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COMMON FIXED POINTS OF BIASED MAPS OF TYPE (A) AND APPLICATIONS

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ABSTRACT. A generalization of compatible maps of type (A) called "biased maps of type (A)" is introduced and used to prove fixed point theorems for certain contractions of four maps. Extensions of known results are thereby obtained, i.e., the results of Pathak, Prasad, Jungck et al. are improved. Some problems on convergence of self-maps and fixed points are also discussed. Further, we use our main results to show the existence of solutions of nonlinear integral equations.

KEY WORDS AND PHRASES: Compatible maps and compatible maps of type (A), A-biased and S-biased maps, weakly A-biased and S-biased maps, common fixed point, simultaneous Volterra-Hammerstein equation.

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1. INTRODUCTION

Self-maps A and S of a metric space (X, d) are said to be compatible of type (A) ([12]) if

$$d(SAx_n, AAx_n) \to 0, \quad d(ASx_n, SSx_n) \to 0$$

whenever $\{x_n\}$ is a sequence in X such that Ax_n and $Sx_n \to t \in X$. Compatible maps of type (A) are, in fact, equivalent to the concept of compatible maps under continuity of maps ([12]). Recall that self-maps A and S of X are said to be compatible ([6]) if

$$d(ASx_n, SAx_n) \to 0$$

whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to t \in X$. It may be remarked that compatible maps were introduced in [6] as a generalization of commuting maps and weakly commuting maps ([21]) and have been proved a sharper tool for obtaining more comprehensive fixed point theorems ([1]-[9], [14], [15], [19], [20]) and in the study of periodic points [10]. We now introduce the concept of biased maps of type (A). Our new concept is an appreciable generalization of compatible maps of type (A) which, as we shall see, proves useful in the "fixed point" arena. Further, we use our main results to show the existence of solutions of nonlinear integral equations.

2. BIASED MAPS OF TYPE (A)

In this section, we show that the concepts of biased maps of type (A) is a legitimate generalization of compatible maps of type (A) and give several properties of biased maps of type (A) for our main results.

Definition 2.1. Let A and S be self-maps of a metric space (X,d). The pair $\{A,S\}$ is said to be S-biased and A-biased of type (A), respectively, if, whenever $\{x_n\}$ is a sequence in X and Ax_n, $Sx_n \to t \in X$,

$$\alpha d(SSx_n, Ax_n) \le \alpha d(ASx_n, Sx_n), \tag{Sb}$$

$$\alpha d(AAx_n, Sx_n) \le \alpha d(SAx_n, Ax_n), \tag{Ab}$$

respectively, where $\alpha = \liminf_{n\to\infty}$ and if $\alpha = \limsup_{n\to\infty}$.

Of course, if the inequality in (Sb) (or (Ab)) holds with $\alpha = \lim_{n\to\infty}$ (one has to presuppose that the indicated limit exists), then $\liminf_{n\to\infty} = \limsup_{n\to\infty} = \lim_{n\to\infty}$ and (Sb) (or (Ab)) is satisfied. We shall frequently use this fact in our further discussion.

As to notations, we shall use N, R^+, Q, I_r and I to denote the positive integers, non-negative real numbers, the rational numbers, the irrational numbers, and $[0,1]$, respectively.

The following example shows why we could not restrict α to "$\lim_{n\to\infty}$" if the concept of biased maps of type (A) is to generalize compatibility of type (A).

Example 2.2. Let $X = I$ and define mappings $A, S : X \to X$ by

$$Ax = Sx = \begin{cases} 1-x & \text{for } x \in \left[0, \frac{1}{2}\right] \\ 0 & \text{for } x \in Q \cap \left(\frac{1}{2}, 1\right] \\ 1 & \text{for } x \in I_r \cap \left(\frac{1}{2}, 1\right]. \end{cases}$$

Let $x_{2n} = \frac{1}{2n}$ and $x_{2n-1} = \frac{\sqrt{3}}{2(2n+1)}$ for $n \in N$. Then $Sx_k \to 1$ as $k \to \infty$, $SSx_{2n} = 0$, $SSx_{2n-1} = 1$, and, therefore, $\lim_{k\to\infty} d(SSx_k, Sx_k)$ does not exist although $\lim_{k\to\infty} d(SSx_k, SSx_k) = 0$. In fact, the pair $\{S,S\}$ is trivially compatible of type (A) for any function S.

Be assured that the concept of "biased maps of type (A)" arises naturally in the context of contractive or relatively nonexpansive maps ([8]). See also Proposition 2.5 below.

Remark 2.3. If the pair $\{A,S\}$ is compatible of type (A), then it is both S-biased and A-biased of type (A). From

$$d(SSx_n, Ax_n) \le d(SSx_n, ASx_n) + d(ASx_n, Sx_n) + d(Sx_n, Ax_n),$$
$$d(AAx_n, Sx_n) \le d(AAx_n, SAx_n) + d(SAx_n, Ax_n) + d(Ax_n, Sx_n)$$

for $n \in N$, it follows that

$$\alpha d(SSx_n, Ax_n) \le 0 + \alpha d(ASx_n, Sx_n) + 0,$$
$$\alpha d(AAx_n, Sx_n) \le 0 + \alpha d(SAx_n, Ax_n) + 0,$$

respectively, if Ax_n, $Sx_n \to t \in X$, $\{A,S\}$ is a compatible pair of type (A) and α is either $\liminf_{n\to\infty}$ or $\limsup_{n\to\infty}$. Therefore, the pair $\{A,S\}$ is both S-biased and A-biased of type (A).

