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TWO THEOREMS ON (ε)-SASAKIAN MANIFOLDS

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ABSTRACT. In this paper, We prove that every (ε)–sasakian manifold is a hypersurface of an indefinite kaehlerian manifold, and give a necessary and sufficient condition for a Riemannian manifold to be an (ε)– sasakian manifold.

KEY WORDS AND PHRASES: (ε)–sasakian manifolds; real hypersurface; indefinite kaehlerian manifolds; (ε)–almost contact structure.

1. INTRODUCTION Let M be a real $(2n+1)$–dimensional differentiable manifold endowed with an almost contact structure (ϕ, ξ, η). This means that ϕ is a tensor field of type (1,1), ξ is a vector field and η is a 1–form on M satisfying:

$$\phi^2 = -I + \eta \otimes \xi; \qquad \eta(\xi) = 1 \tag{1}$$

It follows that

$$\eta \circ \phi = 0; \phi(\xi) = 0; rank\phi = 2n \tag{2}$$

If there exists a semi–Riemannian metric g on M that satisfies (see [1])

$$g(\phi X, \phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y) \qquad \forall X, Y \in \Gamma(TM) \tag{3}$$

Where $\varepsilon = \pm 1$,We call (ϕ, ξ, η, g) an (ε)–almost contact metric structure and M an (ε)–almost contact metric manifold.

From (3), we have

$$\eta(X) = \varepsilon g(X, \xi) \qquad \forall X \in \Gamma(TM) \tag{4}$$

$$g(\xi, \xi) = \varepsilon \tag{5}$$

We say that (ϕ, ξ, η, g) is an (ε)–contact metric structure if we have

$$g(X, \phi Y) = d\eta(X, Y) \qquad \forall X, Y \in \Gamma(TM) \tag{6}$$

In this case, M is an (ε)–contact metric manifold. An (ε)–contact metric structure which is normal is called an (ε)–sasakian structure. A manifold endowed with an (ε)–sasakian structure is called an (ε)– sasakian manifold.

In[1], A. Bejancu and K.L. Duggal give a theorem as following:

THEOREM A (see [1] theorem 6)

Let M be an orientable real hypersurface of an indefinite kaehlerian manifold $\overline{M}$, then the following assertions with respect to the (ε)–almost contact metric structure inherited by M are equivalent:

(1) M is an (ε)–sasakian manifold

(2) The (ε)–characteristic vector field ξ satisfies

$$\nabla_X \xi = -\varepsilon\phi X \qquad \forall X \in \Gamma(TM)$$

(3) The shape operator A satisfies

$$AX = -\varepsilon X + (\varepsilon + \eta(A\xi))\eta(X)\xi \qquad \forall X \in \Gamma(TM)$$

This produces a problem whether an (ε)– sasakian manifold must be a real hypersurface of some indefinite kaehlerian manifold. In sec.2, we prove that the answer to this problem is positive. that is

THEOREM 1.1. Every (ε)–sasakian manifold must be a real hypersurface of some indefinite kaehlerian manifold.

In [2], Hatakeyama, Ogewa and Tanno give the condition for a Riemannian manifold to be a K–contact manifold, they prove

THEOREM B (see [2] or [4]) In order that a $(2n+1)$–dimensional Riemannian manifold M is K–contact, it is necessary and sufficient that the following two conditions are satisfied:

(1) M admits a unit killing vector field ξ;

(2) The sectional curvatures for plane sections containing ξ are equal to 1 at every point of M.

In sec.3, we generalize Theorem B by giving the necessary and sufficient condition for a Riemannian manifold to be an (ε)– sasakian manifold, that is

THEOREM 1.2. In order that a $(2n+1)$–dimensional Riemannian manifold M is (ε)–sasakian manifold, it is necessary and sufficient that the following three conditions are satisfied:

(1) M admits a unit killing vector field ξ;

(2) The sectional curvature for plane sections containing ξ are equal to 1 or -1 at every point on M.

