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FRACTAL MULTIWAVELETS RELATED TO THE CANTOR DYADIC GROUP

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ABSTRACT. Orthogonal wavelets on the Cantor dyadic group are identified with multiwavelets on the real line consisting of piecewise fractal functions. A tree algorithm for analysis using these wavelets is described. Multiwavelet systems with algorithms of similar structure include certain orthogonal compactly supported multiwavelets in the linear double-knot spline space $S^{1,2}$.

KEYWORDS. Wavelets, multiwavelets, fractal functions, Cantor dyadic group, splines
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1. INTRODUCTION.

To study the construction of wavelets and to study abstract harmonic analysis, we consider orthogonal wavelets on the locally compact Cantor dyadic group. In Lang [11], compactly supported wavelets are constructed on this group; the construction proceeds similar to that of Meyer [16], Mallat [14] and Daubechies [2], via scaling filters. (See Holschneider [10] for general information about wavelets on locally compact groups; other constructions of wavelets on groups include Dahlke [1] and Lemarie [13].) The Cantor dyadic group may be identified with the nonnegative real numbers as a measure space; harmonic analysis on the Cantor dyadic group corresponds to analysis using Walsh functions on the line. The wavelets constructed on the Cantor dyadic group turn out to be certain lacunary Walsh series on the line.

Here, we will continue study of these wavelets; we will consider these wavelets as wavelets on the real line. We will describe the form that the natural Mallat tree algorithm for these wavelets takes when used to analyze functions on the line. From the structure of the algorithm, we find that the Cantor dyadic group wavelets may be identified as multiwavelets on the line. In fact, they are multiwavelets consisting of piecewise fractal functions, in the sense of Massopust [15]. It is possible to develop their properties without reference to the Cantor dyadic group.

Other wavelet systems with a tree algorithm with the same structure include certain compactly supported orthogonal multiwavelets in the linear double-knot spline space $S^{1,2}$ described in section 7 below; approximations with these multiwavelets take the form of piecewise linear, not necessarily continuous functions.

2. THE CANTOR DYADIC GROUP.

We describe the locally compact Cantor dyadic group. This group, also known as the 2-series local field, consists of the countably infinite weak direct product of the group of integers modulo

2. We write

$$G = \prod_{i=-\infty}^{\infty}{}^{*} Z/(2) \text{ where } Z/(2) = \{0, 1\}.$$

Thus if $x \in G$ then $x = (x_i)_{i=-\infty}^{\infty}$ where $x_i \in \{0, 1\}$ and where $x_i \neq 0$ for only finitely many $i \geq 0$. So we may identify x with the real number $\sum_{i=-\infty}^{\infty} x_i 2^i$. The Cantor dyadic group is thus identified with the nonnegative real numbers as a measure space (but not algebraically or topologically).

Translation on the Cantor dyadic group is as usual; we will write $T_y(x) = x + y$ for $x, y \in G$. We consider a simple example. Let $y = (y_i)$ where $y_0 = 1$ and $y_i = 0$ for $i \neq 0$. So y corresponds to the real number 1. Translation by this number corresponds to the function

$$T_1(x) = \begin{cases} x+1 & \text{if } 2k \leq x < 2k+1 \text{ for some integer } k \geq 0 \\ x-1 & \text{otherwise} \end{cases} \tag{2.1}$$

when G is identified with $[0, \infty)$.

Dilation on the group G is given by $\rho(x)_i = x_{i-1}$. This corresponds exactly to the map $\rho(x) = 2x$ when G is identified with $[0, \infty)$. We let $\rho^k(x) = 2^k x$ for $k \in Z$.

The group characters of the Cantor dyadic group become Walsh functions when the group is identified with the nonnegative reals. We describe the Walsh functions on the real line. First we define the Rademacher functions r_j for $j \in Z$. We let $r_0(x) = 1$ if $2k \leq x < 2k+1$ for some integer k, and $r_0(x) = -1$ otherwise. We then define $r_j(x) = r_0(2^j x)$. Now consider a real number y. We write $y = \sum y_i 2^i$ where $y_i \in \{0, 1\}$. We then define the Walsh function W_y by $W_y(x) = \prod_{i \in Z} (r_i(x))^{y_i}$; for each x there is only finitely many terms in the product different than 1. (This is the Paley denumeration of the Walsh functions.) For example, $W_3(x) = r_0(x) r_1(x) = 1$ if $x \in [k - 1/4, k + 1/4)$ for some integer k and -1 otherwise. For a function f on G (or a function on the line), we may define the Walsh transform $\hat{f}(y) = \int f(x) W_y(x)\, dx$; this is the natural analogue of the Fourier transform for the group G, but we will make no further reference to this transform here.

