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ON A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS NEAR CRITICAL GROWTH

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ABSTRACT. We use Minimax Methods and explore compact embedddings in the context of Orlicz and Orlicz-Sobolev spaces to get existence of weak solutions on a class of semilinear elliptic equations with nonlinearities near critical growth. We consider both biharmonic equations with Navier boundary conditions and Laplacian equations with Dirichlet boundary conditions.

KEY WORDS AND PHRASES: Elliptic Equations, Variational Methods, Orlicz Spaces.
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1. INTRODUCTION

Our concern in this paper is on finding weak solutions for the problem

$$(-1)^m\Delta^m u = f(x,u) \ \ in \ \ \Omega, \ \ B_m(u) = 0 \ \ on \ \ \partial\Omega \tag{1.1}$$

where Δ^m is the elliptic operator

$$\Delta^m \equiv \sum_{i=1}^{N} \frac{\partial^{2m}}{\partial x_i^{2m}} + (m-1)\sum_{\substack{i,j=1,\\ i\neq j}}^{N} \frac{\partial^{2m}}{\partial x_i^m \partial x_j^m} \quad m = 1,2,$$

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and the boundary operator B_m is given by

$$B_m(u) = (u, (m-1)\Delta u),$$

that is, $B_m(u) = 0$ means either the Dirichlet or the Navier boundary conditions according to $m = 1$ or $m = 2$.

By a weak solution of (1.1) we mean an element $u \in H_m \equiv H_o^1(\Omega) \cap H^m(\Omega)$ satisfying

$$\langle u, v\rangle_m = \int_\Omega f(x,u)v, \ \ v \in H_m$$

with $\Delta u = 0$ on $\partial\Omega$ when $m = 2$, where

$$\langle u, v\rangle_m \equiv (m-1)\int_\Omega \Delta u\Delta v + (2-m)\int_\Omega \nabla u\nabla v \ \ u,v \in H_m.$$

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By the way $\langle .,. \rangle_m$ is an inner product in H_m, we denote by $\|.\|_m$ its corresponding norm and we remark that H_m is a Hilbert space.

Now let $a : [0, \infty) \to I\!R$ be a right continuous, nondecreasing function satisfying the following conditions

$$a(0) = 0, \;\; a(t) > 0 \;\text{ for } t > 0, \quad a(t) \to \infty \;\text{ as }\; t \to \infty \tag{1.2}$$

and let

$$A(t) = \int_0^t a(|s|)ds \;\text{ and }\; p^* = \frac{2N}{(N-2m)}.$$

We shall assume that both

$$|f(x,t)| \le C_1 + C_2\, a(|t|), \;\; (x,t) \in \Omega \times I\!R \tag{1.3}$$

for some $C_1 \ge 0$, $C_2 > 0$ and

$$A(t) = o(t^{p^*}) \;\text{ as }\; t \to \infty. \tag{1.4}$$

Now consider the functional

$$I_m(u) = \frac{1}{2}\|u\|_m^2 - \int_\Omega F(x,u)dx, \;\; u \in H_m$$

where $F(x,t) = \int_0^t f(x,s)ds$. It follows under conditions (1.2)(1.3)(1.4) and condition (1.5) below that $I_m \in C^1(H_m, I\!R)$ and its derivative is given by

$$\langle I_m'(u), v\rangle = \langle u, v\rangle_m - \int_\Omega f(x,u)v \quad u, v \in H_m.$$

We shall look for weak solutions of (1.1) by finding critical points of I_m. Our main result is the following.

THEOREM 1. Assume (1.2)(1.3)(1.4). Assume in addition that

$$a(|t|) \le |t|^{(p^*-1)} \quad t \in I\!R, \tag{1.5}$$

$$f(x,t) = o(t) \quad t \to 0, \;\; \text{uniformly } \; x \in \Omega \tag{1.6}$$

$$0 < \theta F(x,t) \le t f(x,t) \;\; a.e. \;\; x \in \Omega \quad |t| \ge M \tag{1.7}$$

for some $M > 0$, $\theta > 2$.