However, the converse of Remark 2.3 is not necessarily true. To this end, consider the following example:

Example 2.4. Let $X = [0,\infty)$ with the usual metric $d(x,y) = |x-y|$. Define mappings $A, S : X \to X$ by

$$Ax = \begin{cases} 1+x & \text{if } x \in [0,1) \\ 1 & \text{if } x \in [1,\infty), \end{cases} \qquad Sx = \begin{cases} 1-x & \text{if } x \in [0,1) \\ 2 & \text{if } x \in [1,\infty), \end{cases}$$

respectively. Then A and S are not continuous at $x = 1$. Now, we assert that the pair $\{A, S\}$ is not compatible of type (A), but it is S-biased and A-biased of type (A). To show this, we first note that Ax_n, $Sx_n \to t \in X$ iff $t = 1$ and $x_n \to 0^+$. Then, if Ax_n, $Sx_n \to 1$, it follows that $Ax_n = 1+x_n \to 1^+$ and $Sx_n = 1-x_n \to 1^-$. Thus, since $1+x_n > 1$ and $1-x_n < 1$ for all $n \in N$, we have $ASx_n = 2-x_n$, $SAx_n = 2$, $SSx_n = x_n$ and $AAx_n = 1$. Thus $1 = \alpha|SSx_n - Ax_n| \le \alpha|ASx_n - Sx_n| = 1$ and $0 = \alpha|AAx_n - Sx_n| < \alpha|SAx_n - Ax_n| = 1$. Therefore, the pair $\{A, S\}$ is S-biased and A-biased of type (A), but it is not compatible of type (A).

The next result is the analogue to Proposition 1.1 in [13] for S-biased and A-biased maps, and so we omit its proof. Recall that self-maps A and S of a metric space (X, d) are said to be S-biased and A-biased, respectively, iff whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to t \in X$, then

$$\alpha d(SAx_n, Sx_n) \le \alpha d(ASx_n, Ax_n),$$
$$\alpha d(ASx_n, Ax_n) \le \alpha d(SAx_n, Sx_n),$$

respectively, if $\alpha = \liminf$ and if $\alpha = \limsup$.

Proposition 2.5. *Let A and S be self-maps of a metric space (X, d).*

(a) If the pair $\{A, S\}$ is S-biased of type (A) and $Ap = Sp$, then

$$d(SSp, Ap) \le d(ASp, Sp).$$

(b) If A and S are continuous and one of A and S is proper, then the pair $\{A, S\}$ is S-biased of type (A) iff $Ap = Sp$ implies that

$$d(SSp, Ap) \le d(ASp, Sp).$$

In Example 2.2, the pair $\{A, S\}$ was both S-biased and A-biased of type (A). Of course, this need not be the case. Consider the following example:

Example 2.6. Let $I = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. Define mappings $A, S : I \to I$ by

$$Ax = \begin{cases} 1 - 2x & \text{for } x \in [0, \frac{1}{2}) \\ \frac{1}{3} & \text{for } x \in [\frac{1}{2}, 1], \end{cases} \qquad Sx = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}) \\ 0 & \text{for } x \in [\frac{1}{2}, 1], \end{cases}$$

respectively. Now, $Ax = Sx$ iff $x = \frac{1}{4}$. Since $A(\frac{1}{4}) = S(\frac{1}{4}) = \frac{1}{2}$, $SS(\frac{1}{4}) = S(\frac{1}{2}) = 0$, $AA(\frac{1}{4}) = A(\frac{1}{2}) = \frac{1}{3}$, $SA(\frac{1}{4}) = S(\frac{1}{2}) = 0$, $AS(\frac{1}{4}) = A(\frac{1}{2}) = \frac{1}{3}$, we have

$$\Big|SS\Big(\frac{1}{4}\Big) - A\Big(\frac{1}{4}\Big)\Big| = \frac{1}{2}, \quad \Big|AS\Big(\frac{1}{4}\Big) - S\Big(\frac{1}{4}\Big)\Big| = \frac{1}{6};$$
$$\Big|AA\Big(\frac{1}{4}\Big) - S\Big(\frac{1}{4}\Big)\Big| = \frac{1}{6}, \quad \Big|SA\Big(\frac{1}{4}\Big) - A\Big(\frac{1}{4}\Big)\Big| = \frac{1}{2}.$$

Therefore, by Proposition 2.5, the pair $\{A, S\}$ is A-biased of type (A), but it is not S-biased of type (A). Consequently, Remark 2.3 tells that $\{A, S\}$ is not compatible of type (A).

One may note that, in both Examples 2.4 and 2.6, the maps A and S were discontinuous. However, this need not be the case. Consider the following example:

Example 2.7. Define mappings $A, S : [0, 1] \to [0, 1]$ by

$$A(x) = \Big(\frac{1}{2} - x\Big)^2, \quad S(x) = 2\Big(\frac{1}{2} - x\Big)^2$$

for $x \in [0, 1]$, respectively. Then clearly A and S are continuous and proper too. Now, by routine computation, one can see, from Proposition 2.5, that the pair $\{A, S\}$ is A-biased of type (A), but it is not S-biased of type (A). Therefore, Remark 2.3 again says that the pair $\{A, S\}$ is not compatible of type (A).

