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SOME FIXED POINT THEOREMS FOR NONCONVEX SPACES

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ABSTRACT. We give a theorem for nonconvex topological vector spaces which yields the classical fixed point theorems of Ky Fan, Kim, Kaczynski, Kelly and Namioka as immediate consequences, and prove a new fixed point theorem for set-valued maps on arbitrary topological vector spaces.

KEY WORDS AND PHRASES: Fixed points, nonconvex topological vector spaces, multifunctions.

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1. INTRODUCTION.

In 1935, Tychonoff proved the following celebrated result.

THEOREM 1.1. *If X is a nonempty convex and compact subset of a locally convex topological vector space $\mathbb{E}$, then any continuous map $f : X \to X$ has a fixed point.*

Even though Theorem 1.1 has been the subject of extensive and sharp generalizations, the question of whether Theorem 1.1 is true in general topological vector spaces still remains open. Recently, the authors encountered papers that extend Theorem 1.1 to topological vector spaces $\mathbb{E}$ having a separating dual $\mathbb{E}^\star$. The purpose of this paper is to show that this assumption implies local convexity of X and consequently the results follow from known results.

Let $\mathbb{F}$ be the scalar fields $\mathbb{R}$ or $\mathbb{C}$ and $\mathbb{E}$ be a topological vector space over $\mathbb{F}$ with dual $\mathbb{E}^\star$ (possibly $\mathbb{E}^\star = \{0\}$). Let τ and τ_w denote the original and weak topologies of $\mathbb{E}$. Note that τ_w is locally convex and if $\mathbb{E}^\star$ separates points of $\mathbb{E}$, then τ_w is Hausdorff. For a subset $X \subseteq \mathbb{E}$, (X, τ_w) (resp. (X, τ)) denotes X with the relative topology τ_w (resp. τ).

2. SOMETHING OLD, SOMETHING NEW.

In what follows X compact means that X is compact in the topology of $\mathbb{E}$.

PROPOSITION 2.1. *Let X be a nonempty compact subset of $\mathbb{E}$. If (X, τ_w) is Hausdorff, then $(X, \tau_w) = (X, \tau)$.*

PROOF. By definition, $\tau_w \subseteq \tau$. Conversely, let A be a τ-closed subset of X. Then A is τ-compact, and hence τ_w-compact. Since τ_w is Hausdorff, it follows that A is τ_w-closed. Thus $(X, \tau) \subseteq (X, \tau_w)$ and hence $(X, \tau) = (X, \tau_w)$. □

Note that if $\mathbb{E}^\star$ separates the points of X, then (X, τ_w) is Hausdorff.

Proposition 2.1 provides simple proofs of the following results. Original proofs of these results make use of partitions of unity or other techniques.

COROLLARY 2.2 (Ky Fan [2]). *Let X be a nonempty compact and convex subset of $\mathbb{E}$. If $\mathbb{E}^*$ separates the points of X, then every continuous map $f : X \to X$ has a fixed point.*

PROOF. Since $\mathbb{E}^*$ separates the points of X, (X, τ_w) is Hausdorff, hence by Proposition 2.1, $(X, \tau) = (X, \tau_w)$, and thus $f : (X, \tau_w) \to (X, \tau_w)$ is continuous. Since X is τ_w-compact, by Theorem 1.1, f has a fixed point. □

The following result is a generalization of Corollary 2.2.

COROLLARY 2.3. *Let X be a nonempty compact and convex subset of $\mathbb{E}$ and $f : X \to \mathbb{E}$ be a continuous function. If $\mathbb{E}^*$ separates the points of X, then either* (a) *f has a fixed point, or* (b) *there exists $x_0 \in X$ and a τ-continuous seminorm p on $\mathbb{E}$ such that $0 < p(x_0 - f(x_0)) = \min\{p(x - f(x_0)) : x \in X\}$.*

PROOF. Since (X, τ_w) is Hausdorff, by Proposition 2.1 it follows that $(X, \tau) = (X, \tau_w)$. Since $\tau_w \subseteq \tau$, $f : (X, \tau_w) \to (\mathbb{E}, \tau_w)$ is continuous. Since X is τ_w compact, it follows by Ky Fan [2] that either (a) f has a fixed point, or (b) there exists $x_0 \in X$ and a τ_w-continuous seminorm p on $\mathbb{E}$ with $0 < p(x_0 - f(x_0)) = \min\{p(x - f(x_0)) : x \in X\}$. Note that a τ_w is continuous seminorm on $\mathbb{E}$ is a τ-continuous seminorm on $\mathbb{E}$. □

As an immediate consequence of the above corollary, we have

COROLLARY 2.4 (Kaczynski [4]). *Let X be a nonempty compact convex subset of $\mathbb{E}$ and $\mathbb{E}^*$ separate the points of X. If $f : X \to \mathbb{E}$ is a continuous function such that for each $x \in X$, $f(x) \neq x$, there exist λ such that $|\lambda| < 1$ and*

$$\lambda x + (1 - \lambda) f(x) \in X, \tag{1}$$

then f has a fixed point.

