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STABILITY PROBLEM OF SOME NONLINEAR DIFFERENCE EQUATIONS

ALAA E. HAMZA and M.A. EL-SAYED

Department of Mathematics, Faculty of Science, Cairo University,
Giza, 12211, Egypt.

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ABSTRACT. In this paper, we investigate the asymptotic stability of the recursive sequence

$$x_{n+1} = \frac{\alpha + \beta x_n^2}{1 + \gamma x_{n-1}}, \quad n = 0, 1, \ldots$$

and the existence of certain monotonic solutions of the equation

$$x_{n+1} = x_n^p f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots$$

which includes as a special case the rational recursive sequence

$$x_{n+1} = \frac{\beta x_n^p}{1 + \sum_{i=1}^{k} \gamma_i x_{n-i}^{p-r}},$$

where $\alpha \geq 0$, $\beta > 0$, $\gamma > 0$, $\gamma_i \geq 0$, $i = 1, 2, \ldots, k$, $\sum_{i=1}^{k} \gamma_i > 0$, $p \in \{2, 3, \ldots\}$ and $r \in \{1, 2, \ldots, p-1\}$. The case when $r = 0$ has been investigated by Camouzis et. al. [1], and for $r = 0$ and $p = 2$ by Camouzis et. al. [2].

KEY WORDS AND PHRASES: Difference Equations, Monotonic solutions, stability.

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1. INTRODUCTION

Many authors studied the asymptotic behaviour of the recursive sequence

$$x_{n+1} = x_n f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{1.1}$$

which includes as a special case the rational recursive sequence

$$x_{n+1} = \frac{a + b x_n}{1 + \sum_{i=1}^{k} \gamma_i x_{n-i}}, \quad n = 0, 1, \ldots \tag{1.2}$$

See Jaroma et. al. [3]. Also, there are many results about permanence, global attractivity and asymptotic stability of equation (1.2), see Camouzis et. al. [2], Kocic and Ladas [4-5] and Kocic et. al. [6]. The investigation of the behaviour of solutions of the equation

$$x_{n+1} = x_n^p f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{1.3}$$

was suggested by Kocic and Ladas [5]. This equation includes as a special case the equation

$$x_{n+1} = \frac{\alpha + \beta x_n^p}{1 + \gamma x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.4}$$

Our aim in this paper is to study the asymptotic stability of the rational recursive sequence (1.4) when $p = 2$, see section 2. On the basis of the results of section 2 we also investigate the behaviour of solutions of equation (1.3), in section 3. We show that under certain conditions on f, there exists two solutions of (1.3) such that one tends to zero and the other tends to infinity. See theorem 3.1. We apply this theorem to equation (1.4) when $\alpha = 0$.

2. THE RECURSIVE SEQUENCE $x_{n+1} = \left(\alpha + \beta x_n^2\right) / (1 + \gamma x_{n-1})$

In this section we study the asymptotic stability for the rational recursive sequence

$$x_{n+1} = \frac{\alpha + \beta x_n^2}{1 + \gamma x_{n-1}}, \quad n = 0, 1, \ldots \tag{2.1}$$

where $\alpha \geq 0$ and $\beta, \gamma > 0$.

The linearized equation associated with (2.1) about $\bar{x}$ is

$$y_{n+1} - \frac{2\beta\bar{x}}{1 + \gamma\bar{x}} y_n + \frac{\gamma\bar{x}}{1 + \gamma\bar{x}} y_{n-1} = 0, \quad n = 0, 1, \ldots \tag{2.2}$$

The characteristic equation of (2.2) about $\bar{x}$ is

$$\lambda^2 - \frac{2\beta\bar{x}}{1 + \gamma\bar{x}}\lambda + \frac{\gamma\bar{x}}{1 + \gamma\bar{x}} = 0. \tag{2.3}$$

Equation (2.3) can be rewritten in the form

$$(\lambda - l\theta)^2 = l^2\theta^2 - \theta, \tag{2.4}$$

where

$$l = \beta/\gamma \text{ and } \theta = \frac{\gamma\bar{x}}{1 + \gamma\bar{x}}. \tag{2.5}$$

We summarize the results of this section in the following

THEOREM 2.1

(1) If $\beta > \gamma$ and $\alpha = 0$, then equation (2.1) has two equilibria:

$$\bar{x}_1 = 0, \qquad \bar{x}_2 = \frac{1}{\beta - \gamma}$$

and $\bar{x}_1$ is asymptotically stable while $\bar{x}_2$ is unstable. Neither of them is a global attractor.

