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COMPLETE CONVERGENCE FOR B-VALUED L^p-MIXINGALE SEQUENCES

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ABSTRACT. Under weaker conditions of probability,we discuss in this paper the complete convergence for the partial sums and the randomly indexed partial sums of B-valued L^p-mixingale sequences.

KEY WORDS AND PHRASES: Complete convergence,L^p-mixingale sequence, q-smooth Banach space.
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1. INTRODUCTION AND MAIN RESULTS

Since the definition of complete convergence for real random variables was introduced by Hsu and Robbins[1],there have been an extensive literature in the complete convergence for independent and dependent random sequences,see partially the references listed.In particular,Yang[5,6] has discussed the complete convergence for B-valued independent random elements.Yu[11] has considered the complete convergence for martingale difference sequences, Peligrad[7] and Shao[8] have obtained the complete convergence for ϕ-mixing sequences,respectively.However,to our best acknowledgement,there are still few articles on the complete convergence for L^p-mixingale ($1 \leq p < 2$) sequences,which include uniformly mixing (called also ϕ-mixing) sequences,martingale difference sequences,linear processes and other random sequences (see [10]).In this paper,under weaker conditions of probability,we discuss the complete convergence for the partial sums and the randomly indexed partial sums of B-valued L^p-mixingale sequences, and give the complete convergence for B-valued martingale difference sequences as corollary.The methods used here are different from those used in the literature.

Next,let us introduce some notations.

Let B be a real Banach space.B is said to be q-smooth($1 < q \leq 2$) if there exists a constant $C_q > 0$ such that for every B-valued L^q-integrable martingale difference sequence $\{D_i; i \geq 1\}$

$$E\|\sum_{i=1}^{n} D_i\|^q \leq C_q \sum_{i=1}^{n} E\|D_i\|^q,\ n \geq 1.$$

Let $\{X_n, n \geq 1\}$ be a sequence of B-valued L^p-integrable ($1 \leq p \leq 2$) random variables on a probability space $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}_n, -\infty < n < \infty\}$ be an increasing sequence of sub σ-fields of $\mathcal{F}$.Then $\{X_n, \mathcal{F}_n\}$ is called a L^p-mixingale sequence if there exist sequences of nonnegative constants C_n and $\psi(n)$,where $\psi(m) \downarrow 0$ as $m \to \infty$,which satisfy following properties:

(i) $\|E(X_n|\mathcal{F}_{n-m})\|_p \le \psi(m)C_n$ and

(ii) $\|X_n - E(X_n|\mathcal{F}_{n+m})\|_p \le \psi(m+1)C_n$

for all $n \ge 1$ and $m \ge 0$,where $\|X\|_p = (E\|X\|^p)^{1/p}$.

S is the class of all positive non-decreasing function ϕ on $R^+ = [0,\infty)$(see [9],p.228 or [5,6]) satisfying the following conditions:

(i) There exists a constant $k = k(\phi) > 0$ such that

$$\phi(xy) \le k(\phi(x) + \phi(y)),\ \forall x, y \in R^+.$$

(ii) $x/\phi(x)$ is non-decreasing for sufficiently large x.

From now on ,we will use C to denote finite positive constants whose value may change from statement to statement .For real number $x, [x]$ denote the largest integer $k \le x$.$I(A)$ represent indicative function of set A. Put $S_n = \sum_{i=1}^{n} X_i$.

