(1-a_i-\alpha_i,\alpha_i)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n} \\ (1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m},(-\lambda-1,h) \end{matrix}\right]. \tag{4.1}$$

If we denote by $\tilde{a}^*, \tilde{\delta}, \tilde{\Delta}$ and $\tilde{\mu}$ for H_1 instead of those for H, then

$$\tilde{a}^* = -a^* = 0; \quad \tilde{\delta} = \delta; \quad \tilde{\Delta} = \Delta > 0; \quad \tilde{\mu} = -\mu - \Delta - 1. \tag{4.2}$$

We prove that $H_1 \in \mathfrak{L}_{\nu,s}$ for any s $(1 \leqq s < \infty)$. For this, we first apply Theorems 2.4 and 2.5 and Remark 2.1 to $H_1(t)$ to find its asymptotic behavior at zero and infinity. According to (3.4), (3.5) and the assumptions, we find

$$\frac{\mathrm{Re}(b_j)-1}{\beta_j} + 1 \leqq \alpha_0 < \beta_0 \leqq \frac{\mathrm{Re}(a_i)}{\alpha_i} + 1 \quad (j = m+1,\cdots,q;\ i = n+1,\cdots,p);$$

$$\frac{\mathrm{Re}(b_j)-1}{\beta_j} + 1 \leqq \alpha_0 < \nu < \frac{\mathrm{Re}(\lambda)+1}{h} \quad (j = m+1,\cdots,q).$$

Then it follows from here that the poles

$$a_{ik} = \frac{a_i + k}{\alpha_i} + 1 \quad (i = n+1,\cdots,p;\ k = 0,1,2,\cdots), \quad \lambda_n = \frac{\lambda+1+n}{h} \quad (n = 0,1,2,\cdots)$$

of Gamma functions $\Gamma(a_i + \alpha_i - \alpha_i s)$ $(i = n+1,\cdots,p)$ and $\Gamma(1+\lambda - hs)$, and the poles

$$b_{jl} = \frac{b_j - 1 - l}{\beta_j} + 1 \quad (j = m+1,\cdots,q;\ l = 0,1,2,\cdots)$$

of Gamma functions $\Gamma(1 - b_j - \beta_j + \beta_j s)$ $(j = m+1,\cdots,q)$ do not coincide. Hence by Theorem 2.4, (4.1) and Remark 2.1, we have

$$H_1(t) = O(t^{\rho_1}) \quad (|t| \to 0) \quad \text{with} \quad \rho_1 = \min_{m+1 \leqq j \leqq q}\left[\frac{1-\mathrm{Re}(b_j)}{\beta_j}\right] - 1 = -\alpha_0$$

for α_0 being given in (3.4), or

$$H_1(t) = O(t^{-\alpha_0}) \quad (t \to 0) \tag{4.3}$$

with an additional logarithmic multilplier $[\log t]^N$ possibly, if Gamma functions $\Gamma(1 - b_j - \beta_j + \beta_j s)$ $(j = m+1, \cdots, q)$ have general poles of order $N \geqq 2$ at some points.

Further by Theorem 2.5, (4.1) and Remark 2.1,

$$H_1(t) = O(t^{\varrho_1}) \quad (t \to \infty) \quad \text{with} \quad \varrho_1 = \max\left[\beta_0, \frac{-\mathrm{Re}(\mu) - 1/2}{\Delta} - 1, \frac{-\mathrm{Re}(\lambda) - 1}{h}\right]$$

for β_0 being given by (3.5), or

$$H_1(t) = O(t^{-\gamma_0}) \quad (|t| \to \infty) \quad \text{with} \quad \gamma_0 = \min\left[\beta_0, \frac{\mathrm{Re}(\mu) + 1/2}{\Delta} + 1, \frac{\mathrm{Re}(\lambda) + 1}{h}\right] \tag{4.4}$$

and with an additional logarithmic multilplier $[\log(t)]^M$ possibly, if Gamma functions $\Gamma(1 + \lambda - hs), \Gamma(a_i + \alpha_i - \alpha_i s)$ $(i = n+1, \cdots, p)$ have general poles of order $M \geqq 2$ at some points.