We also recall some properties of compatible maps of type (A) in [12].

Proposition 2.8. *Let $A, S : X \to X$ be mappings. If the pair $\{A, S\}$ is compatible of type (A) and $Ap = Sp$ for some $t \in X$, then $ASp = SSp = AAp = SAp$.*

Proposition 2.9. *Let $A, S : X \to X$ be mappings. Let $\{A, S\}$ be a compatible pair of type (A) and let Ax_n, $Sx_n \to p$ for some $p \in X$. Then we have the following:*

(a′) $\lim_{n\to\infty} SAx_n = Ap$ *if A is continuous at p.*

(b′) $ASp = SAp$ *and* $Ap = Sp$ *if A and S are continuous at p.*

In one hand, the equalities among ASp, SSp, SAp, AAp, and, on the other hand, $Ap = Sp$ of Proposition 2.8 implies the conclusion of Proposition 2.5 (a) in the strong sense "$=$", but not in the weak sense "$\leq$". Similar observation can be made while comparing Proposition 2.9 and Proposition 2.5 (b). This comparison and Examples 2.4, 2.6 and 2.7 reveal the fact that "biased maps of type (A)" are a legitimate generalization of the concept of compatible maps of type (A) for continuous and discontinuous functions as well.

The following proposition tells us that biased maps of type (A) arise quite naturally in contractive and relatively nonexpansive maps.

Proposition 2.10. *Let A, B, S and T be self-maps of a metric space (X, d) such that $A(X) \subset T(X)$ and $d(Ax, By) \leq d(Sx, Ty)$ for all $x, y \in X$. If S is continuous, the pair $\{A, S\}$ is A-biased of type (A).*

Proof. Following the proof of [13], we have Ax_n, By_n, Sx_n, $Ty_n \to t$. Now, since $d(AAx_n, Sx_n) \leq d(AAx_n, By_n) + d(By_n, Sx_n)$ for all $n \in N$, the continuity of S implies

$$\begin{aligned}\alpha d(AAx_n, Sx_n) &\leq \alpha d(AAx_n, By_n)\\ &\leq \lim_{n\to\infty} d(SAx_n, Ty_n)\\ &= \lim_{n\to\infty} d(SAx_n, Ax_n),\end{aligned}$$

where $\alpha = \liminf_{n\to\infty}$ or $\limsup_{n\to\infty}$, i.e., the pair $\{A, S\}$ is A-biased of type (A). This completes the proof.

On the other hand, Example 2.7 says that even though $A = B$ and $S = T$ and both A and S are continuous in Proposition 2.5, the pair $\{A, S\}$ need not be S-biased of type (A).

We conclude this section by noting that the concept of biased maps of type (A) appears to be a natural and effective generalization of compatible maps of type (A). It is natural since the resulting development parallels that of biased maps as a generalization of compatible maps ([13]). However, biased maps of type (A) are a variant by name and by definition of the concept "biased maps" introduced in [13]. Hence, we need to demonstrate that our type (A) concept is indeed distinct from the original concept of biased maps. One may note that there are pairs $\{A, S\}$ of maps which are S-biased but not S-biased of type (A), and conversely.

Example 2.11. Let A, $S : [0,1] \to [0,1]$ be defined by

$$Ax = \frac{1}{2} \text{ for } x \in [0,1], \quad Sx = \begin{cases} 1 - x & \text{for } x \in \left[0, \frac{1}{2}\right] \\ 0 & \text{for } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

respectively. Then it is easy to show that the pair $\{A, S\}$ is S-biased, but not S-biased of type (A) (Take a sequence $\{x_n\}$ in $[0,1]$ such that $x_n \to \frac{1}{2}$ and $x_n < \frac{1}{2}$ for all n).

Example 2.12. Let A, $S : [0,1] \to [0,1]$ be defined by

$$Ax = \begin{cases} x & \text{for } x \in \left[0, \frac{1}{2}\right] \\ 1 & \text{for } x \in \left(\frac{1}{2}, 1\right], \end{cases} \quad Sx = \begin{cases} 1 - x & \text{for } x \in \left[0, \frac{1}{2}\right] \\ 1 & \text{for } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

respectively. Then it is easy to show that the pair $\{A, S\}$ is S-biased of type (A), but not S-biased.

3. COMMON FIXED POINT THEOREMS

Let $\mathcal{F}$ be a family of all functions $\phi : (R^+)^5 \to R^+$ such that ϕ is upper semi-continuous, non-decreasing in each coordinate variable and, for any $t \geq 0$,

$$\phi(t, t, 0, \alpha t, 0) \leq \beta t, \qquad \phi(t, t, 0, 0, \alpha t) \leq \beta t,$$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$,

$$\gamma(t) = \phi(t, t, a_1 t, a_2 t, a_3 t) < t,$$

where $\gamma : R^+ \to R^+$ is a mapping and $a_1 + a_2 + a_3 = 4$.