THEOREM 1.1. Let $1 \le t < q \le 2, 0 < \delta < \frac{q}{t} - 1, 1 \le p \le 2, d = 1$ or -1,and let B be a q-smooth Banach space.Supppose $\{X_n, \mathcal{F}_n\}$ is a B-valued L^p-mixingale sequence,$\phi(x) \in S$.If

$$\sum_{i=1}^{n} P(\|X_i\|^t(\phi(\|X_i\|))^{-d} > x) \le Cnx^{-(1+\delta)} \tag{1.1}$$

for sufficiently large x, n and there exists a $\lambda(1 \le \lambda \le p)$ such that $t + (1-t)\lambda > 0$ and

$$\sum_{n=1}^{\infty} \psi^\lambda([n^\beta]) \max_{1\le i\le n} C_i^\lambda < \infty, \tag{1.2}$$

where $0 < \beta < \frac{\delta}{1+q} \wedge \frac{q-t}{t(1+q)}$,then for every $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1\le k\le n} \|S_k\| \ge \epsilon(n(\phi(n))^d)^{1/t}) < \infty, \tag{1.3}$$

in particular

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\|S_n\| \ge \epsilon(n(\phi(n))^d)^{1/t}) < \infty. \tag{1.4}$$

THEOREM 1.2. Under the assumptions of THEOREM 1.1,if there exists a $\lambda(1 \le \lambda \le p)$ satisfying

$$\sum_{m=1}^{\infty} m2^{\lambda m}\psi^\lambda([2^{\beta m}]) \max_{1\le i\le 2^{m+1}} C_i^\lambda < \infty, \tag{1.5}$$

then for every $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\sup_{k\ge n}(\|S_k\|/(k(\phi(k))^d)^{1/t}) \ge \epsilon) < \infty. \tag{1.6}$$

If $\{X_n, \mathcal{F}_n, n \ge 1\}$ is a L^p-integrable B-valued martingale difference sequence,then $C_n = (E\|X_n\|^p)^{1/p}, \psi(m) = 0$ for $m \ge 1$.

COROLLARY 1.1. Let $1 \le t < q \le 2, d = 1$ or -1,and let B be a q-smooth Banach space.Supppose $\{X_n, \mathcal{F}_n, n \ge 1\}$ is a B-valued martingale difference sequence,$\phi(x) \in S$.For $0 < \delta < \frac{q}{t} - 1$ and sufficiently large x and n,if (1.1) is satisfied,then for every $\epsilon > 0$, we obtain that (1.3),(1.4) and (1.6) hold.

REMARK 1.1. For $0 < t < 1$,by C_r-inequality and properties of $\phi(x)$, we can prove that the results of THEOREM 1.1,THEOREM 1.2 and COROLLARY 1.1 hold for any B-valued random variable sequence $\{X_n, n \ge 1\}$ without mixing condition (1.2) and (1.5).

REMARK 1.2. Real uniformly mixing sequence (definition see [7] or [8],[10]) $\{X_n, \mathcal{F}_n\}$ is L^2-mixingale,where $C_n = 2(EX_n^2)^{1/2}, \phi(m) = \phi^{1/2}(m)$,see [10,p.19].

REMARK 1.3. Yang[5] has proved that (1.3) and (1.4) hold for B-valued independent zero mean random element sequence $\{X_n\}$ in type 2 Banach space under moment conditions stronger than the conditions of COROLLARY 1.1.

2. PROOFS OF MAIN RESULTS

We only prove the case in $d = 1$ for Theorem 1.1 and 1.2,the proof of the case in $d = -1$ is analogous.

LEMMA 2.1.([9],Lemma 1) Let $\phi(\cdot) \in S, \delta > 0$,then for any $x \geq 0$,

$$C\phi(x) \leq \phi(x\phi(x)) \leq C\phi(x);$$

$$C\phi(x) \leq \phi(x/\phi(x)) \leq C\phi(x);$$

$$C\phi(x) \leq \phi(x^\delta) \leq C\phi(x).$$

PROOF OF THEOREM 1.1. Notice first

$$\begin{aligned} S_k &= \sum_{i=1}^{k}(X_i - E(X_i|\mathcal{F}_{i+[n^\beta]})) \\ &\quad + \sum_{l=-[n^\beta]+1}^{[n^\beta]} \sum_{i=1}^{k}(E(X_i|\mathcal{F}_{i+l}) - E(X_i|\mathcal{F}_{i+l-1})) \\ &\quad + \sum_{i=1}^{k} E(X_i|\mathcal{F}_{i-[n^\beta]}) \\ &\triangleq A_k^1 + B_k^1 + C_k^1. \end{aligned}$$