Let Gamma functions $\Gamma(1 - b_j - \beta + \beta_j s)$ $(j = m+1, \cdots, q)$ and $\Gamma(1 + \lambda - hs), \Gamma(a_i + \alpha_i - \alpha_i s)$ $(i = n+1, \cdots, p)$ have simple poles. Then from (4.3) and (4.4) we see that for $1 \leqq s < \infty$, $H_1(t) \in \mathfrak{L}_{\nu,s}$ if and only if, for some R_1 and R_2, $0 < R_1 < R_2 < \infty$, the integrals

$$\int_0^{R_1} t^{s(\nu - \alpha_0) - 1} dt, \quad \int_{R_2}^{\infty} t^{s(\nu - \gamma_0) - 1} dt \tag{4.5}$$

are convergent. Since by the assumption $\nu > \alpha_0$, the first integral in (4.5) converges. In view of our assumtions

$$\nu < \beta_0, \quad \nu < \frac{\mathrm{Re}(\mu) + 1/2}{\Delta} + 1, \quad \nu < \frac{\mathrm{Re}(\lambda) + 1}{h}$$

we find $\nu - \gamma_0 < 0$ and the second integral in (4.5) converges, too.

If Gamma functions $\Gamma(1 - b_j - \beta_j + \beta_j s)$ $(j = m+1, \cdots, q)$ or $\Gamma(1 + \lambda - hs), \Gamma(a_i + \alpha_i - \alpha_i s)$ $(i = n+1, \cdots, p)$ have general poles, then the logarithmic multipliers $[\log(t)]^N$ $(N = 1, 2, \cdots)$ may be added in the integrals in (4.5), but they do not influence on the convergence of them. Hence, under the assumptions we have

$$H_1(t) \in \mathfrak{L}_{\nu,s} \quad (1 \leqq s < \infty). \tag{4.6}$$

Let a be a positive number and Π_a denote the operator

$$(\Pi_a f)(x) = f(ax) \quad (x > 0) \tag{4.7}$$

for a function f defined almost everywhere on $(0, \infty)$. It is known in Rooney [11, p.268] that Π_a is a bounded isomorphism of $\mathfrak{L}_{\nu,r}$ onto $\mathfrak{L}_{a\nu,r}$, and if $f \in \mathfrak{L}_{\nu,r}$ $(1 \leqq r \leqq 2)$, there holds the relation for the Mellin transform $\mathfrak{M}$

$$(\mathfrak{M}\Pi_a f)(s) = a^{-s}(\mathfrak{M}f)\left(\frac{s}{a}\right) \quad (\mathrm{Re}(s) = \nu). \tag{4.8}$$

By virtue of Theorem 2.3(c) and (4.6), if $f \in \mathfrak{L}_{\nu,r}$ and $H_1 \in \mathfrak{L}_{\nu,r'}$ (and hence $\Pi_x H_1 \in \mathfrak{L}_{\nu,r'}$), then

$$\int_0^{\infty} H_1(xt)(\boldsymbol{H}f)(t)dt = \int_0^{\infty} (\Pi_x H_1)(t)(\boldsymbol{H}f)(t)dt = \int_0^{\infty} (\boldsymbol{H}\Pi_x H_1)(t) f(t)dt. \tag{4.9}$$

From the assumption $\Delta(1 - \nu) + \mathrm{Re}(\mu) \leqq 1/2 - \gamma(r) \leqq 0$, Theorem 2.3(b) and (4.8) imply that

$$(\mathfrak{M}\boldsymbol{H}\Pi_x H_1)(s) = \mathcal{H}(s)(\mathfrak{M}\Pi_x H_1)(1 - s) = x^{-(1-s)}\mathcal{H}(s)(\mathfrak{M}H_1)(1 - s) \tag{4.10}$$

for $\mathrm{Re}(s) = 1-\nu$. Now from (4.6), $H_1(t) \in \mathfrak{L}_{\nu,1}$. Then by the definitions of the H-function (1.2), (1.3) and the direct and inverse Mellin transforms (see, for example, Samko *et al.* [12, (1.112), (1.113)]), we have

$$(\mathfrak{M}H_1)(s) = \mathcal{H}^{q-m,p-n+1}_{p+1,q+1}\left[\begin{array}{c}(-\lambda,h),(1-a_i-\alpha_i,\alpha_i)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n}\\(1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m},(-\lambda-1,h)\end{array}\Bigg| s\right]$$