We need the following lemma for our main theorems:

Lemma 3.1. ([22]) *For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the n-times composition of γ.*

Let A, B, S and T be mappings from a metric space (X, d) into itself such that

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \tag{3.1}$$

$$\begin{aligned} d^2(Ax, By) \leq \phi(&d^2(Sx, Ty), d(Sx, Ax) \cdot d(Ty, By), \\ &d(Sx, By) \cdot d(Ty, Ax), d(Sx, Ax) \cdot d(Ty, Ax), \\ &d(Sx, By) \cdot d(Ty, By)) \end{aligned} \tag{3.2}$$

for all $x, y \in X$, where $\phi \in \mathcal{F}$. Then, by (3.1), since $A(X) \subset T(X)$, for any arbitrary point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$\begin{cases} y_{2n} = Tx_{2n+1} = Ax_{2n}, \\ y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \end{cases} \tag{3.3}$$

for $n = 0, 1, 2, \cdots$.

Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (3.1) and (3.2). Then we have the following lemmas:

Lemma 3.2. ([12]) *$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, where $\{y_n\}$ is the sequence in X defined by* (3.3).

Lemma 3.3. ([12]) *The sequence $\{y_n\}$ defined by* (3.3) *is a Cauchy sequence in X.*

Proposition 2.5 prompts the following convenient definition:

Definition 3.4. Let A and S be self-maps of a metric space (X, d). The pair $\{A, S\}$ is said to be weakly S-biased of type (A) if $Ap = Sp$ implies

$$d(SSp, Ap) \leq d(ASp, Sp).$$

Of course, if the pair $\{A, S\}$ is S-biased of type (A), it is weakly S-biased of type (A) by Proposition 2.5 (a).

Now, we are ready to prove our main theorem by using preceding lemmas:

Theorem 3.5. *Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions* (3.1), (3.2) *and any one of the following:*
(3.4) *A is continuous and the pairs $\{A, S\}$, $\{B, T\}$ are compatible of type (A) and weakly B-biased of type (A), respectively,*

(3.5) B *is continuous and the pairs* $\{A, S\}$, $\{B, T\}$ *are weakly* A*-biased of type* (A) *and compatible of type* (A)*, respectively,*
(3.6) S *is continuous and the pairs* $\{A, S\}$, $\{B, T\}$ *are compatible of type* (A) *and weakly* T*-biased of type* (A)*, respectively,*
(3.7) T *is continuous and the pairs* $\{A, S\}$, $\{B, T\}$ *are weakly* S*-biased of type* (A) *and compatible of type* (A)*, respectively.*
Then A, B, S *and* T *have a unique common fixed point in* X.

Proof. By virtue of Lemma 3.3, the sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence in X, and so since X is complete, it converges to a point z in X. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to z.

Now suppose that the condition (3.7) holds. Since the pair $\{B, T\}$ is compatible of type (A) and T is continuous, it follows that

$$BTx_{2n+1}, \quad TTx_{2n+1} \to Tz \quad \text{as } n \to \infty. \tag{3.8}$$

Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (3.2), we have

$$\begin{aligned} &d(Ax_{2n}, BTx_{2n+1}) \\ &\leq [\phi(d^2(Sx_{2n}, TTx_{2n+1}), \\ &\quad d(Sx_{2n}, Ax_{2n}) \cdot d(TTx_{2n+1}, BTx_{2n+1}), \\ &\quad d(Sx_{2n}, BTx_{2n+1}) \cdot d(TTx_{2n+1}, Ax_{2n}), \\ &\quad d(Sx_{2n}, Ax_{2n}) \cdot d(TTx_{2n+1}, Ax_{2n}), \\ &\quad d(Sx_{2n}, BTx_{2n+1}) \cdot d(TTx_{2n+1}, BTx_{2n+1}))]^{1/2}. \end{aligned} \tag{3.9}$$

Taking $n \to \infty$ in (3.9) and using (3.8), since $\phi \in \mathcal{F}$, we have

$$\begin{aligned} d(z, Tz) &\leq [\phi(d^2(z, Tz), 0, d^2(z, Tz), 0, 0)]^{1/2} \\ &= [\gamma(d^2(z, Tz)]^{1/2} \\ &< d(z, Tz), \end{aligned}$$

which is a contradiction. Thus, we have $Tz = z$. Again, replacing x by x_{2n} and y by z in (3.2), we have

$$\begin{aligned} d(Ax_{2n}, Bz) \leq [\phi(&d^2(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}) \cdot d(Tz, Bz), \\ &d(Sx_{2n}, Bz) \cdot d(Tz, Ax_{2n}), d(Sx_{2n}, Ax_{2n}) \cdot d(Tz, Ax_{2n}), \\ &d(Sx_{2n}, Bz) \cdot d(Tz, Bz))]^{1/2}. \end{aligned}$$

Taking $n \to \infty$ in the above inequality, we have

$$\begin{aligned} d(z, Bz) &\leq [\phi(0, 0, 0, 0, d^2(z, Bz))]^{1/2} \\ &\leq [\gamma(d^2(z, Bz))]^{1/2} \\ &< d(z, Bz), \end{aligned}$$

which means that $Bz = z$. Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $Bz = Su = z$. By using (3.2) again, we have

$$\begin{aligned} d(Au, z) &= d(Au, Bz) \\ &= [\phi(0, 0, 0, d^2(Au, z), 0)]^{1/2} \\ &\leq [\gamma(d^2(Au, z))]^{1/2} \\ &< d(Au, z), \end{aligned}$$

which is a contradiction and so $Au = z$. But since the pair $\{A, S\}$ is weakly S-biased of type (A) and $Au = Su = z$, by Proposition 2.5,

$$d(SSu, Au) \leq d(ASu, Su)$$

and hence we have $d(Sz, z) \leq d(Az, z)$. By using (3.2), we have

$$\begin{aligned} d(Az, z) &= d(Az, Bz) \\ &\leq [\phi(d^2(Az, z), 0, d^2(Az, z), d^2(Az, z), 0)] \\ &\leq [\gamma(d^2(Az, z))]^{1/2} \\ &< d(Az, z), \end{aligned}$$

which is a contradiction and so $Az = z$. Therefore, $Az = Bz = Sz = Tz = z$, that is, z is a common fixed point of A, B, S and T. The uniqueness of the common fixed point z follow easily from (3.2). Similarly, we can also complete the proof when (3.4) or (3.5) or (3.6) holds. This completes the proof.