Then (1.1) has a non zero weak solution.

Our Theorem improves results by Rabinowitz [15], Gu [7], deFigueiredo, Clement & Mitidieri [3] in the sense that we allow less restrictive growth on $f(x,t)$. It is also related to some results in Brézis & Nirenberg [14], Pucci & Serrin [12], van der Vorst [13].

We employ the Ambrosetti & Rabinowitz Mountain Pass Theorem as in some of the above mentioned papers and the main point here is the use of Orlicz and Orlicz-Sobolev spaces to overcome compactness difficulties.

2. PRELIMINARIES

We shall apply the following variant of the Ambrosetti & Rabinowitz [2] Mountain Pass Theorem (see Mawhin & Willem [6]).

THEOREM 2. Let X be a Banach space and let $I \in C^1(X, \mathbb{R})$ with $I(0) = 0$. Assume in addition that

$$I(u) \geq r \quad \text{when} \quad \|u\| = \rho, \quad \text{for some } r, \rho > 0 \tag{2.1}$$

$$I(e) \leq 0, \quad \text{for some } e \in X \quad \text{with} \quad \|e\| > \rho. \tag{2.2}$$

Then there is a sequence $u_n \in X$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \quad c \geq r$$

and

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = 0 \;\; \gamma(1) = e\}.$$

We shall apply theorem 2 with $I = I_m$ and $X = H_m$. The two lemmas below are crucial in applying theorem 2 to prove theorem 1.

LEMMA 3. (The Mountain Pass Geometry) Assume (1.2)-(1.7). Then (2.1)-(2.2) hold true.

We remark that by lemma 3 there is a sequence $u_n \in H_m$ such that

$$I_m(u_n) \to c \quad \text{and} \quad I'_m(u_n) \to 0.$$

Such a sequence is called a $(PS)_c$ sequence.

We are going to show, (see lemma 5 below), that u_n has a convergent subsequence. The proof of lemma 5 uses a crucial compactness type result (see lemma 4 below).

Prior to stating lemma 4 we shall recall some notations and basic results on Orlicz and Orlicz-Sobolev spaces. We refer the reader to Krasnosels'kii & Rutickii [5], Gossez [4], Adams [1] for an accounting on the subject. In this regard a function A satisfying the set of conditions:

$$A \text{ is convex, even, continuous} \tag{2.3}$$

$$A(t) = 0 \;\; iff \;\; t = 0 \tag{2.4}$$

$$\frac{A(t)}{t} \to \begin{cases} 0 & \text{when } t \to 0 \\ \infty & \text{when } t \to \infty \end{cases} \tag{2.5}$$

is referred to in the literature on Orlicz Spaces as an N-function. An Orlicz space is defined by

$$L_A(\Omega) \equiv \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \int_\Omega A(l|u|) < \infty \; forsome \; l > 0\}$$

and the norm given by

$$|u|_A \equiv \inf_{\alpha \in \mathbb{R}} \left\{ \alpha > 0 \mid \int_\Omega A\left(\frac{|u|}{\alpha}\right) \leq 1 \right\}$$

turns it into a (not necessarily reflexive) Banach space and as a matter of fact $L_A(\Omega) \to L^1(\Omega)$.

Corresponding to A there is an N-function labeled $\overline{A}$ called the conjugate function of A which satisfies the so called Young's inequality

$$st \leq A(t) + \overline{A}(s)$$

and in addition

$$ta(t) = A(t) + \overline{A}(a(t))$$

where

$$A(t) = \int_0^t a(|s|)ds$$

and a satisfies (1.2).

Moreover one also has a Hölder inequality namely

$$\int u \cdot v \leq 2|u|_{L_A}|v|_{L_{\overline{A}}}.$$

Now the Orlicz-Sobolev space is defined by

$$W^m L_A(\Omega) = \{u \in L_A(\Omega) \mid D^\alpha u \in L_A(\Omega), \ |\alpha| \leq m\}$$

and the norm

$$\|u\| \equiv \left[\sum_{|\alpha| \leq m} |D^\alpha u|^2_{L_A}\right]^{\frac{1}{2}}$$

turns it into a Banach space.