(2) If $\beta < \gamma$ and $\alpha = 0$, then the unique equilibrium point $\bar{x} = 0$ of equation (2.1) is globally asymptotically stable.

(3) If $\beta < \gamma$ and $\alpha > 0$, then the unique positive equilibrium point

$$\bar{x} = \frac{\sqrt{1 + 4\alpha(\gamma - \beta)} - 1}{2(\gamma - \beta)}$$

of equation (2.1) is asymptotically stable.

(4) If $\beta = \gamma$, equation (2.1) has the unique equilibrium point $\bar{x} = \alpha$ which is asymptotically stable.

(5) If $\beta > \gamma$, $\alpha > 0$ and $1 > 4\alpha(\beta - \gamma)$, then equation (2.1) has two positive equilibria

$$\bar{x}_1 = \frac{1 - \sqrt{1 - 4\alpha(\beta - \gamma)}}{2(\beta - \gamma)}$$

which is asymptotically stable, and

$$\bar{x}_2 = \frac{1 + \sqrt{1 - 4\alpha(\beta - \gamma)}}{2(\beta - \gamma)}$$

which is unstable. None of them is a global attractor.

(6) If $\beta > \gamma$, $\alpha > 0$ and $1 = 4\alpha(\beta - \gamma)$, then equation (2.1) has the unique equilibrium $\bar{x} = 1/2(\beta - \gamma)$ which is neither stable nor a global attractor.

(7) Assume that $\beta > \gamma$, $\alpha > 0$ and $1 < 4\alpha(\beta - \gamma)$. If the initial conditions $\{x_{-1}, x_0\}$ are such that

$$x_0 \geq x_{-1} \text{ and } x_0 \geq \frac{1}{\beta - \gamma},$$

then $\{x_n\}$ tends to infinity monotonically.

PROOF.

(1) Assume that $\beta > \gamma$ and $\alpha = 0$. The characteristic equation of equation (2.2) about $\bar{x}_1 = 0$ is $\lambda^2 = 0$. Hence $\bar{x}_1$ is asymptotically stable, by Corollary 1.3.2 Kocic and Ladas [5] page 14. The characteristic equation of equation (2.2) about $\bar{x}_2 = 1/(\beta - \gamma)$ is

$$\lambda^2 - 2\lambda + \frac{\gamma}{\beta} = 0,$$

which has two solutions $\lambda = 1 \pm \sqrt{1 - \gamma/\beta}$. Therefore, $\bar{x}_2$ is unstable. The nonattractivity of equilibria $\bar{x}_1$ and $\bar{x}_2$ follows directly from theorems 3.3 and 5.1 of Camouzis et. al. [1].

(2) Assume that $\beta < \gamma$ and $\alpha = 0$. Let $\{x_n\}$ be a positive solution of equation (2.1). We have

$$\frac{x_{n+1}}{x_n} = \frac{\beta x_n}{1 + \gamma x_{n-1}} < \frac{\beta}{\gamma} \frac{x_n}{x_{n-1}}$$

Hence $x_{n+1}/x_n < (\beta/\gamma)^{n+1}(x_0/x_{-1})$ $\forall n \in \mathbf{N}$. Since $\beta/\gamma < 1$, then $(\beta/\gamma)^{n+1}(x_0/x_{-1}) < 1$ $\forall n \geq n_0$ for some $n_0 \in \mathbf{N}$. Therefore, $x_{n+1} < x_n$ $\forall n \geq n_0$. This implies that $\lim_{n\to\infty} x_n = 0$, i.e., $\bar{x} = 0$ is globally asymptotically stable.