See Taibleson [23], Edwards [4] and Hewitt [9] for more information about the harmonic analysis on the Cantor dyadic group. Also see Golubov et al. [5] and Schipp et al. [20] concerning Walsh series and transforms.

3. WAVELETS ON THE CANTOR DYADIC GROUP.

First we describe multiresolution analyses on the Cantor dyadic group. Let Λ be the subgroup of the Cantor dyadic group corresponding to the nonnegative integers. We say that a sequence (V_j) of closed subspaces of $L^2(G)$ is a multiresolution analysis if: $V_j \subset V_{j+1}$ for all $j \in Z$; $f \in V_0 \iff f \circ T_n \in V_0$ for all $n \in \Lambda$; $f \in V_j \iff f \circ \rho \in V_{j+1}$ for all $j \in Z$; $\cap V_j = \{0\}$ and $\cup V_j$ is dense in $L^2(G)$; and there is $f \in V_0$ whose translates by Λ form a Riesz basis of V_0.

We may then construct compactly supported, orthogonal wavelets on the Cantor dyadic group. This may be done by following the method of Meyer, Mallat and Daubechies, using conditions on scaling filters. We omit the details of this construction; see Lang [11] and [12]. If we consider length-4 scaling filters (consisting of trigonometric polynomials of four Walsh functions), we obtain the following wavelets. Let $0 < a \leq 1$ and $a^2 + b^2 = 1$. Let $\phi(x) = f(x/2)$ where

$$f = \tfrac{1}{2} 1_{[0,1)} \left(1 + aW_1 + abW_3 + ab^2 W_7 + ab^3 W_{15} + \cdots\right). \tag{3.1}$$

(Here $1_{[0,1)}$ is the indicator function of $[0, 1)$; it is 1 if x is in that set and 0 otherwise.) Then ϕ is continuous (in the sense of G) and compactly supported, and the translate of ϕ by Λ are orthogonal. Also if V_0 is the space spanned by the translates of ϕ, then V_0 forms a multiresolution analysis as above. The corresponding mother wavelet is the function

$$\psi(x) = 2a_0\phi(T_1(2x)) - 2a_1\phi(2x) + 2a_2\phi(T_3(2x)) - 2a_3\phi(T_2(2x)) \tag{3.2}$$

where $a_0 = (1+a+b)/4$, $a_1 = (1+a-b)/4$, $a_2 = (1-a-b)/4$ and $a_3 = (1-a+b)/4$. The translates of ψ by Λ span a space W_0 where $V_1 = V_0 \oplus W_0$; the translates and dilates of ψ form an orthogonal basis of $L^2(G)$ in the usual way. (See Lang [11] for details.)

If we consider scaling filters of length two, we obtain the familiar Haar wavelets. In Lang [12], length-8 wavelets are detailed; some of these also take the form of lacunary Walsh series.

4. THE ALGORITHM FOR WAVELETS ON THE CANTOR DYADIC GROUP.

Here we detail the Mallat tree-type algorithm for the length-4 wavelets on the Cantor dyadic group. Let a_0, a_1, a_2 and a_3 be as in the previous section and let $b_0 = -a_1$, $b_1 = a_0$, $b_2 = -a_3$ and $b_3 = a_2$. For f a function on G, $j \in Z$, and $k \in \Lambda$, let $c_k^j = \int_G f(x)\phi(\rho^j(x) - k)2^{j/2}\,dx$ and $d_k^j = \int_G f(x)\psi(\rho^j(x) - k)2^{j/2}\,dx$. Then the reconstruction algorithm is

$$c_k^{j+1} = 2^{1/2}\left(\sum_{n\in\Lambda} c_n^j a_{k-\rho(n)} + \sum_{n\in\Lambda} d_n^j b_{k-\rho(n)}\right) \tag{4.1}$$

and the decomposition algorithm is

$$c_k^j = 2^{1/2}\sum_{n\in\Lambda} a_{n-\rho(k)}c_n^{j+1} \text{ and } d_k^j = 2^{1/2}\sum_{n\in\Lambda} b_{n-\rho(k)}c_n^{j+1}. \tag{4.2}$$

Note here that the subscripts of the coefficients are treated as members of the group G. When we identify G as $[0,\infty)$, the algorithm takes the following form:

