Internat. J. Math. & Math. Sci.
VOL. 21 NO. 3 (1998) 493-498

NECESSARY AND SUFFICIENT CONDITIONS FOR THE OSCILLATION OF DELAY DIFFERENTIAL EQUATION WITH A PIECEWISE CONSTANT ARGUMENT

H.A. AGWO
Department of Mathematics
Faculty of Education
Ain Shams University
Roxy ,Cairo , Egypt.

(Received August 13, 1996 and in revised form March 18, 1997)

ABSTRACT: The characteristic equation for an equation with continuous and piecewise constant argument in the form

$$\dot{x}(t) + p\,x(t-\tau) + q\,x([t-k]) = 0 \text{ where } p,q \in \Re\ ,\tau \in \Re^{+} \text{ and } k \in \aleph.$$

is presented , which when $q=0$ reduces to

$$f(\lambda) = \lambda + e^{-\lambda\tau} = 0$$

and when $p=0$ reduces to

$$\lambda - 1 + q\,\lambda^{-k} = 0.$$

Also, the necessary and sufficient conditions for oscillation are obtained.

KEY WORDS: Oscillations , Delay differential equations.

1991 AMS SUBJECT CLASSIFICATION CODES: 34k15 ; 39A10.

1. INTRODUCTION

The study of equations with piecewise constant argument was originated by the work of Wiener and his collaborators. See [1,2,3,4,5 and 6] and the references cited therein . In addition to its own interest this area has stimulated much activity in the study of delay difference equations.

As usual , a solution x(t) is called oscillatory if it has arbitrarily large zeros .Otherwise , the solution is called nonoscillatory.An equation is called oscillatory if all its solutions are oscillatory.

Let [.] denote the greatest-integer function,$\aleph$ the set of non-negative integers and $\Re$ the set of real numbers.

Consider

$$\dot{x}(t) + p\,x(t-\tau) + q\,x([t-k]) = 0 \tag{1.1}$$

where p,q $\in \Re$,$\tau \in \Re^{+}$ and $k \in \aleph$.

By a solution of Eqn.(1.1), we mean a function x which is defined on the set $\{-k, ..., -1, 0\} \cup [-\tau,\infty)$ and satisfies the following properties:

(a) x is continuous on $[-\tau,\infty)$.

(b) the derivative $\dot{x}$ exists at each point $t \in (0,\infty)$ with the possible exception of the points $t \in \aleph$, where one side derivatives exist.

(c) Eqn.(1.1) is satisfied on each interval $[n,n+1)$ for $n \in \aleph$.

Let $\phi \in C([-\tau,0],\Re)$ and $a_{-k},...,a_{-1},a_0$ be given real numbers such that

$$a_{-j} = \phi(-j) \quad \text{for} \quad j \le \tau,\ j=0,1,2,...,k, \tag{1.2}$$

then one can show that Eqn. (1.1) has a unique solution satisfying the initial conditions

$$x(t) = \phi(t) \qquad -\tau \le t \le 0 \tag{1.3a}$$

$$x(-j) = a_{-j} \qquad j=0,1,...,k. \tag{1.3b}$$

When $q=0$, Eqn. (1.1) reduces to

$$\dot{u}(t) + p\,u(t-\tau) = 0 \tag{1.4}$$

which is oscillatory if and only if its characteristic equation

$$f(\lambda) = \lambda + e^{-\lambda\tau} = 0 \tag{1.5}$$

has no real roots, or equivalently, to

$$p\tau > \frac{1}{e}. \tag{1.6}$$

On the other hand, when $p=0$, Eqn.(1.1) reduces to

$$\dot{v}(t) + q\,v([t-k]) = 0 \tag{1.7}$$

which is oscillatory if and only if the following equation

$$\lambda - 1 + q\,\lambda^{-k} = 0 \tag{1.8}$$

has no positive real roots, or equivalently,

$$q > \frac{k^k}{(k+1)^{k+1}} \qquad , k \ge 1 \tag{1.9a}$$

$$q \ge 1 \qquad , k=0 \tag{1.9b}$$

An open question arises (see [4] ,p. 223) for obtaining a characteristic equation for equation (1.1) which reduces to Eqn.(1.5) when $q=0$ and reduces to Eqn. (1.8) when $p=0$ and also obtaining the necessary and sufficient conditions for oscillation of all solutions of

$$\dot{x}(t) + p\,x(t-1) + q\,x([t-1]) = 0 \tag{1.10}$$

2. THE MAIN RESULTS

In the following, a characteristic equation associated with equation (1.1) will be presented in Theorem **2.1**. Also the necessary and sufficient conditions for oscillation are obtained through Theorems **2.2 and 2.3**.