Remark 3.6. Theorem 3.4 extends and improves the results of Pathak ([16], [17]), Prasad ([18]) and Jungck et al. ([12]).

We now give an example which shows the validity of Theorem 3.5 and its superiority over the above cited results.

Example 3.7. Let $X = [0, 1]$ be a metric space with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define

$$Ax = Bx = Sx = \frac{1}{2}, \quad Tx = 1 - x \text{ for } x \in \left[0, \frac{1}{2}\right],$$
$$Ax = Bx = \frac{1}{2}, \quad Sx = 0, \quad Tx = 1 - x \text{ for } x \in \left[0, \frac{1}{2}\right],$$

respectively.

First note that $|Ax - By| = 0$ for all x, y in X. Also,

$$A(X) = \left\{\frac{1}{2}\right\} \subseteq T(X) = \left[\frac{1}{2}, 1\right],$$
$$B(X) = \left\{\frac{1}{2}\right\} \subseteq S(X) = \left\{0, \frac{1}{2}\right\},$$

and A, B are continuous. To see that $\{A, S\}$ is compatible of type (A), suppose that $\{x_n\}$ is a sequence in $[0, 1]$ such that Ax_n, $Sx_n \to t \in X$. Clearly, $t = \frac{1}{2}$ and $x_n \leq \frac{1}{2}$ for large n since $|Ax - Sx| = \frac{1}{2}$ for $x > \frac{1}{2}$. Then $SAx_n = S(\frac{1}{2}) = \frac{1}{2}$, $ASx_n = \frac{1}{2}$, $SSx_n = \frac{1}{2}$ and $AAx_n = \frac{1}{2}$. Thus $|SAx_n - AAx_n| \to 0$ and $|ASx_n - SSx_n| \to 0$. On the other hand, consider the mappings B and T. If $\{x_n\}$ is a sequence in $[0, 1]$ such that Bx_n, $Tx_n \to t \in X$, then $t = \frac{1}{2}$, $x_n < \frac{1}{2}$ and $x_n \leq \frac{1}{2}$ for large n. So $Tx_n \in \{\frac{1}{2}, 1 - x_n\}$, $Bx_n = \frac{1}{2}$, $TBx_n = \frac{1}{2} = BTx_n = BBx_n$ for all large n, $TTx_n = \frac{1}{2}$ if $x_n = \frac{1}{2}$ for all n and $TTx_n = 1$ if $x_n < \frac{1}{2}$ for all large n. Then $\alpha|TTx_n - Bx_n| = \frac{1}{2} > 0 = \alpha|BTx_n - Tx_n|$ and $\alpha|BBx_n - Tx_n| = 0 = \alpha|TBx_n - Bx_n|$. Consequently, $\{B, T\}$ is B-biased of type (A), but not T-biased of type (A) and, thus, in view of Remark 2.3, the pair $\{B, T\}$ is not compatible of type (A). On the other hand, the pair $\{B, T\}$ is weakly B-biased of type (A) as well. Thus, the condition (3.4) holds and $\frac{1}{2}$ is the unique common fixed point of A, B, S and T. By noting the fact that compatibility of type (A) of the pair $\{B, T\}$ is replaced with weakly B-biased of type (A), our Theorem 3.4 improves Theorem 3.4 of Jungck et al. [12]. The interested reader may note that there exist comparably simple examples showing superiority of our Theorem 3.4 over the corresponding Theorems of [17], [18] and [19].

Remark 3.8. The conclusion of Theorem 3.5 remains true even if condition (3.2) is replaced with $\phi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta \max\{t_2, t_3, t_4, t_5\}$, that is,

$$\begin{aligned} d^2(Ax, By) \leq \alpha d^2(Sx, Ty) + \beta \max\{&d(Sx, Ax) \cdot d(Ty, By), \\ &d(Sx, By) \cdot d(Ty, Ax), d(Sx, Ax) \cdot d(Ty, Ax), \\ &d(Sx, By) \cdot d(Ty, By)\} \end{aligned} \tag{3.2'}$$

for all $x, y \in X$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$.

Corollary 3.9. *Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying conditions* (3.1),

$$d^p(Ax, By) \leq \alpha d^p(Sx, Ty) \tag{3.2''}$$

for all $x, y \in X$, where $0 < \alpha < 1$, $p > 0$ and any one of (3.4)∼(3.7). *Then A, B, S and T have a unique common fixed point in X.*

4. CONVERGENCE OF SELF-MAPS AND FIXED POINTS

In this section, we give two theorems on the convergence of self-maps on a metric space and the existence of their fixed points. Since the following theorems follows easily from Theorem 3.4, here we omit the proofs.