$$= \mathcal{H}^{q-m,p-n}_{p,q}\left[\begin{array}{c}(1-a_i-\alpha_i,\alpha_i)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n}\\(1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m}\end{array}\Bigg| s\right]\frac{\Gamma(1+\lambda-hs)}{\Gamma(2+\lambda-hs)}$$

$$= \frac{\mathcal{H}_0(s)}{1+\lambda-hs}$$

for $\mathrm{Re}(s)=\nu$, where $\mathcal{H}_0$ is given by (3.2). It follows from here that for $\mathrm{Re}(s) = 1-\nu$,

$$(\mathfrak{M}H_1)(1-s) = \frac{\mathcal{H}_0(1-s)}{1+\lambda-h(1-s)} = \frac{1}{\mathcal{H}(s)[1+\lambda-h(1-s)]}.$$

Substituting this into (4.10) we obtain

$$(\mathfrak{M}\boldsymbol{H}\Pi_x H_1)(s) = \frac{x^{-(1-s)}}{1+\lambda-h(1-s)} \quad (\mathrm{Re}(s)=1-\nu). \tag{4.11}$$

For $x>0$ let us denote by $g_x(t)$ a function

$$g_x(t) = \begin{cases}\dfrac{1}{h}\,t^{(\lambda+1)/h-1} & \text{if } 0<t<x,\\ 0 & \text{if } t>x,\end{cases} \tag{4.12}$$

then

$$(\mathfrak{M}g_x)(s) = \frac{x^{s+(\lambda+1)/h-1}}{1+\lambda-h(1-s)},$$

and (4.11) takes the form

$$(\mathfrak{M}\boldsymbol{H}\Pi_x H_1)(s) = (\mathfrak{M}[x^{-(\lambda+1)/h}g_x)])(s),$$

which implies

$$(\boldsymbol{H}\Pi_x H_1)(t) = x^{-(\lambda+1)/h}g_x(t). \tag{4.13}$$

Substituting (4.13) into (4.9), we have

$$\int_0^\infty H_1(xt)(\boldsymbol{H}f)(t)dt = x^{-(\lambda+1)/h}\int_0^\infty g_x(t)f(t)dt$$

or, in accordance with (4.12),

$$\int_0^x t^{(\lambda+1)/h-1}f(t)dt = hx^{(\lambda+1)/h}\int_0^\infty H_1(xt)(\boldsymbol{H}f)(t)dt.$$

Differentiating this relation we obtain

$$f(x) = hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H_1(xt)(\boldsymbol{H}f)(t)dt$$

which shows (1.11).

If $\mathrm{Re}(\lambda) < \nu h - 1$, the relation (1.12) is proved similarly to (1.11), by taking the function

$$H_2(t) = H^{q-m+1,p-n}_{p+1,q+1}\left[t\,\Bigg|\begin{array}{c}(1-a_i-\alpha_i,\alpha_i)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n},(-\lambda,h)\\(-\lambda-1,h),(1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m}\end{array}\right] \tag{4.14}$$

instead of the function $H_1(t)$ in (4.1). This completes the proof of the theorem.

In the case $\Delta < 0$ the following statement gives the inversion of $\boldsymbol{H}$-transform on $\mathfrak{L}_{\nu,r}$.

THEOREM 4.2. *Let* $1 < r < \infty, \alpha < 1 - \nu < \beta < \infty, \max[\alpha_0, \{\mathrm{Re}(\mu + 1/2)/\Delta\} + 1] < \nu < \beta_0, a^* = 0, \Delta < 0$ *and* $\Delta(1-\nu) + \mathrm{Re}(\mu) \leqq 1/2 - \gamma(r)$, *where* $\gamma(r)$ *is given by* (2.6). *If* $f \in \mathfrak{L}_{\nu,r}$, *then the relations* (1.11) *and* (1.12) *holds for* $\mathrm{Re}(\lambda) > \nu h - 1$ *and for* $\mathrm{Re}(\lambda) < \nu h - 1$, *respectively.*

This theorem is proved similarly to Theorem 4.1, if we apply Theorem 5.2 from Kilbas *et al.* [6] instead of Theorem 2.3 and take into account the asymptotics of the H-function at zero and infinity (see Srivastava *et al.* [13, §2.2] and Kilbas and Saigo [4, Corollary 4]).

REMARK 4.1. If $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$, then Theorems 8.3 and 8.4 in Rooney [11] follow from Theorems 4.1 and 4.2.

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