LEMMA 4. Assume (1.4). Then $H_m \hookrightarrow L_A(\Omega)$, $m = 1,2$

LEMMA 5. Assume $(1.2) - (1.7)$. Then the sequence u_n has a convergent subsequence.

3. PROOFS.

PROOF OF LEMMA 3.

At first given $\epsilon > 0$ there is by (1.6) some $\delta > 0$ such that

$$\frac{f(x,t)}{t} \leq \epsilon, \quad |t| < \delta \quad a.e. \ x \in \Omega$$

so that

$$F(x,t) \leq \frac{\epsilon}{2}t^2, \quad |t| < \delta \quad a.e. \ x \in \Omega.$$

On the other hand from (1.3), (1.5) we have

$$|f(x,t)| \leq C_1 + C_2|t|^{(p^*-1)}$$

so that

$$F(x,t) \leq C_1|t| + \frac{C_2}{p^*}|t|^{p^*}, \quad \text{a.e. } x \in \Omega, \ t \in \mathbb{R}.$$

Hence

$$F(x,t) \leq \tfrac{\epsilon}{2}t^2 + C_\delta|t|^{p^*}, \quad \text{a.e. } x \in \Omega, \ t \in \mathbb{R}. \tag{3.1}$$

Now observing that

$$(m-1)\int_\Omega |\Delta u|^2 + (2-m)\int |\nabla u|^2 \geq \lambda_{1m}\int_\Omega u^2$$

where λ_{1m} is the first eigenvalue of

$$\begin{cases} (-1)^m \Delta^m u = \lambda u & \text{in } \Omega \\ B_m(u) = 0 & \text{on } \partial\Omega \end{cases}$$

and using (3.1) we get

$$\int_\Omega F(x,u) \le \frac{\epsilon}{2\lambda_{1m}} |u|_m^2 + C_\delta \int_\Omega |u|^{p^*}$$

CLAIM 1. $|u|_{L^{p^*}} \le C\|u\|_m,\ u \in H_m$.

Using CLAIM 1, we get

$$\int_\Omega F(x,u) \le \frac{\epsilon}{2\lambda_{1m}} \|u\|_m^2 + C\|u\|_m^{p^*}$$

so that

$$I_m(u) \ge (\frac{1}{2} - \frac{\epsilon}{2\lambda_{1m}})\|u\|_m^2 - C\|u\|_m^{p^*}.$$

Therefore there are $\rho > 0, r > 0$ such that

$$I_m(u) \ge r, \quad \|u\|_m = \rho.$$

On the other hand using (1.7) it follows that

$$F(x,t) \ge C|t|^\theta, \quad |t| \ge M \quad \text{a.e. } x \in \Omega.$$

Now take $\phi \in C_0^\infty$, $\phi \ge 0$, $\phi \not\equiv 0$ and $\lambda > 0$. Then

$$I_m(\lambda\phi) = \frac{\lambda^2}{2}\|\phi\|_m^2 - \int_{\lambda\phi \le M} F(x,\lambda\phi) - \int_{\lambda\phi > M} F(x,\lambda\phi)$$

Since

$$F(x,\lambda\phi) \ge -C_1\lambda\phi - C_2 A(\lambda\phi)$$

we get

$$\begin{aligned} I_m(\lambda\phi) &\le \tfrac{\lambda^2}{2}\|\phi\|_m^2 + \textstyle\int_{\lambda\phi\le M} (C_1\lambda\phi + C_2A(\lambda\phi)) - \int_{\lambda\phi > M} F(x,\lambda\phi) \\ &\le \tfrac{\lambda^2}{2}\|\phi\|_m^2 + \textstyle\int_{\lambda\phi\le M} (C_1 M + C_2 A(M)) - \int_\Omega \phi^\theta \chi_{\phi > \frac{M}{\lambda}} \\ &\le \tfrac{\lambda^2}{2}\|\phi\|_m^2 + C_M - \lambda^\theta \textstyle\int_\Omega \phi^\theta \chi_{\phi > \frac{M}{\lambda}}. \end{aligned}$$