(3) Suppose that $\beta < \gamma$ and $\alpha > 0$. We can see that $|\lambda| < 1$ for every solution λ of the characteristic equation (2.4), about

$$\bar{x} = \frac{\sqrt{1 + 4\alpha(\gamma - \beta)} - 1}{2(\gamma - \beta)}.$$

Indeed, we have the following two cases

First case: $l^2\theta^2 - \theta < 0$. In this case $\lambda = l\theta \pm ir$, where $r^2 = \theta - l^2\theta^2$. Hence $|\lambda|^2 = l^2\theta^2 + r^2 = \theta < 1$.

Second case: $l^2\theta^2 - \theta \geq 0$. In this case $\lambda = l\theta \pm \sqrt{l^2\theta^2 - \theta}$. Hence, $|\lambda| \leq l\theta + \sqrt{l^2\theta^2 - \theta}$. Since $l < 1$, then $(2l - 1)\theta < 1$. Hence $(1 - l\theta)^2 > l^2\theta^2 - \theta$ whence $l\theta + \sqrt{l^2\theta^2 - \theta} < 1$. Therefore $|\lambda| < 1$. In both cases $|\lambda| < 1$ and thus $\bar{x}$ is asymptotically stable.

(4) Assume that $\beta = \gamma$. For every solution λ of the characteristic equation (2.4) about $\bar{x} = \alpha$, we have $|\lambda|^2 = \gamma\alpha/(1+\gamma\alpha) < 1$. Therefore, $\bar{x}$ is asymptotically stable.

(5) Suppose that $\beta > \gamma$, $\alpha > 0$ and $1 > 4\alpha(\beta - \gamma)$. The characteristic equation of (2.2) about

$$\bar{x}_1 = \frac{1 - \sqrt{1 - 4\alpha(\beta - \gamma)}}{2(\beta - \gamma)}$$

is obtained by setting $\bar{x} = \bar{x}_1$ in equation (2.5). Since $\bar{x}_1 < 1/2(\beta - \gamma) < 1/(\beta - \gamma)$, then $l\theta < 1$. We can see that $|\lambda| < 1$ for every solution λ of equation (2.4). Indeed, we have the following two cases

First case: $l^2\theta^2 - \theta < 0$. In this case $\lambda = l\theta \pm ir$, where $r^2 = \theta - l^2\theta^2$. Hence $|\lambda|^2 = l^2\theta^2 + r^2 = \theta < 1$.

Second case: $l^2\theta^2 - \theta \geq 0$. In this case $\lambda = l\theta \pm \sqrt{l^2\theta^2 - \theta}$. Hence, $|\lambda| \leq l\theta + \sqrt{l^2\theta^2 - \theta}$. Since $\bar{x}_1 < 1/2(\beta - \gamma)$, then $\gamma\bar{x}_1/(1 + \gamma\bar{x}_1) < \gamma/(2\beta - \gamma)$, i.e., $\theta < 1/(2l - 1)$. Hence $(1 - l\theta)^2 > l^2\theta^2 - \theta$ whence $l\theta + \sqrt{l^2\theta^2 - \theta} < 1$. Therefore $|\lambda| < 1$. In both cases $|\lambda| < 1$ and thus $\bar{x}_1$ is asymptotically stable. In a similar manner, it can be shown that

$$\bar{x}_2 = \frac{1 + \sqrt{1 - 4\alpha(\beta - \gamma)}}{2(\beta - \gamma)}$$

is unstable. To show the nonattractivity of $\bar{x}_1$ and $\bar{x}_2$, one chooses the initial conditions $\{x_{-1}, x_0\}$ such that

$$x_0 \geq x_{-1} \text{ and } x_0 \geq \max\{\frac{1}{\beta - \gamma}, \bar{x}_2\}.$$

We show by induction that $\{x_n\}$ is increasing. Indeed, we have

$$x_{n+1} > \frac{\beta x_n^2}{1 + \gamma x_{n-1}}, \quad n = 0, 1, \ldots.$$

Then

$$x_1 > x_0 \frac{\beta x_0}{1 + \gamma x_0} \geq x_0.$$

Assume that there exists $m_0 \geq 0$ such that

$$x_{n+1} > x_n \ \ \forall n \leq m_0.$$

Hence

$$x_{m_0+1} > x_{m_0} \frac{\beta x_{m_0}}{1 + \gamma x_{m_0 - 1}} > x_{m_0} \frac{\beta x_{m_0}}{1 + \gamma x_{m_0}} > x_{m_0} \frac{\beta x_0}{1 + \gamma x_0} > x_{m_0},$$

i.e., $\{x_n\}$ is increasing. The condition $x_0 \geq \bar{x}_2$ implies that x_n tends to infinity.