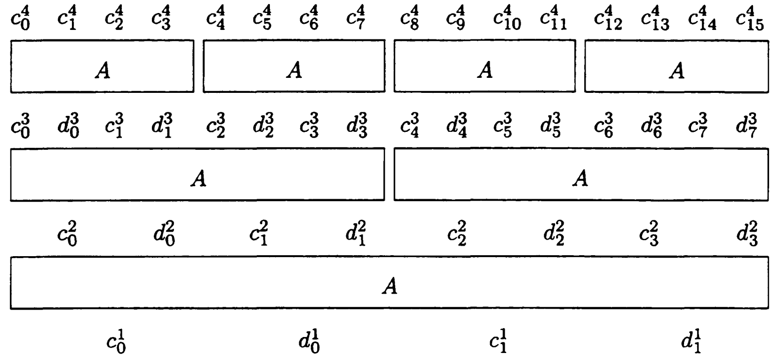
$$\begin{bmatrix} c_{2k}^j \\ d_{2k}^j \\ c_{2k+1}^j \\ d_{2k+1}^j \end{bmatrix} = A \begin{bmatrix} c_{4k}^{j+1} \\ c_{4k+1}^{j+1} \\ c_{4k+2}^{j+1} \\ c_{4k+3}^{j+1} \end{bmatrix}. \tag{4.3}$$

where

$$A = 2^{1/2}\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ a_2 & a_3 & a_0 & a_1 \\ b_2 & b_3 & b_0 & b_1 \end{bmatrix}. \tag{4.4}$$

The decomposition algorithm produces the lower level (smaller j) coefficients from higher level coefficients; the reconstruction algorithm produces higher level coefficients c_k^j from the coefficients d_k^j and lower level c_k^j – multiply both sides of (4.3) by A^{-1}.

The following diagram shows the structure of the algorithm. Each rectangle represents multiplication of the four coefficients c_k^j above it by A to obtain the four coefficients below it. The algorithm has a 'matrix filter' structure reminiscent of Strang and Strela [21], and hence we are led to consider our wavelets as multiwavelets on the line.



5. MULTIWAVELETS ON THE LINE.

With ordinary wavelets, there is a single scaling function ϕ whose translates span a space V_0 which generates a multiresolution analysis. The condition (e.g.) $V_{-1} \subset V_0$ requires that $\phi(x/2) = \sum_{k\in Z} a_k \phi(x-k)$ for some coefficients a_k. In the case of multiwavelets, we would have several scaling functions $\phi_1, \ldots, \phi_n$ whose translates by integers span a space V_0, the dilates of which form a multiresolution analysis. We would write $\vec{\phi} = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix}$. Then the condition $V_{-1} \subset V_0$ becomes $\vec{\phi}(x/2) = \sum_{k\in Z} P_k \vec{\phi}(x-k)$ where the coefficients P_k are now $n \times n$ matrices. See Goodman and Lee [6], Goodman et al. [7], Plonka and Strela [19], and Strang and Strela [21] for more information on multiwavelets.

The Cantor dyadic group wavelets of section 3 can be interpreted as multiwavelets on the real line. Let ϕ and ψ be as in (3.1) and (3.2). Let $\phi_1 = \phi$, $\phi_2 = \phi \circ T_1$, $\psi_1 = \psi$ and $\psi_2 = \psi \circ T_1$, where T_1 is as in (2.1). Define ϕ_k^j by $\phi_{2k}^j(x) = \phi_1(2^j x - 2k)2^{j/2}$ and $\phi_{2k+1}^j(x) = \phi_2(2^j x - 2k)2^{j/2}$, and define ψ_k^j similarly. Suppose f is a function on the real line and let $c_k^j = \int f(x)\phi_k^j(x)\,dx$ and $d_k^j = \int f(x)\psi_k^j(x)\,dx$ for $j,k \in Z$. Then the coefficients are related by (4.3). This follows since every translate of ϕ on G by a member of Λ is, as a function on the line, an (ordinary) translate of either ϕ_1 or ϕ_2.

