SOME RESULTS ON DOMINANT OPERATORS

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ABSTRACT. We show that the Weyl spectrum of a dominant operator satisfies the spectral mapping theorem for analytic functions and then answer a question of Oberai.

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1. INTRODUCTION

Throughout this paper H will denote an infinite dimensional Hilbert space and $B(H)$ the space of all bounded linear operators on H. If $T \in B(H)$, we write $\sigma(T)$ for the spectrum of T, $\pi_0(T)$ for the set of eigenvalues of T, and $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. If K is a subset of $\mathbb{C}$, we write iso K for the set of isolated points of K. An operator $T \in B(H)$ is said to *Fredholm* if its range ran T is closed and both the null space ker T and ker T^* are finite dimensional. The *index* of a Fredholm operator T, denoted by $i(T)$, is defined by

$$i(T) = \dim\ker T - \dim\ker T^*.$$

The *essential spectrum* of T, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Fredholm}\}.,$$

A Fredholm operator of index zero is called a *Weyl* operator. The *Weyl spectrum* of T, denoted by $\omega(T)$, is defined by

$$\omega(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl}\}.$$

It was shown by Berberian [2] that $w(T)$ is a nonempty compact subset of $\sigma(T)$.

An operator $T \in B(H)$ is said to be *dominant* if for every $z \in \mathbb{C}$ there exists a real number $M_z > 0$ such that

$$(T - z)(T - z)^* \leq M_z(T - z)^*(T - z) \tag{1.1}$$

In this case, if $\sup_{z\in\mathbb{C}} M_z = M < \infty$, T is said to be $M-$hyponormal, and if $M = 1$, T is hyponormal. Evidently,

$$T \text{ is hyponormal} \Longrightarrow T \text{ is } M\text{−hyponormal} \Longrightarrow T \text{ is dominant}$$

We also note that an operator T need not be hyponormal even though T and T^* are both $M-$hyponormal. To see this, consider the operator

$$T = \begin{bmatrix} U & K \\ 0 & U^* \end{bmatrix} : l_2 \oplus l_2 \to l_2 \oplus l_2,$$

where U is the unilateral shift on l_2 and $K : l_2 \to l_2$ is given by

$$K(x_1, x_2, x_3, \cdots) = (2x_1, 0, 0, \cdots).$$

Then a direct calculation shows that

$$\frac{1}{2}\|(T-z)x\| \le \|(T-z)^*x\| \le 2\|(T-z)x\|$$

for all $z \in \mathbb{C}$ and for all $x \in l_2 \oplus l_2$, which says that T and T^* are both dominant(even M-hyponormal). But since

$$\begin{bmatrix} I & 0 \\ 0 & I + \frac{3}{2}K \end{bmatrix} = T^*T \neq TT^* = \begin{bmatrix} I + \frac{3}{2}K & 0 \\ 0 & I \end{bmatrix},$$

T is not normal(even hyponormal).

If T is Fredholm then by (1.1)

$$T \text{ dominant} \implies i(T) \le 0. \tag{1.2}$$

It was known by Oberai [8] that the mapping $T \to \omega(T)$ is upper semi-continuous, but not continuous at T. However if $T_n \to T$ with $T_nT = TT_n$ for all $n \in \mathbb{N}$ then

$$\lim \omega(T_n) = \omega(T). \tag{1.3}$$

It was known that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic functions: if f is analytic on a neighborhood of $\sigma(T)$ then

$$\omega(f(T)) \subset f(\omega(T)). \tag{1.4}$$

The inclusion (1.4) may be proper(see Berberian [2, Example 3.3]). If T is normal then $\sigma_e(T)$ and $\omega(T)$ coincide. Thus if T is normal since $f(T)$ is also normal, it follows that $\omega(T)$ satisfies the spectral mapping theorem for analytic functions. We say that *Weyl's theorem holds for T* if

$$\omega(T) = \sigma(T) - \pi_{00}(T).$$

It was known (Berberian [1]) that Weyl's theorem holds for any hyponormal operator – indeed, for any seminormal operator and for any Toeplitz operator. Oberai [9] has raised the following question: Does there exist a hyponormal operator T such that Weyl's theorem does not hold for T^2 ? Note that T^2 may not be hyponormal even if T is hyponormal(Halmos [5, Problem 209]).

In this paper we show that the Weyl spectrum of a dominant operator satisfies the spectral mapping theorem for analytic functions, and that Weyl's theorem holds for $p(T)$ when T is hyponormal and p is any polynomial. The latter result answers the question of Oberai.

2. SPECTRAL MAPPING THEOREM

THEOREM 2.1. If S and T are dominant operators, then

$$S, T \text{ Weyl} \iff ST \text{ Weyl}. \tag{2.1}$$

PROOF. If S, T are Weyl, then S, T are Fredholm and $i(S) = i(T) = 0$. By Conway [3], ST is Fredholm and by the index product theorem, $i(ST) = i(S) + i(T) = 0$. Hence ST is Weyl.