(6) Suppose that $\beta > \gamma$, $\alpha > 0$ and $1 = 4\alpha(\beta - \gamma)$. Substituting by $\bar{x} = 1/2(\beta - \gamma)$ in equation (2.3) one can easily see that $\bar{x} = 1/2(\beta - \gamma)$ is unstable. The nonattractivity of $\bar{x}$ follows directly by considering a solution $\{x_n\}$ with the initial conditions $\{x_{-1}, x_0\}$ satisfying

$$x_0 \geq x_{-1} \text{ and } x_0 \geq \frac{1}{\beta - \gamma}.$$

As in the proof of (5), it is easy to show that $\{x_n\}$ tends to infinity.

(7) Assume that $\beta > \gamma$, $\alpha > 0$ and $1 < 4\alpha(\beta - \gamma)$. Then in a similar way as in (5), one can easily show that the solution $\{x_n\}$ with the initial conditions $\{x_{-1}, x_0\}$ are such that

$$x_0 \geq x_{-1} \text{ and } x_0 \geq \frac{1}{\beta - \gamma}$$

is increasing. Since equation (2.1) has no real equilibria, then x_n tends to infinity.

3. THE EQUATION $x_{n+1} = x_n^p f(x_n, x_{n-1}, \dots, x_{n-k})$

Let $f \in C([0,\infty)^{k+1}, (0,\infty))$ such that f satisfies the following conditions

(C1) $f(x, u_1, \dots, u_k)$ is nonincreasing in $u_1, u_2, \dots, u_k$.

(C2) $x^{p-1} f(x, x, \dots, x)$ is increasing.

(C3) The equation $x^{p-1} f(x, x, \dots, x) = 1$ has a unique positive equilibrium $\bar{x}$.

We show that the asymptotic behaviour of the positive solutions of the difference equation

$$x_{n+1} = x_n^p f(x_n, x_{n-1}, \dots, x_{n-k}) \tag{3.1}$$

depends on the initial conditions, see theorem 3.1. More precisely, we can choose the initial conditions such that the corresponding solution $\{x_n\}$ may tend to zero or infinity.

LEMMA 3.1. Assume that $\{x_n\}$ is a solution of equation (3.1). Under conditions **(C1-C3)** the following statements are true

(a) If for some $n_0 \geq -k$,

$$x_{n_0+k} \leq x_{n_0+j} \quad , j = 0, 1, \dots, k-1 \text{ and } \quad x_{n_0+k} < \bar{x},$$

then

$$x_{n+k+1} < x_{n+k} \ < \bar{x} \ \forall n \geq n_0.$$

(b) If for some $n_0 \geq -k$,

$$x_{n_0+k} \geq x_{n_0+j} \quad , j = 0, 1, \dots, k-1 \text{ and } \bar{x} < x_{n_0+k},$$

then

$$x_{n+k} < x_{n+k+1} \ \forall n \ \geq n_0.$$

PROOF.

(a)Assume that for some $n_0 \geq -k$,

$$x_{n_0+k} \leq x_{n_0+j} \quad , j = 0, 1, \dots, k-1 \text{ and } \quad x_{n_0+k} < \bar{x},$$

Then

$$\begin{aligned} x_{n_0+k+1} &= x_{n_0+k}^p f(x_{n_0+k}, x_{n_0+k-1}, \dots, x_{n_0}) = x_{n_0+k} x_{n_0+k}^{p-1} f(x_{n_0+k}, x_{n_0+k-1}, \dots, x_{n_0}) \\ &\leq x_{n_0+k} x_{n_0+k}^{p-1} f(x_{n_0+k}, x_{n_0+k}, \dots, x_{n_0+k}) < x_{n_0+k}. \end{aligned}$$