Let $\vec{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$ and $\vec{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$. Write A of (4.4) as $A = [a_{ij}]$. Then:

THEOREM 5.1. We have $\vec{\phi}(x/2) = P_0\vec{\phi}(x) + P_2\vec{\phi}(x-2)$ and $\vec{\psi}(x/2) = Q_0\vec{\phi}(x) + Q_2\vec{\phi}(x-2)$ where

$$P_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}, P_2 = \begin{pmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{pmatrix}, Q_0 = \begin{pmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{pmatrix}, Q_2 = \begin{pmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{pmatrix}.$$

6. FRACTAL FUNCTIONS ON THE LINE.

In the previous section, we identified the Cantor dyadic group wavelets of section 3 with multiwavelets on the line. In the present section, we will show that these are piecewise fractal functions in the sense of Massopust [15] p. 137 and p. 258.

We begin by defining fractal functions. (This definition is actually a specialization of the general definition in Massopust [15].) Consider the Read-Bajraktarević operator for real-valued f on $[0,1]$:

$$\Phi f(x) = \begin{cases} \lambda + sf(2x) & \text{if } 0 \le x < 1/2 \\ \mu + tf(2x-1) & \text{if } 1/2 \le x \le 1 \end{cases} \tag{6.1}$$

where λ, μ, s, t are fixed real numbers, with $|s| < 1$ and $|t| < 1$. The domain and range of this operator is $L^\infty([0,1])$; it may be shown (Proposition 6.3 below) that there is a unique fixed point f

for this operator in that space. We call f a fractal function, motivated by the self-similiarity of the graph of f. (The fixed point f of Φ obeys $f(x) = \lambda + sf(2x)$ on $[0, 1/2)$ and $f(x) = \mu + tf(2x-1)$ on $[1/2, 1]$, so the graph of f restricted to $[0, 1/2)$ and the graph of f restricted to $[1/2, 1]$ are each affine linear images of the graph of f.) We note that Read-Bajraktarević operators serve as a framework for studying functions with fractal graphs in terms of iterated function systems; see Massopust [15].

It is possible to write the fixed point f explicitly as a series; we consider the case when $s \neq t$. Let f_0 be the function constantly 1 on $[0, 1]$ and let $f_n = \Psi f_{n-1}$ for $n \geq 1$, where

$$\Psi f(x) = \begin{cases} sf(2x) & \text{if } 0 \leq x < 1/2 \\ tf(2x-1) & \text{if } 1/2 \leq x \leq 1 \end{cases}. \tag{6.2}$$

So each f_n is piecewise constant and f_n is bounded by $\max\{|s|^n, |t|^n\}$. We have by the linearity of Ψ,

PROPOSITION 6.3. The fixed point of (6.1) is the function $f = a + bf_1 + bf_2 + bf_3 + bf_4 + \cdots$, where $a(1-s) + bs = \lambda$ and $a(1-t) + bt = \mu$, provided $s \neq t$.

We now describe integrals involving fractal functions. This is similar to Massopust [15], sections 5.6.1 and 5.6.2.

LEMMA 6.4. Suppose f is the fixed point of the operator (6.1). Then $\int_0^1 f(x)\,dx = (\lambda + \mu)/(2 - s - t)$.

PROOF. We have

$$\begin{aligned} \int_0^1 f(x)\,dx &= \int_0^{1/2} (\lambda + sf(2x))\,dx + \int_{1/2}^1 (\mu + tf(2x-1))\,dx \\ &= \tfrac{1}{2}\left(\lambda + s\int_0^1 f(x)\,dx\right) + \tfrac{1}{2}\left(\mu + t\int_0^1 f(x)\,dx\right). \end{aligned}$$

Now suppose f is a fixed point of the operator

$$\Phi_1 f(x) = \begin{cases} \lambda_1 + sf(2x) & \text{if } 0 \leq x < 1/2 \\ \mu_1 + tf(2x-1) & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

and g is a fixed point of the operator

$$\Phi_2 g(x) = \begin{cases} \lambda_2 + sg(2x) & \text{if } 0 \leq x < 1/2 \\ \mu_2 + tg(2x-1) & \text{if } 1/2 \leq x \leq 1 \end{cases}.$$

Then

LEMMA 6.5. We have

$$\int_0^1 f(x)g(x)\,dx = \left(\frac{1}{2 - s^2 - t^2}\right)\left[\lambda_1\lambda_2 + \mu_1\mu_2 + (s\lambda_2 + t\mu_2)\left(\frac{\lambda_1 + \mu_1}{2 - s - t}\right)(s\lambda_1 + t\mu_1)\left(\frac{\lambda_2 + \mu_2}{2 - s - t}\right)\right].$$

The proof of this uses the previous lemma and follows the technique of the proof of the previous lemma.

The next lemma says that the two 'halves' of a fractal function are fractal functions.