THEOREM 2.1. The characteristic equation associated with equation (1.1) is

$$f(\lambda) = \lambda - 1 + \frac{p\lambda^{-\tau}}{\ln\lambda}(\lambda - 1) + q\lambda^{-k} = 0 \tag{2.1}$$

which reduces to Eqn. (1.5) when q=0 and reduces to Eqn. (1.8) when p=0.

PROOF: Consider Eqn.(1.1) and assume that the initial conditions (1.3a) and (!.3b) are satisfied .For $t \in [n,n+1)$, we have $[t-k] = n-k$ and one can write

$$\dot{x}(t) + p\,x(t-\tau) + q\,a_{n-k} = 0 \qquad , \; t \in [n,n+1) \tag{2.2a}$$

$$x(n) = a_n \qquad , \; n \in \aleph \tag{2.2b}$$

Integrating (2.2a) from n to t , we get

$$x(t) - a_n + p\int_n^t x(s-\tau)\,ds + q\,a_{n-k}(t-n) = 0\,. \tag{2.3}$$

By using the continuity of x(t) as $t \to n+1$, we find

$$a_{n+1} - a_n + p\int_n^{n+1} x(s-\tau)\,ds + q\,a_{n-k} = 0\,. \tag{2.4}$$

Assume that $x(t) = e^{\lambda t}$, $t \in [n,n+1)$, then from (2.4) ,we get

$$f(\lambda) = e^{\lambda} - 1 + \frac{p e^{-\lambda\tau}}{\lambda}(\lambda - 1) + q e^{-\lambda k} = 0 \tag{2.5}$$

Putting $e^{\lambda} = \gamma$ in Eqn.(2.5),then

$$F(\gamma) = \gamma - 1 + \frac{p\gamma^{-\tau}}{\ln\gamma}(\gamma - 1) + q\gamma^{-k} = 0\,. \tag{2.6}$$

and consequently Eqn.(2.6) has no positive real roots if and only if Eqn.(2.5) has no real roots.Assume that Eqn.(2.5) has no real roots , then $\lambda \neq 0$, and consequently $\gamma \neq 1$. If p= 0, then Eqn. (2.6) reduces to Eqn.(1.8),also if q=0 ,then Eqn.(2.6) reduces to Eqn.(1.5) .

THEOREM 2.2. Equation (1.1) is oscillatory if and only if its characteristic equation (2.6) has no positive real roots.

PROOF: Assume that the characteristic equation (2.6) has a positive real root γ_0, then γ_0^t is a solution of Eqn.(2.4) which is a nonoscillatory solution and consequently Eqn.(1.1) is not oscillatory .On the other hand , assume that $x(t) > 0 \ \forall\ t \in [n,n+1)$ for sufficiently large n and $F(\gamma)$ has no positive real roots .As $F(\infty) = \infty$, it follows that $F(\gamma) > 0 \ \forall\ \gamma \in (0,\infty)$.For seeking the contradiction , choose :

(i) $p \le 0$ and $q \le 0$ then $F(\gamma) < 0 \ \forall\ \gamma \in (0,1)$,

(ii) $p \ge 0$, $q < 0$ with $p < |q|$ and $\tau \le k$ then $F(\gamma) < 0 \ \forall\ \gamma \in (0,1)$,

(iii) $p < 0$, $q \ge 0$ with $q < |p|(1 - 1/e)$ and $\tau = k$ then $F(1/e) < 0$,

(iv) $p \ge 0$, $q \ge 0$ with $p + q \le 1/8e^k$ and $\tau \le k$ then $F(1/e) < 0$,

which is a contradiction.