Theorem 4.1. *Let $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ be sequences of self-maps of a complete metric space (X, d) such that $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to self-maps A, B, S and T on X, respectively.*

Suppose that, for $n = 1, 2, \cdots$, x_n is a common fixed point of A_n and S_n, and y_n is a common fixed point of B_n and T_n, respectively. Further, let self-maps A, B, S and T on X satisfy the conditions (3.1), (3.2) *and any one of the conditions* (3.4)∼(3.7). *If x is a common fixed point of A, B, S and T, $\sup\{d(x_n, x)\} < \infty$ and $\sup\{d(y_n, x)\} < \infty$, then $x_n \to x$ and $y_n \to x$ as $n \to \infty$.*

Theorem 4.2. *Let $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ be sequences of mappings from a complete metric space (X, d) into itself such that, for $n = 1, 2, \cdots$,*

(4.1) *$A_n(X) \subset T_n(X)$, $B_n(X) \subset S_n(X)$ and one of the conditions* (4.2)∼(4.5) *holds,*

(4.2) *A_n is continuous and the pairs $\{A_n, S_n\}$, $\{B_n, T_n\}$ are compatible of type (A) and weakly B_n-biased of type (A), respectively,*

(4.3) *B_n is continuous and the pairs $\{A_n, S_n\}$, $\{B_n, T_n\}$ are weakly A_n-biased of type (A) and compatible of type (A), respectively,*

(4.4) *S_n is continuous and the pairs $\{A_n, S_n\}$, $\{B_n, T_n\}$ are compatible of type (A) and weakly T_n-biased of type (A), respectively,*

(4.5) *T_n is continuous and the pairs $\{A_n, S_n\}$, $\{B_n, T_n\}$ are weakly S-biased of type (A) and compatible of type (A), respectively, and*

$$\begin{aligned} d(A_n x, B_n y) \leq [\phi(d^2(S_n x, T_n y), &\, d(S_n x, A_n x) \cdot d(T_n y, B_n y), \\ &\, d(S_n x, B_n y) \cdot d(T_n y, A_n y), d(S_n x, A_n x) \cdot d(T_n y, A_n), \\ &\, d(S_n x, B_n y) \cdot d(T_n y, B_n y))]^{1/2} \end{aligned} \tag{4.6}$$

for all $x, y \in X$, where $\phi \in \mathcal{F}$.

If $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to self-maps A, B, S and T on X, respectively, then A B, S and T satisfy the conditions (3.1), (3.2) *and one of the corresponding conditions* (3.4)∼(3.7).

Further, the sequence $\{x_n\}$ of unique common fixed points x_n of A_n, B_n, S_n and T_n converges to a unique common fixed point x of A, B, S and T if $\sup\{d(x_n, x)\} < \infty$.

5. APPLICATIONS TO NONLINEAR INTEGRAL EQUATIONS

In the section 5, we apply our results in the section 3 to two different classes of nonlinear integral equations to establish the solvability of such equations. In this section, we consider pairs of simultaneous Volterra-Hammerstein equations and of simultaneous Hammerstein-type equations. It is interesting to note that when the control functions are identical, the existence of the solution can very well be established by using the classical Jungck's contraction principle ([5]).

(I) First, we give an existence theorem for a pair of simultaneous Volterra-Hammerstein equations in $L_\infty[a, b]$.

Let us consider the following pair of equations:

$$x(t) = w(t) + \mu \int_a^t m(t,s) g_i(s, x(s)) ds + \lambda \int_a^b k(t,s) h_i(s, x(s)) ds \tag{5.1}$$

for all $a \le t \le b$ and $i = 1, 2$, where $w \in L_\infty[a,b]$ is known and assume the following conditions: There exists a sequence $\{x_n(t)\}$ in $L_\infty[a,b]$ such that

$$\mu \int_a^t m(t,s) g_1(s, x_n(s)) ds, \quad x_n(t) - w(t) - \lambda \int_a^b k(t,s) h_1(s, x_n(s)) ds \\ \to \Gamma(t) \in L_\infty[a,b]$$

implies

$$\lim_{n\to\infty} \sup_{a\le t\le b} \Bigg| \mu \int_a^t m(t,s) \left[g_1(s, x_n(s)) - g_1\left(s, \mu \int_a^s m(s,\tau) g_1(\tau, x_n(\tau)) d\tau\right)\right] ds \\ - w(t) - \lambda \int_a^b k(t,s) h_1\left(s, \mu \int_a^s m(s,\tau) g_1(\tau, x_n(\tau)) d\tau\right) ds \Bigg| = 0$$

and

$$\lim_{n\to\infty} \sup_{a\le t\le b} \Bigg| \mu \int_a^t m(t,s) \left[g_1\left(s, x_n(s)) - w(s) - \lambda \int_a^b k(s,\tau) h_1(\tau, x_n(\tau)) d\tau\right)\right] ds \\ - x_n(t) + 2w(t) + \lambda \int_a^b k(t,s) \Big[h_1(s, x_n(s)) \\ + h_1\left(s, x_n(s) - w(s) - \lambda \int_a^b k(s,\tau) h_1(\tau, x_n(\tau)) d\tau\right)\Big] ds \Bigg| = 0 \tag{5.2}$$

and there exists a $x(t) \in L_\infty[a,b]$ such that

$$\mu \int_a^t m(t,s) g_2(s, x(s)) ds = x(t) - w(t) - \lambda \int_a^b k(t,s) h_2(s, x(s)) ds$$