LEMMA 6.6. Suppose f is the fixed point of the operator (6.1). Let $f_1(x) = f(x/2)$ and $f_2(x) = f((x+1)/2)$ for $0 \leq x \leq 1$. Then f_1 is the fixed point of the operator

$$\Phi_1 f_1(x) = \begin{cases} \lambda + sf_1(2x) & \text{if } 0 \leq x < 1/2 \\ (1-t)\lambda + s\mu + tf_1(2x-1) & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

and f_2 is the fixed point of the operator

$$\Phi_2 f_2(x) = \begin{cases} (1-s)\mu + t\lambda + s f_2(2x) & \text{if } 0 \le x < 1/2 \\ \mu + t f_2(2x-1) & \text{if } 1/2 \le x \le 1 \end{cases}.$$

PROOF. Let Ψ be the operator (6.2). Now $f_1(x) = f(x/2) = \lambda + sf(x)$ for $0 \le x < 1$ Applying Ψ to this equation and solving for f_1 gives the first result; the second result is similar.

We are now ready to describe the Cantor dyadic group wavelets of section 3 as piecewise fractal functions. Let ϕ be as in (3.1) and let $\phi_1 = \phi$, $\phi_2 = \phi \circ T_1$, as in section 5.

THEOREM 6.7. If $f(x) = \phi(2x)$ then

$$f(x) = \begin{cases} \frac{1+a-b}{2} + bf(2x) & \text{if } 0 \le x < 1/2 \\ \frac{1-a+b}{2} - bf(2x-1) & \text{if } 1/2 \le x \le 1 \end{cases}.$$

Furthermore $\phi_1(x) = \begin{cases} f_1(x) & \text{if } 0 \le x < 1 \\ f_2(x-1) & \text{if } 1 \le x \le 2 \end{cases}$ and $\phi_2(x) = \begin{cases} f_2(x) & \text{if } 0 \le x < 1 \\ f_1(x-1) & \text{if } 1 \le x \le 2 \end{cases}$ where

$$f_1(x) = \begin{cases} \frac{1+a-b}{2} + bf_1(2x) & \text{if } 0 \le x < 1/2 \\ \frac{1+a+b}{2} - bf_1(2x-1) & \text{if } 1/2 \le x \le 1 \end{cases}$$

and

$$f_2(x) = \begin{cases} \frac{1-a-b}{2} + bf_2(2x) & \text{if } 0 \le x < 1/2 \\ \frac{1-a+b}{2} - bf_2(2x-1) & \text{if } 1/2 \le x \le 1 \end{cases}.$$

Also, f_1 and f_2 (and hence ϕ_1 and ϕ_2) are orthogonal.

PROOF. Let Ψ be the operator $\Psi f(x) = \begin{cases} f(2x) & \text{if } 0 \le x < 1/2 \\ -f(2x-1) & \text{if } 1/2 \le x \le 1 \end{cases}$. Let $g_0 \equiv 1$ on $[0,1]$ and let $g_n = \Psi g_{n-1}$ for $n \ge 1$. Let $g = \frac{1}{2}(1 + ag_1 + abg_2 + ab^2g_3 + \cdots)$. Applying Ψ to this equation, we find

$$g(x) = \begin{cases} \frac{1+a-b}{2} + bg(2x) & \text{if } 0 \le x < 1/2 \\ \frac{1-a+b}{2} - bg(2x-1) & \text{if } 1/2 \le x \le 1 \end{cases}.$$

We may show $g_n = W_{2^n-1}$ for $n \ge 1$. Consequently $g(x) = \phi(2x)$. The remaining assertions follow from lemmas 6.4, 6.5 and 6.6.

We remark that we were able to show that ϕ_1 and ϕ_2 were orthogonal, using lemma 6.5. Of course, this was already known in Lang [11], using Fourier analysis on the Cantor dyadic group. Other properties of these wavelets may be developed the techniques of this section, such as the scaling relations of theorem 5.1; but we do not pursue this here.

7. OTHER MULTIWAVELET SYSTEMS WITH SIMILAR ALGORITHMS.

The Cantor dyadic group wavelets of section 3 have an algorithm with a particular structure as described by the diagram in section 4. That structure in part reflects the arithmetic of the Cantor dyadic group. Other multiwavelet systems unrelated to the Cantor dyadic group have algorithms with the same structure. We will describe one example, composed of multiwavelets in the double-knot spline space $S^{1,2}$. This space consists of the functions, not necessarily continuous. whose restrictions to each interval $[k, k+1)$ (k an integer) is a first degree polynomial; our notation resembles de Boor [3] and Plonka [17]. (Here, the first superscript refers to the degree of the basis functions and the second superscript refers to the decrement for regularity; thus the splines belong to C^{-1}, meaning no continuity is required. We may describe this space as a linear spline space where pairs of 'knots' coincide. See de Boor [3].) The multiwavelets will be compactly supported, piecewise linear and orthogonal; they fit into the general treatment of Plonka [17] and Plonka [18].