We can see by induction that

$$x_{n+k+1} < x_{n+k} \ < \bar{x} \ \forall n \geq n_0.$$

(b) Assume that for some $n_0 \geq -k$,

$$x_{n_0+k} \geq x_{n_0+j} \quad , j = 0, 1, \dots, k-1 \text{ and } \bar{x} < x_{n_0+k},$$

Then

$$\begin{aligned} x_{n_0+k+1} &= x_{n_0+k}^p f(x_{n_0+k}, x_{n_0+k-1}, \dots, x_{n_0}) = x_{n_0+k} x_{n_0+k}^{p-1} f(x_{n_0+k}, x_{n_0+k-1}, \dots, x_{n_0}) \\ &\geq x_{n_0+k} x_{n_0+k}^{p-1} f(x_{n_0+k}, x_{n_0+k}, \dots, x_{n_0+k}) > x_{n_0+k}. \end{aligned}$$

By induction we see that

$$x_{n+k} < x_{n+k+1} \ \forall n \ \geq n_0.$$

THEOREM 3.1. Under conditions **(C1-C3)** the following statements are true

If $\{x_n\}$ is a solution of equation (3.1) with initial conditions $\{x_{-k}, \ldots, x_0\}$ that satisfy

$$x_{-j} \geq x_0 > 0, \ \ j = 1, \ldots, k \text{ and } \bar{x} > x_0,$$

then x_n tends to zero monotonically.

If the initial conditions $\{x_{-k}, \ldots, x_0\}$ are such that

$$x_{-j} \leq x_0, \ \ j = 1, \ldots, k \text{ and } \bar{x} < x_0,$$

then x_n tends to infinity monotonically.

PROOF.

(1) From Lemma 3.1 we see that the solution $\{x_n\}$ is decreasing whence it converges to a nonnegative number, say l. Since $l < \bar{x}$, then $l = 0$, because of condition **(C3)**.

(2) We can see in a similar manner that $\{x_n\}$ is increasing and $x_n > \bar{x} \quad \forall n \in \mathbf{N}$. Therefore, $x_n \to \infty$ as $n \to \infty$ by condition **(C3)**.

As a direct consequence we obtain the following result

COROLLARY 3.1. Under conditions **(C1-C3)**, equation (3.1) is not permanent.

4. MONOTONE SOLUTIONS OF $x_{n+1} = \beta x_n^p / (1 + \sum_{i=1}^k \gamma_i x_{n-i}^{p-r})$

We apply theorem (3.1) to the rational recursive sequence

$$x_{n+1} = \frac{\beta x_n^p}{1 + \sum_{i=1}^k \gamma_i x_{n-i}^{p-r}}, \tag{4.1}$$

where $\beta > 0, \ \gamma_i \geq 0 \ \forall i = 1, \ldots, k, \ p \in \{2, 3, \ldots\}, \ r \in \{1, 2, \ldots, p-1\}$ and $\gamma = \sum_{i=1}^k \gamma_i > 0$.

We verify that the function $f(x, u_1, \ldots, u_k) = \beta / (1 + \sum_{i=1}^k \gamma_i u_i^{p-r})$ satisfies conditions **(C1-C3)**. We can see easily that conditions **(C1-C2)** are satisfied. The equation

$$x = \frac{\beta x^p}{1 + \gamma x^{p-r}} \tag{4.2}$$

has a unique positive solution if and only if the function

$$h(x) = \beta x^{p-1} - \gamma x^{p-r} - 1$$

has a unique positive zero. Since

$$h'(x) = x^{p-r-1}[\beta(p-1)x^{r-1} - \gamma(p-r)],$$

then we have the following two cases

If $r \in \{2, \ldots, p-1\}$, then h has a unique positive zero $\bar{x} > [\gamma(p-r)/\beta(p-1)]^{1/r-1} = x_0$ which is the unique equilibrium point of (4.1). Indeed, the function h is decreasing for $0 < x < x_0$ and increasing for $x > x_0$. Moreover, $\lim_{x \to \infty} h(x) = \infty$ and $h(0) = -1 < 0$. Then equation (4.1) has a unique positive equilibrium $\bar{x}$.