(3) $R(X,Y)\xi = 0 \qquad \forall X, Y \perp \xi$

2. THE PROOF OF THEOREM 1.1

Let M be a $(2n+1)$–dimensional (ε)–sasakian manifold with (ε)–sasakian structure (ϕ, ξ, η, g). Let R be real line with coordinate t and unit tangent vector $\frac{d}{dt}$. Denote $M \times R$ by $\overline{M}$, then vector fields on $\overline{M}$ are given by $\overline{X} = (X, f\frac{d}{dt}), \overline{Y} = (Y, h\frac{d}{dt}), \cdots,$

Where X, Y ..., are vector fields tangent to M and $f, h, \ldots$, are function on M, we define a linear map J on the tangent space of $\overline{M}$ by [5]

$$J\overline{X} = J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{7}$$

From (1) and (2), we have

$$J^2\overline{X} = J(\phi X - f\xi, \eta(X)\frac{d}{dt}) = (\phi^2 X - \eta(X)\xi, -f\frac{d}{dt}) = -\overline{X}$$

It shows that J is almost complex structure on $\overline{M}$, but M is an (ε)–sasakian manifold, this means $N(J) = 0$, then J is a complex structure on $\overline{M}$, thus $\overline{M}$=$M \times R$ is a complex manifold.

Let $\pi : \overline{M}=M \times R \rightarrow M$ be the projection map, we introduce a metric G on $\overline{M}$ by

$$G = e^{\varepsilon t}(\pi^* g + \varepsilon dt \otimes dt) \tag{8}$$

As an induced metric of g, we have

$$G((X,0),(Y,0)) = g(X,Y) \qquad (t=0) \tag{9}$$

For any vector fields $\overline{X} = (X, f\frac{d}{dt}), \overline{Y} = (Y, h\frac{d}{dt})$on $\overline{M}$, we obtain from (7)(8)

$$G(\overline{X}, \overline{Y}) = e^{\varepsilon t}(g(X,Y) + \varepsilon fh) \tag{10}$$

$$G(J\overline{X}, \overline{Y}) = e^{\varepsilon t}(g(\phi X, Y) - \varepsilon f\eta(Y) + \varepsilon h\eta(X)) \tag{11}$$

$$G(\overline{X}, J\overline{Y}) = e^{\varepsilon t}(g(X, \phi Y) - \varepsilon h\eta(X) + \varepsilon f\eta(Y)) \tag{12}$$

$$\begin{aligned} G(J\overline{X}, J\overline{Y}) &= G((\phi X - f\xi, \eta(X)\frac{d}{dt}), (\phi Y - h\xi, \eta(Y)\frac{d}{dt})) \\ &= e^{\varepsilon t}(g(\phi X, \phi Y) + \varepsilon fh + \varepsilon\eta(X)\eta(Y)) \end{aligned} \tag{13}$$

From (10)–(13),we see

$$G(\overline{X}, J\overline{Y}) = -G(J\overline{X}, \overline{Y}), \qquad G(J\overline{X}, J\overline{Y}) = G(\overline{X}, \overline{Y})$$

Thus G is a Hermitian metric on $\overline{M}$.

Define a 2–fore on $\overline{M}$ by

$$\Phi = e^{\varepsilon t}(\pi^* d\eta + \varepsilon dt \wedge (\pi^*\eta)) \tag{14}$$

using $\pi^* \circ d = d \circ \pi^*$, we get

$$\begin{aligned} d\Phi &= \varepsilon e^{\varepsilon t} dt \wedge (\pi^* d\eta + \varepsilon dt \wedge \pi^*\eta)) + \\ &\quad e^{\varepsilon t}[\pi^* d^2\eta + \varepsilon d^2 t \wedge (\pi^*\eta) - \varepsilon dt \wedge \pi^* d\eta] = 0 \end{aligned} \tag{15}$$

therefore, ϕ is a closed 2–form on $\overline{M}$, by a direct computation, we get

$$\begin{aligned} \Phi(\overline{X}, \overline{Y}) &= \Phi((X, f\frac{d}{dt}), (Y, h\frac{d}{dt})) \\ &= e^{\varepsilon t}(d\eta(X,Y) + \varepsilon(dt \wedge \pi^*\eta)(\overline{X}, \overline{Y})) \\ &= e^{\varepsilon t}(d\eta(X,Y) + \varepsilon f\eta(Y) - \varepsilon h\eta(X)) \end{aligned} \tag{16}$$