Let $\phi_1 = 1_{[0,1)}$ and let $\phi_2(x) = \begin{cases} 2x-1 \text{ on } [0,1) \\ 0 \text{ otherwise} \end{cases}$. Also let

$$\psi_1(x) = \begin{cases} 1-6x & \text{on } [0,1/2) \\ 5-6x & \text{on } [1/2,1) \\ 0 & \text{otherwise} \end{cases} \text{ and } \psi_1(x) = \begin{cases} 14x-3 & \text{on } [0,1/2) \\ 1-2x & \text{on } [1/2,1) \\ 0 & \text{otherwise} \end{cases}.$$

Define ϕ_k^j by $\phi_{2k}^j(x) = \phi_1(2^j x - 2k)$ and $\phi_{2k+1}^j(x) = \phi_2(2^j x - 2k)$, and define ψ_k^j similarly. We find that these are related by the scaling relations in theorem 5.1, when the matrix A is replaced by the matrix

$$A = \frac{1}{2}\begin{bmatrix} 2 & 0 & 2 & 0 \\ -1 & -3 & 1 & -3 \\ -1 & 1 & 1 & 1 \\ 1 & 7 & -1 & -1 \end{bmatrix}. \tag{7.1}$$

Let V_0 be the space spanned by the integer translates of ϕ_1 and ϕ_2. Then $V_0 = S^{1,2}$. Let V_j be the span of $\{\phi_k^j : k \in Z\}$, so $V_j = \{f(2^j \cdot) : f \in V_0\}$. Since ϕ_1 and ϕ_2 are orthogonal, $\{\phi_k^j : k \in Z\}$ forms an orthogonal basis for V_j. We define level j approximation to be the projection of a function onto V_j, i.e., $P^j f = \sum_k c_k^j \phi_k^j$ where c_k^j is obtained as in section 4 above by integrating f against (normalized) ϕ_k^j. These approximations take the following form: over each interval $I = [k2^{-j}, (k+1)2^{-j})$, $P^j f$ is the least squares best fit line to f. That is, $P^j f$ is given by $mx + b$ where m and b are chosen to minimize $\int_I |f(x) - mx - b|^2\,dx$. (This follows since P^j restricted to functions on I is orthogonal projection onto the subspace V_j restricted to I, which is just the space of degree-one polynomials on I.)

We are then able to show the following: Suppose the first and second derivatives of f are bounded in absolute value by a on $[0,1]$. Then

$$||P^j f - f||_\infty < (4a/3)(2^{-j})^2, \tag{7.2}$$

where the supremum norm is taken over $[0,1]$. (This follows from the elementary result that $||f - mx - b||_\infty < a/2$ if $|f''(x)| < a$ on $[0,1]$, where m, b are chosen so that $\int_{[0,1]} |f(x) - mx - b|^2\,dx$ is least.) This compares with the estimate $||Q^j f - f||_\infty < a2^{-j}$, where $Q^j f$ is the ordinary Haar approximation (i.e., $Q^j f$ is constantly equal to the average value of f on each appropriate dyadic interval). Thus $P^j f$ is in this sense a good approximation to f even though it is not necessarily continuous.

If we define W_0 to be the span of the integer translates of ψ_1 and ψ_2, we find that $V_1 = W_0 \oplus V_0$. (This follows from the orthogonality of ψ_1, ψ_2 and ϕ_1, ϕ_2.) This, the scaling relations (5.1) with the matrix (7.1), and the estimate (7.2), imply that $\{\psi_k^j\}$ is an orthogonal basis of $L^2(R)$.

The algorithm of these multiwavelets compares in speed and complexity with the ordinary Haar algorithm (note that the matrices A and A^{-1} have entries that are integers divided by 4). This with the approximation result (7.2) above suggests the utility of these multiwavelets for applications such as image compression.

For more information about multiresolution analyses and multiscale relations on spline spaces with higher-order defects, or spline spaces with multiple knots, see Plonka [17] and [18].

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