Internat. J. Math. & Math. Sci.
VOL. 21 NO. 1 (1998) 201-204

A NOTE ON COMMUTATIVITY OF AUTOMORPHISMS

M.S. SAMMAN
M. ANWAR CHAUDHRY
A.B. THAHEEM

Department of Mathematical Science
King Fahd University of Petroleum and Minerals
Dhahran 31261, SAUDI ARABIA

(Received March 26, 1996)

ABSTRACT. Let α and β be automorphisms of a ring satisfying the equation $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ In this paper we prove some results where this equation itself implies the commutativity of α and β.

KEY WORDS AND PHRASES: Unital ring, nilpotent element, automorphism
1991 AMS SUBJECT CLASSIFICATION CODES: Primary 16W20; Secondary 46L40

1. INTRODUCTION

The equation

$$\alpha+\alpha^{-1}=\beta+\beta^{-1}, \qquad (*)$$

where α and β are automorphisms of a ring R, has been of considerable interest during recent years (see e.g [1,2]). The study of this equation becomes simpler when an additional assumption of commutativity of α and β is made. However, some situations have been identified where equation $(*)$ itself implies the commutativity of α and β. For instance, it has been shown in [1, Corollary 3] that if R is a semiprime unital ring containing the element $1/2$ and α,β are inner automorphisms os R satisfying the equation $(*)$, then α and β commute.

The purpose of this note is precisely to address the commutativity problém and prove certain results in this context. The main result (Theorem 2.1) is, in fact, a generalization of a result of Cater and Thaheem [3] proved for complex algebras, where the equation $(*)$ appears in a more general form $\alpha+m\alpha^{-1}=\beta+m\beta^{-1}$ for an appropriate integer m The mappings of the type $\alpha+m\alpha^{-1}$ occur in the studies of automorphisms of certain C^*-algebras (see for instance [3]). We show here (Theorem 2 1) that if R is a unital ring and α and β are inner automorphisms of R induced by u and v, respectively, such that (i) $\alpha(x)+m\alpha^{-1}(x)=\beta(x)+m\beta^{-1}(x)$, for $x=u,v$, and (ii) $\alpha\beta(x)=\beta\alpha(x)$, for $x=u,v$, where R is (m^2-1)-torsion free, then α and β commute As a corollary (Corollary 2 2) we provide an alternate proof of a special case of Brešar's result [1, Corollary 3] when R has no nontrivial nilpotent elements. However as in Theorem 2.1, the equation $(*)$ here need not hold for all the elements of the ring

We remark that the equation $(*)$ has been extensively studied for von Neumann algebras and C^*-algebras and for more information in this context we may refer to [4,5], which contain further references.

2. COMMUTATIVITY RESULTS

We begin with the following theorem which generalizes a result of Cater and Thaheem [3] proved for complex algebras Our approach here is almost analogous to that of [3]

THEOREM 2.1. Let R be a unital ring and α,β be inner automorphisms of R induced by u and v, respectively, such that

$$\alpha(x) + m\alpha^{-1}(x) = \beta(x) + m\beta^{-1}(x), \quad \text{for} \quad x = u, v, \tag{i}$$

and

$$\alpha\beta(x) = \beta\alpha(x), \quad \text{for} \quad x = u, v. \tag{ii}$$

If R is $(m^2 - 1)$-torsion free, then α and β commute.

PROOF. Put $k = u^{-1}vuv^{-1}$ We first show that k commutes with u and v Substituting $x = v$ in (ii), we get $uvu^{-1} = vuvu^{-1}v^{-1}$ We may rewrite this equation as

$$vu = uvk \tag{1}$$

or

$$vk = u^{-1}vu. \tag{2}$$

Also,

$$kv = u^{-1}vuv^{-1}v = u^{-1}vu. \tag{3}$$

It follows from (2) and (3) that $kv = vk$ Thus k and v commute By symmetry, k and u also commute Thus we can write (1) as