PROOF. Assume that f has no fixed points. Then by Corollary 2.3, there exists $x_0 \in X$ and a τ-continuous seminorm p on $\mathbb{E}$ satisfying $0 < p(x_0 - f(x_0)) = \min\{p(x - f(x_0)) : x \in X\}$. By assumption, there exists λ with $|\lambda| < 1$ such that $u = \lambda x + (1 - \lambda) f(x) \in X$. This implies that

$$0 < p(x_0 - f(x_0)) \leq p(u - f(x_0)) = |\lambda| p(x_0 - f(x_0)) < p(x_0 - f(x_0)),$$

a contradiction. Hence f has a fixed point. □

DEFINITION 2.5. Let $X \subseteq \mathbb{E}$. A mapping $f : X \to \mathbb{E}$ is weakly continuous if for every $x^* \in \mathbb{E}^*$, $x^*(f) : (X, \tau_w) \to \mathbb{F}$ is continuous.

PROPOSITION 2.6. *If $f : X \to \mathbb{E}$ is weakly continuous, then $f : (X, \tau_w) \to (\mathbb{E}, \tau_w)$ is continuous.*

PROOF. For $\varepsilon > 0$ let $N(0, \varepsilon)$ denote the open neighborhood of 0 of radius ε in $\mathbb{F}$. If V is a τ_w-basic neighborhood of 0 in $\mathbb{E}$, then $V = \bigcap_{i=1}^n (x_i^*)^{-1}(N(0, \varepsilon_i))$ for some $x_1^*, \cdots, x_n^* \in \mathbb{E}^*$, and $\varepsilon_1, \cdots, \varepsilon_n > 0$. Hence $f^{-1}(V) = \bigcap_{i=1}^n f^{-1}(x_i^*)^{-1}(N(0, \varepsilon_i)) = \bigcap_{i=1}^n (x_i^*(f))^{-1}(N(0, \varepsilon_i)) \in \tau_w$. Thus $f : (X, \tau_w) \to (\mathbb{E}, \tau_w)$ is continuous. □

Let $X \subseteq \mathbb{E}$ and $x \in X$. The *inward set* of X is defined to be

$$I_X(x) = \{z \in \mathbb{E} : z = x + \lambda(y - x) : y \in X, \lambda > 0\}$$

COROLLARY 2.7 (Kim [6]; also see Singh [7], Theorem 4.53, p. 206). *Let X be a nonempty compact convex subset of $\mathbb{E}$ and $\mathbb{E}^*$ separate the points of X. If $f : X \to \mathbb{E}$ is weakly continuous such that for each $x \in X$ with $f(x) \neq x$, $f(x) \in \tau -$ closure$(I_X(x))$, then f has a fixed point.*

PROOF. By Proposition 2.1, (X, τ_w) is a compact convex subset of the locally convex space $(\mathbb{E}, \tau_w)$ and by Proposition 2.6 $f : (X, \tau_w) \to (\mathbb{E}, \tau_w)$ is continuous. Since the $\tau -$closure$(I_X(x)) \subseteq \tau_w-$closure$(I_X(x))$, it follows that for any $x \neq f(x)$, $f(x) \in \tau_w-$closure$(I_X(x))$. The result now follows from Halpren [3]. □

COROLLARY 2.8 (Kelly-Namioka [5, p. 124]). *Let X be a compact convex subset of $\mathbb{E}$. If for each nonzero $x \in X - X$, there exists $x^* \in \mathbb{E}^*$ with $x^*(x) \neq 0$ and $f : X \to X$ is a continuous mapping satisfying*

$$f(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i f(x_i) \tag{2}$$

for each positive integer n, $x_i \in X$ for $i \in \{1, 2, \cdots, n\}$, and $a_i \geq 0$ with $\sum_{i=1}^{n} a_i = 1$, then f has a fixed point.

PROOF. Since $\mathbb{E}^*$ separates the points of X, (X, τ_w) is Hausdorff, and hence $(X, \tau) = (X, \tau_w)$. Since (X, τ_w) is compact and $f : (X, \tau_w) \to (X, \tau_w)$ is continuous the result follows from Theorem 1.1. □

REMARK. Note that the condition (2) is redundant in the present proof.