Now, by Lebesgue Theorem

$$\int_\Omega \phi^\theta \chi_{\phi > \frac{M}{\lambda}} \to \int_\Omega \phi^\theta.$$

Thus

$$I_m(\lambda\phi) \to -\infty \quad \text{as} \quad \lambda \to \infty.$$

VERIFICATION OF CLAIM 1. If $m = 1$ CLAIM 1 holds by the Sobolev inequality. So let us assume $m = 2$. Letting

$$\|u\|_{2,2} \equiv \max_{|\alpha|\le 2} |D^\alpha u|_{L^2},$$

it is an easy matter to check that the space H_2 endowed with $\|.\|_{2,2}$ is complete. We claim that

$$\|u\|_2 \le C\|u\|_{2,2}.$$

Indeed,

$$\|u\|_2^2 = \int_\Omega |\Delta u|^2 \le C\left(\max_i \int_\Omega |\frac{\partial^2 u}{\partial x_i^2}|^2\right) \le C\left(\max_{|\alpha|\le 2} |D^\alpha u|_{L^2}^2\right) = C\|u\|_{2,2}^2$$

Hence we also have

$$\|u\|_{2,2} \le C\|u\|_2 \tag{3.2}$$

and by Sobolev embedding we get $|u|_{L^{p^*}} \le C\|u\|_2$, showing CLAIM 1 and thus proving lemma 3.

The proof of lemma 4 is a consequence of a general result due to Donaldson & Trudinger [9] (see also Adams [1, Theorem 8.40]). For the sake of completeness we recall that result in an Appendix. (see THEOREM A.1)

PROOF OF LEMMA 4.

Case $m = 2$. Applying the notations of theorem A.1 let

$$B_0^{-1}(t) = \sqrt{2}t^{\frac{1}{2}},\ t \ge 0$$

and

$$(B_k)^{-1}(t) \equiv \int_0^t \frac{(B_{k-1})^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau \quad t \ge 0,\ \ k = 1, 2.$$

We claim that

$$\int_1^\infty \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau = \infty \ \text{ for } \ k = 0, 1. \tag{3.3}$$

and

$$\int_1^\infty \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty \ \text{ for some } \ k \ge 2. \tag{3.4}$$

By (3.3) and (3.4) J is defined and $2 \le J \le N$.

Indeed by computing we find that

$$B_1^{-1}(t) = \frac{\sqrt{2}(N-2)}{2N} t^{\frac{N-2}{2N}} \tag{3.5}$$

and

$$B_2^{-1}(t) = \frac{\sqrt{2}(N-2)}{2N} t^{\frac{N-4}{2N}}. \tag{3.6}$$

Now using (3.5) and (3.6) and computing again we get (3.3). Thus $J \ge 2$.
In order to show (3.4) it suffices to evaluate

$$\int_0^t \frac{(B_{N-1})^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau.$$

But

$$B_k^{-1}(t) = C_{N,k} t^{\frac{N-2k}{2N}} \quad t \ge 0,\ \ C_{N,k} > 0,\ \ k \ge 1$$

and from this

$$\int_1^\infty \frac{B_N^{-1}(\tau)}{\tau^{\frac{N+1}{N}}} d\tau < \infty.$$

By computing again we find that

$$\int_0^1 \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty, \quad k = 1, 2.$$

Therefore by theorem A.1 we have

$$W^2 L_{B_0}(\Omega) \hookrightarrow L_A(\Omega),$$

since as we have shown above $J \geq 2$ and yet by (1.4)

$$\frac{B_2(\lambda t)}{A(t)} = \frac{C_{N,\lambda}|t|^{2^*}}{A(t)} \to \infty \text{ as } t \to \infty, \quad \lambda > 0.$$

The case $m = 1$ that is

$$W^1 L_{B_0}(\Omega) \hookrightarrow L_A(\Omega)$$

is similar and even more direct.