Obviously

$$\begin{aligned} &\sum_{n=1}^{\infty} \tfrac{1}{n}P(\max_{1\leq k\leq n} \|S_k\| \geq \epsilon(n\phi(n))^{1/t}) \\ \leq &\sum_{n=1}^{\infty} \tfrac{1}{n}P(\max_{1\leq k\leq n} \|A_k^1\| \geq \tfrac{\epsilon}{3}(n\phi(n))^{1/t}) \\ &+ \sum_{n=1}^{\infty} \tfrac{1}{n}P(\max_{1\leq k\leq n} \|B_k^1\| \geq \tfrac{\epsilon}{3}(n\phi(n))^{1/t}) \\ &+ \sum_{n=1}^{\infty} \tfrac{1}{n}P(\max_{1\leq k\leq n} \|C_k^1\| \geq \tfrac{\epsilon}{3}(n\phi(n))^{1/t}) \\ \triangleq &\ I_1 + I_2 + I_3. \end{aligned}$$

By the Markovian inequality,L^p-mixingale property and the properties of $\phi(x)$,we have

$$\begin{aligned} I_1 &\leq C \sum_{n=1}^{\infty} \tfrac{1}{n}(n\phi(n))^{-\lambda/t}E(\sum_{i=1}^{n} \|X_i - E(X_i|\mathcal{F}_{i+[n^\beta]})\|)^\lambda \\ &\leq C \sum_{n=1}^{\infty} \tfrac{1}{n}(n\phi(n))^{-\lambda/t}(\sum_{i=1}^{n} \|X_i - E(X_i|\mathcal{F}_{i+[n^\beta]})\|_\lambda)^\lambda \\ &\leq C \sum_{n=1}^{\infty} (\phi(n))^{-\lambda/t} \cdot n^{\lambda-1-\lambda/t}\psi^\lambda([n^\beta]) \max_{1\leq i\leq n} C_i^\lambda \\ &\leq C \sum_{n=1}^{\infty} \psi^\lambda([n^\beta]) \max_{1\leq i\leq n} C_i^\lambda < \infty. \end{aligned}$$

Similarly,we can obtain

$$I_3 \leq C \sum_{n=1}^{\infty} \psi^\lambda([n^\beta]) \max_{1\leq i\leq n} C_i^\lambda < \infty.$$

Let $Y_{i,n} = X_i I(\|X_i\| \leq (n\phi(n))^{1/t}), Z_{i,n} = X_i - Y_{i,n}, W_{l,i} = E(X_i|\mathcal{F}_{i+l}) - E(X_i|\mathcal{F}_{i+l-1}), U_{l,i} = E(Y_{i,n}|\mathcal{F}_{i+l}) - E(Y_{i,n}|\mathcal{F}_{i+l-1}), V_{l,i} = E(Z_{i,n}|\mathcal{F}_{i+l}) - E(Z_{i,n}|\mathcal{F}_{i+l-1}), 1 \leq k \leq n, 1 \leq i \leq k, -[n^\beta] + 1 \leq l < [n^\beta]$.

Clearly,$X_i = Y_{i,n} + Z_{i,n}, W_{l,i} = U_{l,i} + V_{l,i}$.For fixed $l, \{U_{l,i}, \mathcal{F}_{i+l}, 1 \leq i \leq n\}$ and $\{V_{l,i}, \mathcal{F}_{i+l}, 1 \leq$