Conversely if ST is Weyl, then ST is Fredholm and $i(ST) = 0$. Since S and T are dominant, $\ker S \subset \ker S^*$ and $\ker T \subset \ker T^*$. Since $\ker S^* \subseteq \ker(ST)^*$, $\dim\ker S \le \dim\ker S^* \le$

$\dim\ker(ST)^* < \infty$. Thus $\ker S$ and $\ker S^*$ are finite dimensional. By Schechter [10, Chap. 5 Theorem 3.5], S and T are Fredholm. Since $0 = i(ST) = i(S) + i(T)$ by the index product theorem, by (1.2) $i(S) = i(T) = 0$. Hence S and T are Weyl.

If the "dominant" condition is dropped in the above theorem, then the backward implication may fail even though T_1 and T_2 commute: For example, if U is the unilateral shift on l_2, consider the following operators on $l_2 \oplus l_2 : T_1 = U \oplus I$ and $T_2 = I \oplus U^*$.

THEOREM 2.2. If T is dominant and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.

PROOF. Suppose that p is any polynomial. Let

$$P(T) - \lambda I = a_0(T - \mu_1 I) \cdots (T - \mu_n I).$$

Since T is dominant, $T - \mu_i I$ are dominant operators for each $i = 1, 2, \cdots, n$. It thus follows from Theorem 2.1 that

$$\begin{aligned}\lambda \notin \omega(p(T)) &\Longleftrightarrow p(T) - \lambda I = \text{Weyl} \\ &\Longleftrightarrow a_0(T - \mu_1 I) \cdots (T - \mu_n I) = \text{Weyl} \\ &\Longleftrightarrow T - \mu_i I = \text{Weyl for each } i = 1, 2, \cdots, n \\ &\Longleftrightarrow \mu_i \notin \omega(T) \text{ for each } i = 1, 2, \cdots, n \\ &\Longleftrightarrow \lambda \notin p(\omega(T))\end{aligned}$$

which says that $\omega(p(T)) = p(\omega(T))$. If f is analytic on a neighborhood of $\sigma(T)$, then there is a sequence (p_n) of polynomials such that $f_n \to f$ uniformly on $\sigma(T)$. Since $p_n(T)$ commutes with $f(T)$, by Oberai [8]

$$f(\omega(T)) = \lim p_n(\omega(T)) = \lim \omega(p_n(T)) = \omega(f(T)).$$

Recall that $T \in B(H)$ is said to be *isoloid* if iso $\sigma(T) \subset \pi_0(T)$ (Oberai [9]).

LEMMA 2.3. (Oberai [9]) Let $T \in B(H)$ be isoloid. Then for any polynomial $p(t)$, $p(\sigma(T) - \pi_{00}(T)) = \sigma(p(T)) - \pi_{00}(p(T))$.

Let T be an $M-$hyponormal operator which satisfies the additional property that for all z in the complex plane, all integers n and all x in H,

$$\|(T - z)^n x\|^2 < M\|(T - z)^{2n} x\| \cdot \|x\|.$$

T is said to be an operator of *$M-$power class* (N) (Istrătescu [7]). The following $M-$ hyponormal operator T which is not hyponormal is of $M-$power class (N) (Istrătescu [7]): Let $\{e_i\}$ be an orthonormal basis for H, and define

$$Te_i = \begin{cases} e_2, & \text{if } \ i = 1 \\ 2e_3, & \text{if } \ i = 2 \\ e_{i+1}, & \text{if } \ i \geq 3 \end{cases}$$

i.e., T is a weighted shift. From the definition of T we see that T is similar to the unilateral shift U(Halmos [5], Problem 90). Thus there exists an S such that $T = SUS^{-1}$. In our case $\|S\| = 2$, $\|S^{-1}\| = 1$. Since U is the unilateral shift, U is a hyponormal operator, and thus for every n and $z \in \mathbb{C}$ the operator $(U - z)^n$ is of class (N). It follows that

$$\|(U - z)^n x\|^2 \leq \|(U - z)^{2n} x\|$$

for all $x \in H$ with $\|x\| = 1$, and hence T is of $M-$power class with $M = 4$. Thus our class is strictly larger than the class of hyponormal operators. Since $w(T) = w(U) = D$(the closed unit disc) and $\pi_0(T) = \emptyset$, $\sigma(T) = w(T)$ and so Weyl's theorem holds for T.

THEOREM 2.4. If $T \in B(H)$ is an operator of $M-$power class (N), then for any polynomial p on a neighborhood of $\sigma(T)$ Weyl's theorem holds for $p(T)$.

PROOF. By Istrătescu [7], T is isoloid and Weyl's theorem holds for any operator of $M-$power class (N). Hence by Theorem 2.2 and Lemma 2.3,

$$w(p(T)) = p(w(T)) = p(\sigma(T) - \pi_{00}(T)) = \sigma(p(T)) - \pi_{00}(p(T))$$

Therefore Weyl's theorem holds for $p(T)$.

Since every hyponormal operator is of 1-power class (N), we obtain the following result which answers the question of Oberai.

COROLLARY 2.5. If $T \in B(H)$ is hyponormal, then for any polynomial p on a neighborhood of $\sigma(T)$ Weyl's theorem holds for $p(T)$.

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