Hence

$$W^m L_{B_0}(\Omega) \hookrightarrow L_A(\Omega) \quad m = 1, 2.$$

Using (3.2) we finally get

$$H_m \hookrightarrow L_A(\Omega) \quad m = 1, 2.$$

This completes the proof of lemma 4.

Before proceding to the proof of lemma 5 we consider the function $a^*(t) \equiv 2C_2 a(t)$. We remark that $a^*(t)$ has the same properties of $a(t)$ and in addition its potential $A^*(t) \equiv \int_0^t a^*(\tau) d\tau$ is an N-function having the same properties as $A(t)$. In particular A^* satisfies (1.4) and moreover

$$|f(x,t)| \leq C_1 + \frac{1}{2} a^*(t).$$

PROOF OF LEMMA 5.

Using (1.7) we have

$$C \geq \tfrac{1}{2}\|u\|_m^2 - \textstyle\int_\Omega F(x, u_n) \geq \tfrac{1}{2}\|u\|_m^2 - C - \tfrac{1}{\theta}\textstyle\int_\Omega u_n f(x, u_n). \tag{3.7}$$

Now since $I'_m(u_n) \to 0$ we have

$$|\langle I'_m(u_n), u_n\rangle| \leq \epsilon \|u\|_m \text{ for large} n$$

that is

$$\|u\|_m^2 - \int_\Omega u_n f(x, u_n)| \leq \epsilon \|u\|_m \text{ for large} n.$$

Hence

$$\begin{aligned} C &\geq \tfrac{1}{2}\|u_n\|_m^2 - C - \tfrac{1}{\theta}\|u_n\|_m^2 - \tfrac{1}{\theta}\epsilon\|u_n\|_m \\ &= \left(\tfrac{1}{2} - \tfrac{1}{\theta}\right)\|u_n\|_m^2 - \tfrac{1}{\theta}\epsilon\|u_n\|_m - C \end{aligned}$$

showing that u_n is bounded in H_m. Hence by lemma 4 there is some $u \in H_m$ such that

$$u_n \rightharpoonup u \text{ in } H_m \text{ and } u_n \to u \text{ in } L_{A^*}(\Omega).$$

On the other hand, since $I'_m(u_n) \to 0$ we have

$$\langle u_n, \phi\rangle_m - \int_\Omega f(x, u_n)\phi = o(1), \quad \phi \in H_m.$$

We claim that

$$|f(x,u_n)|_{L_{A^*}} \leq C, \quad \text{for some } C > 0. \tag{3.8}$$

Assume (3.8) for a while. Using Hölder inequality in Orlicz spaces for L_{A^*} and $L_{\bar{A}^*}$ where $\bar{A}^*$ is the conjugate function of A^* (see e.g. Adams [1, pg 234]) we get

$$|\langle u_n, \phi\rangle_m| \leq o(1) + |f(x,u_n)|_{L_{\bar{A}^*}} |\phi|_{L_{A^*}} \tag{3.9}$$

Now replacing ϕ by $u_n - u$ in (3.9) and using (3.8) we have

$$0 = \lim \langle u_n, u_n - u\rangle_m = \lim \langle u_n, u_n\rangle_m = \lim \langle u_n, u_n\rangle_m - \langle u, u\rangle_m$$

showing that $u_n \to u \ \ in \ \ H_m$.

VERIFICATION OF (3.8). We have

$$\begin{aligned} \int_\Omega \overline{A^*}(|f(x,u_n)|) &\leq \int_\Omega \overline{A^*}\left(C_1 + \tfrac{1}{2}a^*(|u_n|)\right) \\ &\leq \tfrac{1}{2}\int_\Omega \overline{A^*}(2C_1) + \tfrac{1}{2}\int_\Omega \overline{A^*}(a^*(|u_n|)) \\ &\leq C + \tfrac{1}{2}\int_\Omega A^*(|u_n|) + \int_\Omega |u_n| a^*(|u_n|) \\ &\leq C + C_1\left[\int_\Omega |u_n|^{p^*} + \int_\Omega |u_n|^{p^*}\right] \leq C \end{aligned}$$

showing (3.8) and consequently lemma 5.