From (12) and (16) we see that

$$\Phi(\overline{X},\overline{Y}) = G(\overline{X}, J\overline{Y}) \tag{17}$$

Then from (15) and (17), we know,the Φ defined by (14) is the closed fundamental 2-form, thus the G defined by (8) is an indefinite kaehlerian metric on $\overline{M}^{[3]}$ and hence $\overline{M} = M \times R$ is an indefinite kaehlerian manifold.

3. THE PROOF OF THEOREM 1.2

First of all, we state some results which we shall need later.

LEMMA 3.1. (see [1] p. 548). An (ε)–almost contact metric structure (ϕ, ξ, η, g) is (ε)–sasakian if and only if

$$(\nabla_X\phi)Y = g(X,Y)\xi - \varepsilon\eta(Y)X, \qquad \forall X,Y \in \Gamma(TM) \tag{18}$$

Where ∇ is the Levi–civita connection with respect to g.

If we replace Y by ξ in (18) and from (1) (2) we get

$$\nabla_X\xi = -\varepsilon\phi X \qquad \forall X \in \Gamma(TM) \tag{19}$$

Because

$$\begin{aligned}
(L_\xi g)(X,Y) &= \xi g(X,Y) - g([\xi,X],Y) - g(X,[\xi,Y]) \\
&= \xi g(X,Y) - g(\nabla_\xi X - \nabla_X\xi, Y) - g(X, \nabla_\xi Y - \nabla_Y\xi) \\
&= (\xi g(X,Y) - g(\nabla_\xi X, Y) - g(X, \nabla_\xi Y + g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) \\
&= (\nabla_\xi g)(X,Y) + g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) \\
&= g(-\varepsilon\phi X, Y) + g(X, -\varepsilon\phi Y) \\
&= -\varepsilon(g(\phi X, Y) + g(X, \phi Y)) = 0 \qquad \forall X,Y \in \Gamma(TM)
\end{aligned}$$

Then we get

PROPOSITION 3.1. The characteristic vector field ξ on an (ε)–sasakian manifold is a killing vector field.

LEMMA 3.2.([6] p.265) Let M be a contact metric manifold with contact metric structure (ϕ, ξ, η, g). Then $N^{(3)} \equiv (L_\xi\phi)X$ vanishes if and only if ξ is a killing vector field with respect to g.

PROPOSITION 3.2. Let M be an (ε)–sasakian manifold. then the sectional curvature for plane sections containing ξ are equal to 1 or -1 at every point on M.

PROOF. Let X be an unit vector field on M and $X \perp \xi$, then from (19) we have

$$\begin{aligned}
R(\xi,X)\xi &= \nabla_\xi \nabla_X \xi - \nabla_X \nabla_\xi \xi - \nabla_{[\xi,X]}\xi \\
&= -\varepsilon \nabla_\xi (\phi X) + \varepsilon\phi([\xi,X]) \\
&= -\varepsilon(\nabla_\xi(\phi X) - \phi(\nabla_\xi X - \nabla_X\xi)) \\
&= -\varepsilon((\nabla_\xi\phi)X + \phi(\nabla_X\xi))
\end{aligned}$$