If $r = 1$ and $\beta > \gamma$, then equation (4.1) has the unique positive equilibrium

$$\bar{x} = \left[\frac{1}{\beta - \gamma}\right]^{\frac{1}{p-1}}.$$

Now we can apply theorem (3.1) to equation (4.1) to obtain the following result.

COROLLARY 4.1. Assume that either

$$r \in \{2, \dots, p-1\}$$

or

$$r = 1 \text{ and } \beta > \gamma.$$

Let $\bar{x}$ be the unique positive equilibrium point of equation (4.1) and let $\{x_n\}$ be a solution of equation (4.1).

If for some $n_0 \geq -k$

$$x_{n_0+k} \leq x_{n_0+j} \quad, j = 0, 1, \dots, k-1 \text{ and } \quad x_{n_0+k} < \bar{x},$$

then

$$x_{n+k+1} < x_{n+k} \ \forall n \geq n_0.$$

If for some $n_0 \geq -k$,

$$x_{n_0+k} \geq x_{n_0+j} \quad, j = 0, 1, \dots, k-1 \text{ and } \bar{x} < x_{n_0+k},$$

then

$$x_{n+k} < x_{n+k+1} \ \forall n \geq n_0.$$

If the initial conditions $\{x_{-k}, \dots, x_0\}$ are such that

$$x_{-j} \geq x_0 > 0 \quad, j = 1, \dots, k \text{ and } \bar{x} > x_0,$$

then x_n tends to zero monotonically.

If the initial conditions $\{x_{-k}, \dots, x_0\}$ are such that

$$x_{-j} \leq x_0 \quad, j = 1, \dots, k \text{ and } \bar{x} < x_0,$$

then $x_n \to \infty$ monotonically.

Now, we consider the equation

$$x_{n+1} = \frac{\beta x_n^p}{1 + \gamma x_{n-1}^{p-1}}, \tag{4.2}$$

where $\beta > 0$, $\gamma > 0, p \in \{2, 3, \dots\}$. We prove that there exists a solution $\{x_n\}$ which tends monotonically to $\bar{x}$. We follow the proof by Camouzis et. al. [2].

THEOREM 4.1. If $\beta > \max\{\gamma, 2\sqrt{\gamma}\}$, then equation (4.2) has two solutions $\{x_n\}$ and $\{y_n\}$ such that $\{x_n\}$ increases to $\bar{x}$ and $\{y_n\}$ decreases to $\bar{x}$

PROOF. First, define the functions f_{-1} and f_0 on $[0, \infty)$ by

$$f_{-1}(x) = x^2 \quad , \quad f_0(x) = x$$

and

$$f_{n+1} = \frac{\beta f_n^p}{1+\gamma f_{n-1}^{p-1}}, \quad n = 0, 1, \ldots.$$

Let

$$A = \{x \in [0,\infty) : \sup_{n\geq 0} f_n(x) < \bar{x}\}.$$

We show that $A \neq \emptyset$. Indeed, let θ be a positive number such that

$$\theta < \min\left\{\bar{x}, \ \left(\frac{\beta}{2\gamma} - \frac{1}{2\gamma}\sqrt{\beta^2 - 4\gamma}\right)^{\frac{1}{p-1}}\right\}.$$

We have

$$f_1(\theta) = \frac{\beta\theta^p}{1+\gamma\theta^{2p-2}}.$$

One can easily show that $f_1(\theta) < f_0(\theta) = \theta < \bar{x}$. By Corollary 4.1 (3), $f_{n+1}(\theta) < f_n(\theta)$ $\forall n \geq 0$. This implies that $\sup_{n\geq 0} f_n(\theta) = f_0(\theta) < \bar{x}$.