$$kuv = vu. \tag{4}$$

Substituting $x = v$ in (i), we get

$$uvu^{-1} + mu^{-1}vu = (1+m)v. \tag{5}$$

Multiplying (5) on the right by u, we get

$$uv + mu^{-1}vuu = (1+m)vu. \tag{6}$$

It follows from (4) and (6) that

$$uv + mkvu = (1+m)vu. \tag{7}$$

It follows from (4) and (7) that

$$uv + mk^2uv = (1+m)kuv, \tag{8}$$

or what is the same

$$(mk-1)(k-1)uv = \big(mk^2 - mk - k + 1\big)uv = 0. \tag{9}$$

Since uv is invertible, we get from (9) that

$$mk^2 - mk - k + 1 = 0. \tag{10}$$

We observe from (4) that $k^{-1}vu = uv$. Repeating the above procedure with v in place of u, u in place of v and k^{-1} in place of k, we get

$$\big(mk^{-1} - 1\big)\big(k^{-1} - 1\big) = 0. \tag{11}$$

Multiplying (11) on the left by mk and on the right by k, we obtain

$$\big(m^2 - mk\big)(1-k) = m^2 - m^2k - mk + mk^2 = 0, \tag{12}$$

or what is the same

$$-mk^2 + mk + m^2k - m^2 = 0. \tag{13}$$

Adding (10) and (13), we get

$$(m^2 - 1)(k - 1) = 0. \tag{14}$$

Since R is $(m^2 - 1)$-torsion free, we get from (14) that $k - 1 = 0$ or $k = 1$ This implies $uv = vu$ and hence α and β commute

The following corollary gives an alternate proof of Brešar's result [1, Corollary 3] in the special case when R has no nontrivial nilpotent elements with an additional assumption that $(\alpha\beta)(u) = (\beta\alpha)(u)$ and $(\alpha\beta)(v) = (\alpha\beta)(v)$ However, in our setting it is sufficient for equation $(*)$ to hold for some specific elements rather than all the elements of the ring to ensure the commutativity of α and β

COROLLARY 2.2. Let R be a unital ring with no nonzero nilpotent elements and α,β be inner automorphisms of R induced by u and v respectively such that

$$\alpha(v) + \alpha^{-1}(v) = \beta(v) + \beta^{-1}(v) \tag{iii}$$

and

$$\alpha\beta(x) = \beta\alpha(x), \quad \text{for} \quad x = u, v. \tag{iv}$$

Then α and β commute

PROOF. As in the proof of Theorem 2 1, put $k = u^{-1}vuv^{-1}$ Then k commutes with u and v follows from (iv) and consequently equation (4) holds Then using (iii) and following a procedure similar to Theorem 2 1, we obtain that $(k-1)^2 = 0$ Since R has no nonzero nilpotent elements, we get $k - 1 = 0$ or $k = 1$. This proves that $uv = vu$ and hence α and β commute

REMARK 2.3. (a) We observe that the main argument in proving the commutativity of α and β has been to show that $k = 1$. In fact, somewhat weaker condition, namely that k belongs to the center of R would also ensure the commutativity of α and β Therefore, it would also be interesting to prove that k is in the center of R

(b) It would be interesting to prove or disprove Theorem 2.1 for the case $m = -1$.

ACKNOWLEDGMENT. The authors gratefully acknowledge the support provided by King Fahd University of Petroleum and Minerals during this research.

REFERENCES

[1] BREŠAR, M , On certain pairs of automorphisms of rings, *J. Austral. Math. Soc.* (Series A), **54** (1993), 29-38.

[2] BREŠAR, M , On the composition of (α, β)-derivations of rings and applications to von Neumann algebras, *Acta Sci. Math.*, **56** (1992), 369-375

[3] CATER, F.S. and THAHEEM, A B., On a pair of automorphisms of C^*-algebras, To appear in *Ren. Sem. Mat. Univ. Padova.*

[4] THAHEEM, A.B., On certain decompositional properties of von Neumann algebras, *Glasgow Math. J.*, **29** (1987), 177-179

[5] THAHEEM, A B , On a functional equation on C^*-algebras, *Funkcial. Ekvac.*, **31** (1988), 411-413.