The theorems of this section readily imply that the Hardy spaces H^p, $0 < p < 1$, have the fixed point property.

3. A FIXED POINT THEOREM FOR MULTIFUNCTIONS.

Let $2^{\mathbb{E}}$ denote the family of nonempty subsets of $\mathbb{E}$ and let $f : X \to 2^{\mathbb{E}}$ be a multifunction. f is upper semicontinuous if for any closed set $F \subseteq \mathbb{E}$, $f^{-1}(F) = \{x \in X : f(x) \cap F \neq \emptyset\}$ is a closed subset of X; f is closed (resp. compact) valued if $f(x)$ is closed (resp. compact) subset of $\mathbb{E}$ for each $x \in X$. It is easy to show that if f is closed-valued and if a net $x_\alpha \in X$, $x_\alpha \to x_0 \in \mathbb{E}$, and $y_\alpha \in f(x_\alpha)$ with $y_\alpha \to y_0$, then the upper semicontinuity of f implies $y_0 \in f(x_0)$. Furthermore, if $f : X \to 2^{\mathbb{E}}$ is upper semicontinuous and compact-valued, then for any compact set K, the image $f(K) = \bigcup\{f(x) : x \in K\}$ is compact. For additional properties of multifunctions see Dugundji and Granas [1] or Smithson [8], for example. The following proposition, which clearly implies Theorem 11.4 of Dugundji and Granas (see [1], p. 97), is equivalent to their theorem.

PROPOSITION 3.1. *Let $\mathbb{E}$ be a topological vector space and let X be a nonempty compact convex subset of $\mathbb{E}$. If $\mathbb{E}^*$ separates the points of X, and $f : X \to 2^X$ is a closed-valued, upper semicontinuous and convex-valued multifunction, then f has a fixed point.*

The following result is motivated by Corollary 2.8.

THEOREM 3.2. *Let $\mathbb{E}$ be a topological vector space and let X be a closed subset of $\mathbb{E}$ and $f : X \to 2^{\mathbb{E}}$ be a closed-valued, upper semicontinuous, multifunction. If*

(i) *$S = \{x - y : y \in f(x), x \in X\}$ is convex,*

(ii) *there exists a sequence $(x_n) \subseteq X$ with $x_{n+1} \in f(x_n)$ for $n = 1, 2, \cdots$,*

(iii) *$f(X)$ is compact,*

then f has a fixed point.

REMARK. Condition (i) may be replaced by the somewhat more general hypothesis that $\frac{1}{n}\sum_{k=1}^{n} x_k - x_{k+1} \in S$ for every $n \in \mathbb{N}$, where (x_k) are as in (ii).

PROOF. We first show that S is a closed subset of $\mathbb{E}$. Let u_α be a net in S with $u_\alpha \to u_0 \in \mathbb{E}$ in the τ-topology of $\mathbb{E}$. Then by definition of S, $u_\alpha = x_\alpha - y_\alpha$ where $y_\alpha \in f(x_\alpha)$ and $x_\alpha \in X$. Since $(y_\alpha) \subseteq f(X)$ and $f(X)$ is compact, it follows that (y_α) has a subnet $y_\beta \to y_0 \in E$. Hence $x_\beta = u_\beta + y_\beta \to u_0 + y_0$, by definition. However, $(x_\beta) \subseteq X$ and X is closed. Hence $u_0 + y_0 \in X$. Now if $x_0 = u_0 + y_0$, $x_0 \in X$, and since $x_\beta \to x_0$, $y_\beta \to y_0$ and $y_\beta \in f(x_\beta)$, it follows that $y_0 \in f(x_0)$. Consequently, $u_0 = x_0 - y_0 \in S$ and thus S is closed. By the definition of (x_n) in (ii), $x_n - x_{n+1} \in S$, and since S is convex, it follows that for any positive integer n,

$$\frac{1}{n}\sum_{k=1}^{n}(x_k - x_{k+1}) \in S;$$

that is,

$$\frac{1}{n}(x_1 - x_{n+1}) \in S. \tag{3}$$

By (ii), $x_{n+1} \in f(X)$ for $n = 1, 2, \cdots$, and hence $x_1 - x_{n+1} \in x_1 - f(X)$ which is a compact subset of $\mathbb{E}$. Consequently, for any neighborhood U of 0, $x_1 - f(X) \subseteq kU$ for some $k \in \mathbb{N}$. Thus $\frac{1}{n}(x_1 - f(X)) \subseteq U$ for all $n \geq k$. In particular, $\frac{1}{n}(x_1 - f(x_n)) \subseteq U$ for all $n \geq k$. Letting $n \to \infty$, since S is closed, by (3) we have $0 \in S$. Consequently, there is some $x_0, y_0 \in f(x_0)$ with $x_0 - y_0 = 0$. This implies $x_0 = y_0 \in f(x_0)$. □