We define the function S by

$$S(x) = \sup_{n\geq 0} f_n(x).$$

We claim that S is continuous on A and A is open. Fix $x \in A$. There exists $N \geq 0$ such that

$$f_0(x) \leq f_1(x) \leq \ldots \leq f_N(x) < \bar{x} \quad \text{and} \quad f_{N+1}(x) < f_N(x).$$

If this were not true, then

$$f_0(x) \leq f_1(x) \leq \ldots \leq S(x) < \bar{x},$$

whence $f_n(x) \to S(x) = \bar{x}$ which is a contradiction. This implies that

$$S(x) = f_N(x) \quad \text{and} \quad f_{N+1}(x) < f_N(x).$$

Let $\epsilon > 0$ be such that $\epsilon < \min\{\bar{x} - f_N(x), (f_N(x) - f_{N+1}(x))/2\}$. From the continuity of $f_0, \ldots, f_{N+1}$, there exists $\delta > 0$ such that for $x' \in A$ we have

$$|x - x'| < \delta \Rightarrow \sup_{0\leq n\leq N+1} |f_n(x) - f_n(x')| < \epsilon.$$

Since $f_{N+1}(x') < f_{N+1}(x) + \epsilon < f_N(x) - \epsilon < f_N(x') < f_N(x) + \epsilon < \bar{x}$, then

$$S(x) - \epsilon = f_N(x) - \epsilon < f_N(x') \leq S(x'),$$

and

$$S(x') = \sup_{0\leq n\leq N} f_n(x') < \sup_{0\leq n\leq N} (f_n(x) + \epsilon)$$
$$= f_N(x) + \epsilon < f_N(x) + \bar{x} - f_N(x) = \bar{x}.$$

Therefore, S is continuous and A is open. Set $\lambda = \sup A$. Then $\lambda \notin A$ whence $S(\lambda) \geq \bar{x}$. The continuity of f_m for every $m \geq 0$ implies that $S(\lambda) \leq \bar{x}$. Hence $S(\lambda) = \bar{x}$. Now, we claim that $f_0(\lambda) < f_1(\lambda) < \ldots < \bar{x}$. Indeed, we can see that $f_1(\lambda) > f_0(\lambda)$. If not, then $f_0(\lambda) \geq f_1(\lambda) \geq f_2(\lambda) \ldots$, because of corollary 4.1. Hence $S(\lambda) = f_0(\lambda) = \lambda = \bar{x}$ whence

$$f_1(\lambda) = f_1(\bar{x}) = \frac{\beta\bar{x}^p}{1+\gamma\bar{x}^{2p-2}} = \frac{\beta(\bar{x})^p}{1+\gamma(\bar{x})^{p-1}} \frac{1+\gamma(\bar{x})^{p-1}}{1+\gamma\bar{x}^{2p-2}} > \bar{x}.$$

Note that $\bar{x} < 1$. Now assume that $f_0(\lambda) < f_1(\lambda) < \ldots < f_N(\lambda)$ and $f_N(\lambda) \geq f_{N+1}(\lambda)$ for some $N \geq 1$. Then $S(\lambda) = f_N(\lambda) = \bar{x}$ whence

$$f_{N+1}(\lambda) = \frac{\beta f_N^p(\lambda)}{1+\gamma f_{N-1}^{p-1}(\lambda)} > \frac{\beta f_N^p(\lambda)}{1+\gamma f_N^{p-1}(\lambda)}$$
$$= \frac{\beta \bar{x}^p}{1+\gamma \bar{x}^{p-1}(\lambda)} = \bar{x},$$

which is a contradiction. Therefore, $f_n(\lambda)$ is increasing to $\bar{x}$.

Next, we define the functions f_{-1} and f_0 on $[0,\infty)$ by

$$f_{-1}(x) = x \quad , \quad f_0(x) = x^2$$

and

$$f_{n+1} = \frac{\beta f_n^p}{1+\gamma f_{n-1}^{p-1}}, \quad n = 0, 1, \ldots.$$

We denote by

$$A = \{x \in [0,\infty) : \inf_{n\geq 0} f_n(x) > \bar{x}\}.$$

We can see that $A \neq \emptyset$. Indeed, let θ be such that

$$\theta > \max\left\{\sqrt{\bar{x}}\ ,\ (\frac{\gamma}{2\beta} + \frac{1}{2\beta}\sqrt{\gamma^2+4\beta})^{\frac{1}{p-1}}\right\}.$$