From Lemma 3.1, we get

$$(\nabla_\xi\phi)X = g(\xi,X)\xi - \varepsilon\eta(X)\xi = 0$$

thus we have

$R(\xi,X)\xi = -\varepsilon\phi(\nabla_X\xi) = \phi^2X = -X$ then

$$g(R(\xi,X)X,\xi) = -g(R(\xi,X)\xi,X) = \pm 1$$

From (18) and (19), let any $X,Y \in \Gamma(TM)$ and $X,Y \perp \xi$ we have

$$\begin{aligned} R(X,Y)\xi &= \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi \\ &= \nabla_X(-\varepsilon\phi Y) - \nabla_Y(-\varepsilon\phi X) + \varepsilon\phi[X,Y] \\ &= \varepsilon((\nabla_Y\phi)X - (\nabla_X\phi)Y) \\ &= \varepsilon(g(X,Y)\xi - \varepsilon\eta(X)Y - g(X,Y)\xi + \varepsilon\eta(Y)X) \\ &= \eta(Y)X - \eta(X)Y = o \end{aligned}$$

Then, by Proposition 3.1; 3.2, we get the necessary condition of Theorem 2.

Conversely, first, we define a 1-form η and a tensor field of type (1.1) by

$$\eta(X) = g(X,\xi) \qquad \phi X = -\nabla_X \xi$$

We know from [4] (ϕ,ξ,η,g) be an almost contact metric structure, satisfying

$$\phi^2 = -I + \eta\otimes\xi, \qquad g(X,\phi Y) = d\eta(X,Y)$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

Let $\bar\xi = \varepsilon\xi, \bar\eta = \varepsilon\eta, \bar g = \varepsilon g$, then

$$\bar\eta(X) = \varepsilon\bar g(X,\bar\xi), \qquad \phi X = -\varepsilon\nabla_X\bar\xi$$

$$\phi^2 = -I + \bar\eta\otimes\bar\xi, \qquad \bar g(X,\phi Y) = d\bar\eta(X,Y)$$

$$\bar g(\phi X,\phi Y) = \bar g(X,Y) - \varepsilon\bar\eta(X)\bar\eta(Y)$$

Thus $(\phi,\bar\xi,\bar\eta,\bar g)$ be an (ε)–contact metric structure.

Now we show that $N^{(1)} = 0$, from condition (3) of Theorem 2, we obtain

$(\nabla_X\phi)Y = (\nabla_Y\phi)X, \qquad \forall X,Y \perp \bar\xi,$ thus

$$\begin{aligned} N_\phi(X,Y) &= [\phi,\phi](X,Y) \\ &= (\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X + \phi[(\nabla_Y\phi)X - (\nabla_X\phi)Y] \\ &= (\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X \qquad \forall X,Y \perp \bar\xi \end{aligned}$$

By using Lemma 3.1, we get

$$N_\phi(X,Y) = -2\bar g(X,\phi Y)\bar\xi$$

then

$$N^{(1)}(X,Y) = N_\phi(X,Y) + 2\bar g(X,\phi Y)\bar\xi = 0$$

If $X \perp \bar{\xi}$, we have by Lemma 3.2

$$N^{(1)}(X,\bar{\xi}) = N_{\phi}(X,\bar{\xi}) = \varepsilon\phi(L_{\bar{\xi}}\phi)X = 0$$

Thus, for any vector field X, Y on M $N^{(1)}(X,Y) = 0$

Hence, the (ε)–contact metric structure $(\phi,\bar{\xi},\bar{\eta},\bar{g})$ is normal, that is, M is an (ε)– sasakian manifold with an (ε)–sasakian structure $(\phi,\bar{\xi},\bar{\eta},\bar{g})$.

Theorem 2 can be improved.

THEOREM 2′. In order that a $(2n+1)$–dimensional Riemannian manifold M is (ε)–sasakian manifold, it is necessary and sufficient that the following two conditions are satisfied

(1) M admits a unit killing vector field ξ

(2) $R(X,Y)\xi = \eta(Y)X - \eta(X)Y \qquad \forall X,Y \in \Gamma(TM)$

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