Note that condition (ii) is satisfied if $f(x) \cap X \neq \emptyset$ for all $x \in X$; in particular, if $f(x) \subseteq X$. Further note that $\mathbb{E}^*$ in Theorem 3.2 may be just $\{0\}$. That is to say, no assumption on $\mathbb{E}^*$ separating points of $\mathbb{E}$ is made here. The following corollary follows immediately.

COROLLARY 3.3. *Let X be a closed convex subset of a topological vector space and $f : X \to X$ be continuous. If $f(X)$ is compact and f satisfies (2) (see Corollary 2.8), then f has a fixed point.* □

It is interesting to note that the conditions on convexity of S can be replaced by various other useful conditions on f and X. The remark and note following Theorem 3.2 and Corollary 3.3 provide some such conditions. The following proposition gives a condition on f which is equivalent to the convexity assumption on S.

PROPOSITION 3.4. *Let X be a closed convex subset of the topological vector space $\mathbb{E}$, and $f : X \to 2^{\mathbb{E}}$ be a closed-valued, upper semicontinuous, multifunction such that $f(X)$ is compact. Then $S = \{x - y : y \in f(x), x \in X\}$ is convex if and only if*

$$\frac{1}{2}f(x) + \frac{1}{2}f(y) \subseteq f(\frac{1}{2}x + \frac{1}{2}y) \qquad \forall x, y \in X. \tag{4}$$

PROOF. Suppose S is convex. Then for $z_1, z_2 \in S$, $\frac{1}{2}z_1 + \frac{1}{2}z_2 \in S$. In particular, if $z_i = x_i - y_i$, with $x_i \in X$, and $y_i \in f(x_i)$ for $i = 1, 2$, we have $\frac{1}{2}(x_1 + x_2) - \frac{1}{2}(y_1 + y_2) \in S$. That is, $\frac{1}{2}(y_1 + y_2) \in f(\frac{1}{2}(x_1 + x_2))$. Now, since this holds for every choice of $y_i \in f(x_i)$, $i \in \{1, 2\}$, (4) readily follows. Conversely, suppose f satisfies (4), and let $z_1, z_2, x_1, x_2, y_1, y_2$ be chosen as above. Then since S is closed (by proof of Theorem 3.2), to show S is convex it is sufficient to show S is midpoint convex. Since $\frac{1}{2}z_1 + \frac{1}{2}z_2 = \frac{1}{2}(x_1 + x_2) - \frac{1}{2}(y_1 + y_2)$, by (4), $\frac{1}{2}(y_1 + y_2) \in f(\frac{1}{2}(x_1 + x_2))$, and thus $\frac{1}{2}z_1 + \frac{1}{2}z_2 \in S$. Hence S is midpoint convex and thus convex. □

Note that a multifunction is midpoint convex if $f(\frac{1}{2}x + \frac{1}{2}y) \subseteq \frac{1}{2}f(x) + \frac{1}{2}f(y)$ and f is midpoint concave if it satisfies (4). We conclude with a simple example.

EXAMPLE 3.5. Let $X = [0,1] \subseteq \mathbb{R}$, $\varphi : [0,\infty[\to [0,\infty[$ be a nondecreasing, continuous function such that $\frac{1}{2}(\varphi(x) + \varphi(y)) \leq \varphi(\frac{1}{2}(x+y))$ for all $x, y \in [0,\infty[$. For example, let $\varphi(x) = \log(x+1)$. Define $f(x) = [0, \varphi(x)]$ for all $x \in X$. Then $fX = [0, \varphi(1)]$ is compact, and clearly f is a closed-valued and upper semicontinuous multifunction. By the hypothesis on φ, for $x, y \in X$,

$$\frac{1}{2}f(x) + \frac{1}{2}f(y) = [0, \frac{1}{2}(\varphi(x) + \varphi(y))] \subseteq [0, \varphi(\frac{1}{2}(x+y))] = f(\frac{1}{2}(x+y)),$$

hence by Proposition 3.4, S is convex. Since $f(x) \cap X \neq \emptyset$ for any $x \in X$, by Theorem 3.2, f has a fixed point.

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