$i \le n\}$ are martingale difference sequences. Then

$$\begin{aligned} I_2 &\le \sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=-[n^\beta]+1}^{[n^\beta]} P(\max_{1\le k\le n}\|\sum_{i=1}^{k}W_{l,i}\| \ge \tfrac{\epsilon}{6}(n\phi(n))^{1/t}\cdot n^{-\beta}) \\ &\le \sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=-[n^\beta]+1}^{[n^\beta]} P(\max_{1\le k\le n}\|\sum_{i=1}^{k}U_{l,i}\| \ge \tfrac{\epsilon}{12}(n\phi(n))^{1/t}\cdot n^{-\beta}) \\ &\quad + \sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=-[n^\beta]+1}^{[n^\beta]} P(\max_{1\le k\le n}\|\sum_{i=1}^{k}V_{l,i}\| \ge \tfrac{\epsilon}{12}(n\phi(n))^{1/t}\cdot n^{-\beta}) \\ &\triangleq I_4 + I_5. \end{aligned}$$

Since B is q-smoothable,therefore using Doob inequality,the monotone property of $x/\phi(x)$ and Lemma 2.1 we have

$$\begin{aligned} I_4 &\le C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=-[n^\beta]+1}^{[n^\beta]}(n\phi(n))^{-q/t}\cdot n^{\beta q}\sum_{i=1}^{n}E\|Y_{i,n}\|^q \\ &\le C\sum_{n=1}^{\infty}\frac{1}{n}\cdot[n^\beta]\cdot(n\phi(n))^{-q/t}\cdot n^{\beta q}\sum_{i=1}^{n}\int_0^{n\phi(n)}x^{q/t-1}P(\|X_i\|^t > x)dx \\ &\le C\sum_{n=1}^{\infty}(\phi(n))^{-q/t}\cdot n^{-[q/t-(1+q)\beta]} \\ &\quad + C\sum_{n=1}^{\infty}(\phi(n))^{-q/t}\cdot n^{-[q/t-(1+q)\beta]-1}\int_C^{n\phi(n)}x^{q/t-1}\sum_{i=1}^{n}P(\|X_i\|^t/\phi(\|X_i\|) \ge Cx/\phi(x))dx \\ &\le C + C\sum_{n=1}^{\infty}n^{(1+q)\beta-(1+\delta)} < \infty. \end{aligned}$$

By applying the definition of $\phi(x)$ and Lemma 2.1,we can obtain

$$\phi(x^\alpha) \le Cx^\beta \tag{2.1}$$

for $\alpha, \beta > 0$ and sufficiently large x.

By the Markovian inequality,the definition of $\phi(x)$,Lemma 2.1 and (2.1) we have

$$\begin{aligned} I_5 &\le C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=-[n^\beta]+1}^{[n^\beta]}(n\phi(n))^{-1/t}\cdot n^{\beta}\sum_{i=1}^{n}E\|Z_{i,n}\| \\ &\le C\sum_{n=1}^{\infty}\frac{1}{n}\cdot[n^\beta]\cdot(n\phi(n))^{-1/t}\cdot n^{\beta}\sum_{i=1}^{n}[(n\phi(n))^{1/t}P(\|X_i\| > (n\phi(n))^{1/t}) \\ &\quad + \int_{(n\phi(n))^{1/t}}^{\infty}P(\|X_i\| > x)dx] \\ &\le C\sum_{n=1}^{\infty}n^{2\beta-1}\sum_{i=1}^{n}[P(\|X_i\|^t/\phi(\|X_i\|) \ge Cn) + \int_1^{\infty}P(\|X_i\| > y(n\phi(n))^{1/t})dy] \\ &\le C\sum_{n=1}^{\infty}n^{2\beta-(1+\delta)} + C\sum_{n=1}^{\infty}n^{2\beta-1}\int_1^{\infty}\sum_{i=1}^{n}P(\frac{\|X_i\|^t}{\phi(\|X_i\|)} \ge \frac{Cy^t(n\phi(n))}{\phi(y^t)+C\phi(n)})dy \\ &\le C + C\sum_{n=1}^{\infty}n^{2\beta-(1+\delta)}\int_1^{\infty}[\frac{(\phi(y^t))^{1+\delta}}{y^{t(1+\delta)}} + \frac{1}{y^{t(1+\delta)}}]dy < \infty. \end{aligned}$$

The proof is completed.