implies

$$\sup_{a\le t\le b} \Bigg| \mu \int_a^t m(t,s) g_2\left(s, \mu \int_a^s m(s,\tau) g_2(\tau, x(\tau)) d\tau\right) ds \\ - x(t) + w(t) + \lambda \int_a^b k(t,s) h_2(s, x(s)) ds \Bigg| \\ \le \sup_{a\le t\le b} \Bigg| -w(t) + \lambda \int_a^b k(t,s) h_2\left(s, \mu \int_a^s m(s,\tau) g_2(\tau, x(\tau)) d\tau\right) ds \Bigg|, \tag{5.3}$$

m is measurable in both t and s and

$$\sup_{a\le t\le b} \int_a^b |m(t,s)| ds \equiv M_1 < \infty, \tag{5.4}$$

k is measurable in both t and s and

$$\sup_{a\le t\le b} \int_a^b |k(t,s)| ds \equiv M_2 < \infty, \tag{5.5}$$

$g_i(s,u)$ are continuous in s and u and satisfy

$$|g_1(s, x(s) - g_2(s, y(s))| \le L_1 \|x - y\| \tag{5.6}$$

for some $L_1 > 0$ and $\|\cdot\|$ denotes the supremum norm,
$h_i(s,u)$ are continuous in s and u and satisfy

$$|h_1(s, x(s) - h_2(s, y(s))| \le L_2 \|x - y\| \tag{5.7}$$

for some $L_2 > 0$.

Theorem 5.1. *Under the assumptions* (5.2)$\sim$(5.7), *the simultaneous equations* (5.1) *has a unique solution in* $L_\infty[a,b]$ *for each pair of real numbers* μ *and* λ *with* $|\lambda|L_2M_2 < 1$ *and* $\frac{|\mu|L_1M_1}{1-|\lambda|L_2M_2} < 1$.

Proof. Define

$$\begin{aligned}
Ax(t) &\equiv \mu \int_a^t m(t,s)g_1(s,x(s))ds,\\
Bx(t) &\equiv \mu \int_a^t m(t,s)g_2(s,x(s))ds,\\
Cx(t) &\equiv w(t) + \lambda \int_a^b k(t,s)h_1(s,x(s))ds,\\
Dx(t) &\equiv w(t) + \lambda \int_a^b k(t,s)h_2(s,x(s))ds,\\
Sx(t) &\equiv (I-C)x(t),\\
Tx(t) &\equiv (I-D)x(t).
\end{aligned}$$

Then we have

$$\begin{aligned}
\|Ax - By\| &= \sup_{a\le t\le b}\left|\mu \int_a^t m(t,s)\{g_1(s,x(s)) - g_2(s,y(s))\}ds\right|\\
&\le \sup_{a\le t\le b} |\mu| L_1 \int_a^b |m(t,s)||x(s)-y(s)|ds\\
&\le |\mu|L_1M_1\|x-y\|.
\end{aligned} \tag{5.8}$$

On the other hand, we have

$$\begin{aligned}
Sx(t) &= (I-C)x(t)\\
&= x(t) - w(t) - \lambda \int_a^b k(t,s)h_1(s,x(s))ds,\\
Ty(t) &= (I-D)y(t)\\
&= y(t) - w(t) - \lambda \int_a^b k(t,s)h_2(s,y(s))ds
\end{aligned}$$

and so

$$\begin{aligned}
&\|Sx - Ty\|\\
&\ge \|x-y\| - \sup_{a\le t\le b}\left|\lambda \int_a^b k(t,s)\{h_1(s,x(s)) - h_2(s,y(s))\}ds\right|\\
&\ge \|x-y\| - |\lambda|L_2M_2\|x-y\|\\
&= (1-|\lambda|L_2M_2)\|x-y\|.
\end{aligned} \tag{5.9}$$

From (5.8) and (5.9), we have

$$\|Ax - By\|^p \le \frac{|\mu|^pL_1^pM_1^p}{(1-|\lambda|L_2M_2)^p}\|Sx - Ty\|^p, \quad p > 0.$$

Hence, if λ, μ, L_i and M_i $(i = 1,2)$ satisfy $\frac{|\mu|L_1M_1}{1-|\lambda|L_2M_2} < 1$, then the condition (3.2″) in the section 3 holds for all $x(t)$, $y(t)$ in $L_\infty[a,b]$. Since $h_i(\cdot, x(\cdot)) \in L_\infty[a,b]$ whenever $x \in L_\infty[a,b]$,

$$(I-C)x, \quad (I-D)x \in L_\infty[a,b] + C[a,b].$$

Thus $A(L_\infty[a,b]) \subset T(L_\infty[a,b])$ and $B(L_\infty[a,b]) \subset S(L_\infty[a,b])$. Moreover, (5.2) and (5.3) imply that the pairs $\{A,S\}$ and $\{B,T\}$ are compatible of type (A) and weakly B-biased of type (A), respectively. Since $h_1(\cdot,x(\cdot))$ is continuous, it follows that A is continuous. Thus all conditions of Corollary 3.7 are satisfied. Hence there exists a unique common fixed point $x \in L_\infty[a,b]$ such that $Ax = Bx = Sx = Tx = x$, which proves the existence of a unique solution of (5.1). This completes the proof.

(II) Secondly, we give an existence theorem for the Hammerstein-type simultaneous equations in $L_\infty[a,b]$.