THEOREM 2.3. If $p, q \in \Re^+$, then all solutions of equation (1.1) are oscillatory if and only if

$$pe\tau + q\frac{(k+1)^{k+1}}{k^k} > 1 \quad , \; k \ge 1 \tag{2.7}$$

PROOF: Assume that Eqn.(1.1) has a nonoscillatory solution ,then the characteristic Eqn.(2.6) has a positive real root $\gamma_0 \in (0,1)$. Otherwise $F(\gamma_0) > 0 \ \forall\ \gamma_0 \in [1,\infty)$ and therefore , we have

$$F(\gamma_0) = \gamma_0 - 1 + \frac{p\gamma_0^{-\tau}}{\ln\gamma_0}(\gamma_0 - 1) + q\gamma_0^{-k} = 0,\ \gamma_0 \in (0,1)$$

and then

$$0 = (\gamma_0 - 1)\{1 + \frac{p\gamma_0^{-\tau}}{\ln\gamma_0} + q\gamma_0^{-k}/(\gamma_0 - 1)\}$$

$$0 = 1 + \frac{p\gamma_0^{-\tau}}{\ln\gamma_0} + q\gamma_0^{-k}/(\gamma_0 - 1)$$

$$\le 1 - pe\tau - q\frac{(k+1)^{k+1}}{k^k}$$

which is a contradiction .On the other hand ,assume that

$$pe\tau + q\frac{(k+1)^{k+1}}{k^k} \le 1\ ,k \ge 1\ .$$

Now , we study the following cases:

(1) $q=0$,p>0.

Since $F(\gamma) > 0, \forall\ \gamma \in (1,\infty)$ and $F(e^{-\frac{1}{\tau}}) \le 0$,then there exists $\gamma_1 \in \Re^+$ such that $F(\gamma_1) = 0$. i.e. the characteristic equation has a positive real root and consequently equation (1.1) is not oscillatory.

(2) $p=0$, $q>0$.

In this case , $F(\gamma) > 0, \forall\ \gamma \in (1,\infty)$ and $F(\frac{k}{k+1}) \le 0$.Therefore , the characteristic equation has a positive real root and then equation(1.1) has a nonoscillatory solution.

(3) $p>0$,$q>0$.

Since $pe\tau + q\frac{(k+1)^{k+1}}{k^k} \le 1$,

then,

$$q\frac{(k+1)^{k+1}}{k^k} + \frac{p\frac{(k+1)^{\tau+1}}{k^\tau}}{\ln(\frac{k}{k+1})} < pe\tau + q\frac{(k+1)^{k+1}}{k^k} \le 1,\ k \ge 1\ . \tag{2.8}$$

It is clear that the characteristic equation has no real roots in $(1,\infty)$ and $F(\gamma) > 0$,but

$$F(\frac{k}{k+1}) = \frac{k}{k+1} - 1 + \frac{p\frac{(k+1)^\tau}{k^\tau}}{\ln(\frac{k}{k+1})} + q\frac{(k+1)^k}{k^k}$$

$$= -\frac{1}{k+1} + \frac{q}{(k+1)}\frac{(k+1)^{k+1}}{k^k} + \frac{p}{k+1}\frac{\frac{(k+1)^{\tau+1}}{k^\tau}}{\ln(\frac{k}{k+1})}$$ From (2.8),it follows that $F(\frac{k}{k+1}) \le 0$ and consequently equation (1.1) has a nonoscillatory solution.

REMARK. If $\tau = k = 1$ and $p,q \in \Re^+$,then $pe + 4q > 1$ is a necessary and sufficient condition for oscillation of

$$\dot{x}(t) + p\,x(t-1) + q\,x([t-1]) = 0\ .$$

REFERENCES

[1] **COOKE, K.L.** and **WIENER,J.**, Retarded differential equations with piecewise constant delays , *J. Math. Anal. and Appl.* **99** (1984), 265- 294.

[2] **COOKE, K.L.** and **WIENER,J.**, Neutral differential equations with piecewise constant argument, *Bolletino Unione Mathematica Italiana* **1-B** (1987), 321-345.

[3] **GROVE, E.A.** , **GYÖRI,I.**,and **LADAS,G.**,On the characteristic equations for equations with continuous and piecewise constant arguments ,*Radovi Mathematicki* **5** (1990), 271- 281.

[4] **GYÖRI,I.**,and **LADAS,G.** , Oscillation Theory of Delay Differential Equations with Applications , *Clarendon Press, Oxford* ,**1991.**

[5] **WIENER,J.** and **COOKE, K.L.**, Oscillations in systems of differential equations with piecewise constant argument , *J. Math. Anal. and Appl.* **137** (1989) , 221-239 .

[6] **WIENER, J.**, Generalized Solutions of Functional Differential Equations , *World Scientific* , *Singapore* , 1993 .