PROOF OF THEOREM 1.2. First,by the monotone property of $\phi(x)$ we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{1}{n}P(\sup_{k\ge n}(\|S_k\|/(k\phi(k))^{1/t}) \ge \epsilon) \\ \le\ &\sum_{i=0}^{\infty}P(\sup_{k\ge 2^i}(\|S_k\|/(k\phi(k))^{1/t}) \ge \epsilon) \\ \le\ &C\sum_{m=1}^{\infty}mP(\max_{2^m\le k<2^{m+1}}(\|S_k\| \ge \epsilon(2^m\phi(2^m))^{1/t}). \end{aligned} \tag{2.2}$$

Observe that for $2^m \le k < 2^{m+1}$,

$$\begin{aligned} S_k &= \sum_{i=1}^{k}(X_i - E(X_i|\mathcal{F}_{i+[2^{\beta m}]})) \\ &\quad + \sum_{l=-[2^{\beta m}]+1}^{[2^{\beta m}]}\sum_{i=1}^{k}(E(X_i|\mathcal{F}_{i+l}) - E(X_i|\mathcal{F}_{i+l-1})) \\ &\quad + \sum_{i=1}^{k}E(X_i|\mathcal{F}_{i-[2^{\beta m}]}) \\ &\triangleq A_k^2 + B_k^2 + C_k^2. \end{aligned}$$

Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} \tfrac{1}{n} P(\sup_{k\geq n}(\|S_k\|/(k\phi(k))^{1/t}) \geq \epsilon) \\
\leq\ & C \sum_{m=1}^{\infty} mP(\max_{2^m\leq k<2^{m+1}}(\|A_k^2\| \geq \tfrac{\epsilon}{3}(2^m\phi(2^m))^{1/t}) \\
&+C \sum_{m=1}^{\infty} mP(\max_{2^m\leq k<2^{m+1}}(\|B_k^2\| \geq \tfrac{\epsilon}{3}(2^m\phi(2^m))^{1/t}) \\
&+C \sum_{m=1}^{\infty} mP(\max_{2^m\leq k<2^{m+1}}(\|C_k^2\| \geq \tfrac{\epsilon}{3}(2^m\phi(2^m))^{1/t}) \\
\triangleq\ & I_6 + I_7 + I_8.
\end{aligned}$$

By analogizing the proof of I_1 we have

$$I_6 \leq C \sum_{m=1}^{\infty} m2^{\lambda m}\psi^{\lambda}([2^{\beta m}]) \max_{1\leq i\leq 2^{m+1}} C_i^{\lambda} < \infty.$$

By analogizing the proof of I_3,similarly, we can obtain $I_8 < \infty$.

Let $Y_{i,m} = X_i I(\|X_i\| \leq (2^m\phi(2^m))^{1/t}), Z_{i,m} = X_i - Y_{i,m}, W_{l,i} = E(X_i|\mathcal{F}_{i+l}) - E(X_i|\mathcal{F}_{i+l-1}), U_{l,i} = E(Y_{i,m}|\mathcal{F}_{i+l}) - E(Y_{i,m}|\mathcal{F}_{i+l-1}), V_{l,i} = E(Z_{i,m}|\mathcal{F}_{i+l}) - E(Z_{i,m}|\mathcal{F}_{i+l-1}), 2^m \leq k < 2^{m+1}, 1 \leq i \leq k, -[2^{\beta m}] + 1 \leq l \leq [2^{\beta m}]$. Then

$$\begin{aligned}
I_7 \leq\ & C \sum_{m=1}^{\infty} m \sum_{l=-[2^{\beta m}]+1}^{[2^{\beta m}]} P(\max_{2^m\leq k<2^{m+1}} \|\sum_{i=1}^{k} U_{l,i}\| \geq \tfrac{\epsilon}{12}(2^m\phi(2^m))^{1/t}\cdot 2^{-\beta m}) \\
&+C \sum_{m=1}^{\infty} m \sum_{l=-[2^{\beta m}]+1}^{[2^{\beta m}]} P(\max_{2^m\leq k<2^{m+1}} \|\sum_{i=1}^{k} V_{l,i}\| \geq \tfrac{\epsilon}{12}(2^m\phi(2^m))^{1/t}\cdot 2^{-\beta m}) \\
\triangleq\ & I_9 + I_{10}.
\end{aligned}$$