Let us consider the following Hammerstein-type simultaneous equations:

$$x(t) = w(t) + g_i(t, x(t)) + \lambda \int_a^b k(t,s)\psi_i(s, x(s))ds \tag{5.10}$$

for all $a \leq t \leq b$ and $i = 1, 2$, where w is a known element in $L_\infty[a, b]$. It may be remarked that these simultaneous equations appear in the problem associated with free surface seepage from nonlinear channels constructed by two different types of materials under the control functions $g_i(\cdot, x(\cdot))$ and $\psi_i(\cdot, x(\cdot))$ $(i = 1, 2)$. We assume the following conditions:
There exists a sequence $\{x_n(t)\}$ in $L_\infty[a, b]$ such that

$$x_n(t) - w(t) - \lambda \int_a^b k(t,s)\psi_1(s, x_n(s))ds, \quad g_1(t, x_n(t)) \to \Gamma(t) \in L_\infty[a, b]$$

implies

$$\begin{aligned}\lim_{n\to\infty} \sup_{a\leq t\leq b} \Bigg| g_1\left(t, x_n(t) - w(t) - \lambda \int_a^b k(t,s)\psi_1(s, x_n(s)ds\right) \\ - x_n(t) + 2w(t) + \lambda \int_a^b k(t,s)\Bigg[\psi_1(s, x_n(s)) \\ + \psi_1\left(s, x_n(s) - w(s) - \lambda \int_a^b k(s,\tau)\psi_1(\tau, x_n(\tau))d\tau\right)\Bigg]ds\Bigg| = 0\end{aligned} \tag{5.11}$$

and

$$\lim_{n\to\infty} \sup_{a\leq t\leq b} \left| g_1(t, x_n(t) - w(t) - \lambda \int_a^b k(t,s)\psi_1(s, g_1(s, x_n(s)))ds - g_1(t, g_1(t, x_n(t)))\right| = 0$$

and there exists a $x(t) \in L_\infty[a, b]$ such that

$$x(t) - w(t) - \lambda \int_a^b k(t,s)\psi_2(s, x(s))ds = g_2(t, x(t))$$

implies

$$\begin{aligned}\sup_{a\leq t\leq b} \Bigg| x(t) - 2w(t) - \lambda \int_a^b k(t,s)\Bigg[\psi_2(s, x(s)) + \psi_2\Bigg(s, x(s) - w(s) \\ - \lambda \int_a^b k(t,\tau)\psi_2(\tau, x(\tau))d\tau\Bigg)\Bigg]ds - g_2(t, x(t))\Bigg| \\ \leq \sup_{a\leq t\leq b} \Bigg| g_2\left(t, x(t) - w(t) - \lambda \int_a^b k(t,s)\psi_2(s, x(s))ds\right) \\ - x(t) + w(t) + \lambda \int_a^b k(t,s)\psi_2(s, x(s))ds\Bigg|,\end{aligned}$$

$$\sup_{a\leq t\leq b} \int_a^b |k(t,s)|ds \equiv M_1 < \infty, \tag{5.12}$$

$\psi_i(s, x)$ are continuous in s and x and satisfy

$$|\psi_i(s, x(s)) - \psi_2(s, y(s))| \leq L_1 \|x(s) - y(s)\| \tag{5.13}$$

for all $s \in [a, b]$ and $x, y \in L_\infty[a, b]$ with $L_1 > 0$,
$g_i(s, x)$ are continuous in s and x and satisfy

$$|g_1(s, x(s)) - g_2(s, y(s))| \geq L_2 \|x(s) - y(s)\| \tag{5.14}$$

for all $s \in [a, b]$ and $x, y \in L_\infty[a, b]$ with $L_2 > 1$.

Theorem 5.2. *Under the assumptions* (5.11)–(5.14), *the simultaneous equations* (5.10) *has a unique solution in* $L_\infty[a,b]$ *provided that* $\frac{1+|\lambda|L_1M_1}{L_2} < 1$.

Proof. Define

$$Ax(t) \equiv x(t) - w(t) - \lambda \int_a^b k(t,s)\psi_1(s,x(s))ds,$$
$$Bx(t) \equiv x(t) - w(t) - \lambda \int_a^b k(t,s)\psi_2(s,x(s))ds,$$
$$Sx(t) \equiv g_1(t,x(t)),$$
$$Tx(t) \equiv g_2(t,x(t)).$$

Then

$$\begin{aligned}\|Ax - By\| &\le \|x-y\| + |\lambda|L_1M_1\|x-y\| \\ &= (1+|\lambda|L_1M_1)\|x-y\| \\ &\le \frac{(1+|\lambda|L_1M_1)}{L_2}\|Sx-Ty\|,\end{aligned}$$

that is,

$$\|Ax - By\|^p \le \frac{(1+|\lambda|L_1M_1)^p}{L_2^p}\|Sx-Ty\|^p, \quad p>0.$$

Since $A(L_\infty[a,b]) \subset T(L_\infty[a,b]) = L_\infty[a,b]$ and $B(L_\infty[a,b]) \subset S(L_\infty[a,b]) = L_\infty[a,b]$ by Corollary 3.7, there exists a unique common fixed point $x \in L_\infty[a,b]$ such that $x = Ax = Bx = Sx = Tx$. Hence the simultaneous equations (5.10) have a unique solution in $L_\infty[a,b]$. This completes the proof.

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