PROOF OF THEOREM 1.

We have already shown using the lemmata above that I_m has a critical point $u \in H_m$ so that

$$\langle u, v\rangle_m = \int_\Omega f(x,u)v, \quad v \in H_m.$$

In the case $m = 1$, we have $H_1 = H_o^1$ and so u is a weak solution of $(*)_1$.

In the case $m = 2$ it remains to show that $\Delta u = 0$ *on* $\partial\Omega$. We use here an argument of [4].

By (1.3) and (1.5), we have

$$f(x,u) \in L^{p^{*'}}(\Omega) \text{ with } \frac{1}{p^*} + \frac{1}{p^{*'}} = 1.$$

Letting $g(x) = f(x,u)$ using the fact that $p^* > 2$ it follows that $W \equiv W^{2,p^*}(\Omega) \cap W_0^{1,p^*}(\Omega) \subset H_2$ and we have

$$\int_\Omega \Delta u \Delta z = \int_\Omega g(x)z, \quad z \in W.$$

Since $g(x) \in L^{p^{*'}}(\Omega)$ there is a unique $w \in W^{2,p^{*'}}(\Omega) \cap W_0^{1,p^{*'}}(\Omega)$ such that

$$\Delta w = g(x), \quad x \in \Omega.$$

Hence

$$\int_\Omega \Delta u \Delta z = \int_\Omega \Delta w z = \int_\Omega w \Delta z, \quad z \in W.$$

On the other hand given $h \in L^{p^*}(\Omega)$, there is a unique $z \in W$, such that

$$\Delta z = h(x), \quad x \in \Omega.$$

Thus

$$\int_\Omega (\Delta u - w)h = 0, \quad h \in L^{p^*}(\Omega)$$

showing that

$$\Delta u = w \quad in \quad \Omega$$

and so

$$\Delta u = 0, \quad \text{on} \quad \partial\Omega.$$

This proves theorem 1.

4. APPENDIX

At first we recall a general result due to Donaldson & Trudinger [9] (see also Adams [1, theorem 8.40]).

Let C be an N-function and consider the sequence of N-functions

$$B_0(t) \equiv C(t), \quad t \geq 0$$

$$(B_k)^{-1}(t) \equiv \int_0^t \frac{(B_{k-1})^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau, \quad k = 1, 2, \cdots, \quad t \geq 0.$$

It follows that

$$\int_1^\infty \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty \quad \text{for some} \quad k \geq 1.$$

Let us label $J \equiv J(C)$ the least such k.

THEOREM A.1. Assume $\Omega \subset \mathbb{R}^N$ is a bounded domain with the cone property. Assume also that

$$\int_0^1 \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty, \quad k = 1, 2, \ldots.$$

Then

$$W^m L_{B_0}(\Omega) \to L_{B_m}(\Omega) \tag{3.10}$$

provided $J \geq m$,

$$W^m L_{B_0}(\Omega) \hookrightarrow L_A(\Omega) \tag{3.11}$$

provided both $J \geq m$ and A is an N-function such that

$$\frac{B_m(\lambda t)}{A(t)} \to \infty \quad \text{as} \quad t \to \infty, \quad \lambda > 0.$$

Next we present an example to illustrate our assumptions $(1.2) - (1.5)$.

EXAMPLE A.2. Let $a : [0, \infty) \to \mathbb{R}$ be given by $a(t) = t^{p^*-1}$ if $0 \leq t < 1$, $a(t) = t^{(p^*-1) - \frac{1}{log(log(2))}}$ if $1 \leq t < 3$ and $a(t) = t^{(p^*-1) - \frac{1}{log(log(n))}}$ if $n \leq t < (n+1)$ for $n = 3, 4, \ldots$.

Then a satisfies $(1.2), (1.5)$ and it is a straightfoward calculation to show that A satisfies (1.4).

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