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THE DIOPHANTINE EQUATION $x^2 + 3^m = y^n$

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ABSTRACT. The object of this paper is to prove the following

THEOREM. Let m be odd. Then the diophantine equation $x^2 + 3^m = y^n, n \geq 3$ has only one solution in positive integers x, y, m and the unique solution is given by $m = 5 + 6M$, $x = 10.3^{3M}$, $y = 7.3^{2M}$ and $n = 3$.

KEY WORDS AND PHRASES: Diphantine equation.
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INTRODUCTION

It is well known that there is no general method for determining all integral solutions x and y for a given diophantine equation $ax^2 + bx + c = dy^n$, where a, b, c and d are integers, $a \neq 0$, $b^2 - 4ac \neq 0$, $d \neq 0$, but we know that it has only a finite number of solutions when $n \geq 3$ This was first shown by Thue [1]

The first result for the title equation for general n is due to Lebesgue [2] who proved that when $m = 0$ there is no solution, for $m = 1$, Nagell [3] has proved that it has no solution and in 1993 Cohn [4] has given another proof for this case.

The proof of the theorem is divided into two main cases $(3, x) = 1$ and $3|x$. It is sufficient to consider x a positive integer.

To prove the theorem we need the following

LEMMA (Nagell [5]). The equation $3x^2 + 1 = y^n$, where n is an odd integer ≥ 3 has no solution in integers x and y for y odd and ≥ 1.

PROOF OF THEOREM. Suppose $m = 2k + 1$. Since the result is known for $m = 1$ we shall lassume that $k > 0$. The case when x is odd, can be easily eliminated since $y^n \equiv 0 \pmod 8$, so we assume that x is even.

CASE 1: Let $(3, x) = 1$. First let n be odd, then there is no loss of generality in considering $n = p$ an odd prime. Thus $x^2 + 3^{2k+1} = y^p$. Then from [6, Theorem 1] we have only two possibilities and they are

$$x + 3^k\sqrt{-3} = \left(a + b\sqrt{-3}\right)^p \tag{1}$$

where $y = a^2 + 3b^2$ and

$$x + 3^k\sqrt{-3} = \left(\frac{a + b\sqrt{-3}}{2}\right)^3, \quad a \equiv b \equiv 1 \pmod 2 \tag{2}$$

where $y = \frac{a^2+3b^2}{4}$, for some rational integers a and b.

In (1) since $y = a^2 + 3b^2$ and y is odd so only one of a or b is odd and the other is even. Equating imaginary parts we get

$$3^k = b\sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r+1} a^{p-2r-1}(-3b^2)^r.$$

So b is odd Since 3 does not divide the term inside $\sum$ we get $b = \pm 3^k$ Hence

$$\pm 1 = \sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r+1} a^{p-2r-1}\left(-3^{2k+1}\right)^r.$$

This is equation (1) in [6], and Lemmas 4 and 5 in [6] show that both the signs are impossible. Hence (1) gives rise to no solutions

Now consider equation (2). By equating imaginary parts we obtain

$$8.3^k = b\left(3a^2 - 3b^2\right). \tag{3}$$

If $b = \pm 1$ in (3) we get

$$\pm 8.3^k = 3a^2 - 3.$$

The case $k = 1$ can be easily eliminated, so suppose $k > 1$ then

$$\pm 8.3^{k-1} = a^2 - 1.$$

This equation has the only solution $a = \pm 5$, $k = 2$ and so $y = \frac{a^2+3b^2}{4} = (25+3)/4 = 7$. Hence from (2) $x = \left|\frac{a^3-9ab^2}{8}\right| = 10$

If $b = \pm 3^\lambda$, $0 < \lambda < k$, then (3) becomes $\pm 8.3^{k-\lambda-1} = a^2 - 3^{2\lambda}$, and this is not possible modulo 3 if $k - \lambda - 1 > 0$. So $k - \lambda - 1 = 0$, that is $\pm 8 = a^2 - 3^{2(k-1)}$, and we can reject the positive sign modulo 3. So we have $a^2 - 3^{2(k-1)} = -8$, which has the only solution $a = \pm 1, k = 2$ and $x = 10$ Finally if $b = \pm 3^k$ then $\pm 8 = 3a^2 - 3^{2k+1}$, and this is not true modulo 3.

Now if n is even, then from the above it is sufficient to consider $n = 4$, hence $(y^2+x)(y^2-x) = 3^{2k+1}$ Since $(3, x) = 1$, we get

$$y^2 + x = 3^{2k+1} \quad \text{and} \quad y^2 - x = 1,$$

by adding these two equations we get $2y^2 = 3^{2k+1} + 1$, which is impossible modulo 3.

CASE 2. Let $3|x$. Then of course $3|y$. Suppose that $x = 3^u X$, $y = 3^\nu Y$ where $u > 0$, $\nu > 0$ and $(3, X) = (3, Y) = 1$ Then $3^{2u}X^2 + 3^{2k+1} = 3^{n\nu}Y^n$ There are three possibilities.

1 $2u = \min(2u, 2k+1, n\nu)$. Then by cancelling 3^{2u} we get $X^2 + 3^{2(k-u)+1} = 3^{m\nu-2u}Y^n$, and considering this equation modulo 3 we deduce that $n\nu - 2u = 0$, then $x^2 + 3^{2(k-u)+1} = Y^n$, with $(3, X) = 1$. If $k - u = 0$, this equation has no solution [3,4] and if $k - u > 0$, as proved above this equation has a solution only if $k - u = 2$ and $n = 3$, so $n\nu = 3\nu = 2u$ that is $3|u$, let $u = 3M$ then $k = 2 + 3M$ and $m = 5 + 6M$. So this equation has a solution only if $m = 5 + 6M$ and the solution is given by $X = 10$, $Y = 7$. Hence the solution of our title equation is $x = 10.3^u = 10.3^{3M}$ and $y = 7.3^\nu = 7.3^{2M}$.

2 $2k + 1 = \min(2u, 2k+1, n\nu)$ Then $3^{2u-2k-1}X^2 + 1 = 3^{n\nu-2k-1}Y^n$ and considering this equation modulo 3 we get $n\nu - 2k - 1 = 0$, so n is odd and $3(3^{u-k-1}X)^2 + 1 = Y^n$, by the lemma this equation has no solution.

3. $n\nu = \min(2u, 2k+1, n\nu)$. Then $3^{2u-n\nu}X^2 + 3^{2k+1-n\nu} = Y^n$ and this is possible modulo 3 only if $2u - n\nu = 0$ or $2k + 1 - n\nu = 0$ and both of these cases have already been discussed This concludes the proof.

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