We have

$$f_1(\theta) = \frac{\beta f_0^p(\theta)}{1+\gamma f_{-1}^{p-1}(\theta)} = \frac{\beta \theta^{2p}}{1+\gamma\theta^{p-1}}.$$

Set $a = \theta^{p-1}$. Then $a > (\gamma + \sqrt{\gamma^2+4\beta})/2\beta$ whence $\beta a^2 > 1 + a\gamma$ i. e. $(\beta\theta^{2p-2})/(1+\gamma\theta^{p-1}) > 1$. Hence $f_1(\theta) > f_0(\theta) = \theta^2 > \bar{x}$. This implies that $\inf_{n\geq 0} f_n(\theta) = f_0(\theta) > \bar{x}$.

Define the function S by

$$S(x) = \inf_{n\geq 0} f_n(x).$$

We show that S is continuous on A and A is open. In fact, fix $x \in A$, there exists a natural number N such that

$$f_0(x) \geq f_1(x) \geq \ldots \geq f_N(x) > \bar{x} \text{ and } f_{N+1}(x) > f_N(x)$$

Otherwise,

$$f_0(x) \geq \ldots \geq f_n(x) \geq \ldots \geq S(x) > \bar{x}$$

and therefore

$$\lim_{n\to\infty} f_n(x) \geq S(x) > \bar{x}$$

which is a contradiction. Hence

$$S(x) = f_N(x) \text{ and } f_{N+1}(x) > f_N(x).$$

Choose

$$0 < \epsilon < \min\left\{f_N(x) - \bar{x}, \frac{f_{N+1}(x) - f_N(x)}{2}\right\}.$$

From the continuity of f_n, there exists $\delta > 0$ such that

$$\forall x' \in [0,\infty) \ \left(|x - x'| < \delta \Rightarrow |f_n(x) - f_n(x')| < \epsilon\right),$$

where $n = 0, 1, \ldots, N+1$. Hence for $x' \in (x - \delta, x + \delta) \cap [0, \infty)$ we have

$$f_{N+1}(x') > f_{N+1}(x) - \epsilon > f_N(x) + \epsilon > f_N(x') > f_N(x) - \epsilon > \bar{x}$$

Therefore

$$S(x) + \epsilon = f_N(x) + \epsilon > f_N(x') \geq \inf_{n \geq 0} f_n(x') = S(x').$$

Also

$$S(x) \leq f_n(x) < f_n(x') + \epsilon \qquad 0 \leq n \leq N.$$

Hence

$$S(x') + \epsilon > S(x)$$

and

$$S(x) < S(x') + f_N(x) - \bar{x}.$$

This implies that $S(x') > \bar{x}$ and

$$|S(x) - S(x')| < \epsilon,$$

i.e., S is continuous and A is open.

Let $\lambda = \inf A$. Then $\lambda \notin A$. The continuity of f_n for every n implies that $S(\lambda) = \bar{x}$. Now, we show that $\{f_n(\lambda)\}_{n \geq 0}$ is decreasing to $\bar{x}$. We can see that $f_1(\lambda) < f_0(\lambda)$. Assume for the sake of contradiction that $f_0(\lambda) \leq f_1(\lambda)$. Then $\bar{x} \leq f_0(\lambda) \leq f_1(\lambda) \leq \ldots$ whence $S(\lambda) = f_0(\lambda) = \lambda^2 = \bar{x}$. Hence

$$f_1(\lambda) = \frac{\beta \lambda^{2p}}{1 + \gamma \lambda^{p-1}} = \frac{\beta \bar{x}^p}{1 + \gamma \bar{x}^{\frac{p-1}{2}}} = \bar{x} \frac{1 + \gamma \bar{x}^{p-1}}{1 + \gamma \bar{x}^{\frac{p-1}{2}}} < \bar{x},$$

which is a contradiction. By induction we can show that

$$f_0(\lambda) > f_1(\lambda) > \ldots > \bar{x}.$$

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