By analogizing the proof of I_4 we have

$$I_9 \leq C \sum_{m=1}^{\infty} m2^{(\beta+q\beta+1-q/t)m}(\phi(2^m))^{-q/t} + C \sum_{m=1}^{\infty} m2^{(\beta+q\beta-\delta)m} < \infty.$$

By analogizing the proof of I_5 we have

$$I_{10} \leq C \sum_{m=1}^{\infty} m2^{(2\beta-\delta)m} + C \sum_{m=1}^{\infty} m2^{(2\beta-\delta)m} \int_1^{\infty} [\frac{(\phi(y^t))^{1+\delta}}{y^{t(1+\delta)}} + \frac{1}{y^{t(1+\delta)}}]dy < \infty.$$

3. RANDOMLY INDEXED PARTIAL SUMS

Throughout this section let $\{X_n, \mathcal{F}_n\}$ be a B-valued L^p-mixingale sequence ($1 \leq p \leq 2$), and let $\{\tau_n, n \geq 1\}$ be a sequence of nonnegative,integer valued random variables.τ is a positive random variable.All random variables are defined on the same probability space.

THEOREM 3.1. Under the assumptions of THEOREM 1.2,if there exists some constant $\epsilon_0 > 0$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\frac{\tau_n}{n} < \epsilon_0) < \infty, \tag{3.1}$$

then for every $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\|S_{\tau_n}\| \geq \epsilon(\tau_n(\phi(\tau_n))^d)^{1/t}) < \infty. \tag{3.2}$$

THEOREM 3.2. Under the assumptions of THEOREM 1.1,if there exist constants $a, b, \epsilon_0 (0 < \epsilon_0 < a \leq b < \infty)$ such that $P(a \leq \tau \leq b) = 1$ and

$$\sum_{n=1}^{\infty} \frac{1}{n} P(|\frac{\tau_n}{n} - \tau| > \epsilon_0) < \infty, \tag{3.3}$$

then for every $\epsilon > 0$ we obtain that (3.2) holds.

THEOREM 3.3. Under the assumptions of THEOREM 1.1,if there exist constants $b > 0$ and $\epsilon_0 > 0$ such that $P(\tau \leq b) = 1$ and (3.3) is satisfied,then for every $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\|S_{\tau_n}\| \geq \epsilon(n(\phi(n))^d)^{1/t}) < \infty.$$

Obviously,suppose $P(\tau \geq a) = 1$ for some $a > 0$,then for any $\epsilon > 0(\epsilon < a)$ we have

$$P(\frac{\tau_n}{n} < a - \epsilon) \leq P(|\frac{\tau_n}{n} - \tau| > \epsilon),$$

therefore,if condition (3.3),where $P(\tau \geq a) = 1$ for some $a > \epsilon_0 > 0$ replaces condition (3.1),then THEOREM 3.1 still holds.

Similarly,using COROLLARY 1.1,we can obtain the complete convergence for the randomly indexed partial sums of B-valued martingale difference sequences,respectively.

REMARK 3.1. Condition (3.1) and (3.3) are just ones which are usually employed in literature.

REMARK 3.2. Note that if $\phi(x) = 1, L_k(x)(L_0(x) = \max(1, \log x), L_k = \log[\max(e, L_{k-1}(x)]$, $k = 1, 2, \cdots$),we can derive many significative results from the results of this paper.In addition, since real space is a 2-smooth Banach space,the THEOREM and COROLLARY in this paper are suitable for real valued random variable.

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