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RESEARCH NOTES

THE SOLUTION OF A SINGULAR INTEGRAL EQUATION WITH SOME APPLICATIONS IN POTENTIAL THEORY

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ABSTRACT. An analytical solution is derived for a singular integral equation which governs some two-dimensional potential boundary value problems in a region exterior to n-infinite co-axial circular strips An application in electrostatics is discussed

KEY WORDS AND PHRASES: Singular integral equation, electrostatics, Laplace's equation
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1 INTRODUCTION

In this paper we derive a solution to the Fredholm singular integral equation

$$\sum_{k=0}^{n-1}\int_{-\alpha}^{\alpha} f(\phi)\left[q+\log\left|\sin\left(\frac{1}{2}(\theta-\phi-\frac{2k\pi}{n})\right)\right|\right]d\phi = F(\theta) \quad , \tag{1.1}$$

where q and α are constants and $F(\theta)$ is a differentiable function for $|\theta-(2k\pi/n)|<\alpha$, $k=0,\dots,n-1$ This integral equation governs the solution of various two-dimensional Dirichlit and Neumann potential boundary value problems for the region consisting of the whole (r,θ) plane outside the circular strips

$$r=a \quad , \left|\theta-\frac{2k\pi}{n}\right|<\alpha \; , k=0,\dots,n-1 \; , \quad |\alpha|<\frac{\pi}{n} \tag{1.2}$$

The previous investigations in potential problems of circular strips [1-8] were concerned mainly with the case of two strips, where Greens function approach leads to a singular integral equation with kernel $q+\log\left|\sin\frac{1}{2}(\theta-\phi)\right|$ Shail [3] transformed this equation into a well known Carleman type

A different technique has been used by Sampath and Jain [4] based on decoupling the equation into two singular equations which can be solved using eigenfunctions expansion metod The same technique has been used to solve various boundary value problems involving two circular strips[5-8]

In section 2 we use the approach of [4] to solve equation(1.1),and in section 3 we apply the results to solve laplace equation associated with Dirichlit and Neumann boundary conditions As an illustrative example we consider in section 4 the electrostatics problem in a region external to perfectly conducting n- circular strips Formulae for the surface charge density and concentration factor are derived,and these are believed to be new

2.SOLUTION OF THE INTEGRAL EQUATION

To obtain the solution of the integral equation (1 1) we first reduce it to simpler singular integral equations, so interchanging the sum and integral signs and using some properties of the function in the kernel we obtain

$$\int_{-\alpha}^{\alpha} I(\phi)\left[qn - \frac{n}{2}\log 2 + \frac{1}{2}\log\prod_{k=0}^{n-1}(1-\cos(\theta-\phi-\frac{2k\pi}{n}))\right]d\phi = F(\theta), \tag{2 1}$$

Using the identity [9],

$$\prod_{k=0}^{n-1}(1-\cos(\theta-\phi-\frac{2k\pi}{n})) = \frac{1}{2^{n-1}}(1-\cos(n(\theta-\phi))), \tag{2. 2}$$

in the equation (2 1) we get

$$\int_{-\alpha}^{\alpha} I(\phi)[Q+\log(1-\cos n(\theta-\phi)]d\phi = 2F(\theta), \qquad |\theta|<\alpha \tag{2 3}$$

where $Q = 2qn - (2n-1)\log 2$

Now we write each of the functions $I(\theta)$ and $F(\theta)$ as

$$I(\theta) = I_e(\theta) + I_o(\theta) \ , \tag{2 4a}$$

$$F(\theta) = F_e(\theta) + F_o(\theta) \ , \tag{2 4b}$$

where the subscripts e and o stand for the even and odd parts respectively . Substituting (2 4) into (2 3) we obtain the following decoupled singular integral equations

$$\int_{0}^{\alpha} I_e(\phi)\left[Q+\log\{\cos(n\theta)-\cos(n\phi)\ \}\right]d\phi = F_e(\theta) \ , \qquad 0<\theta<\alpha \tag{2 5}$$

$$\int_{0}^{\alpha} I_o(\phi)\log\left|\frac{\sin\frac{n}{2}(\phi-\theta)}{\sin\frac{n}{2}(\phi+\theta)}\right|d\phi = F_o(\theta) \quad , \qquad 0<\theta<\alpha \tag{2 6}$$

In order to solve eq (2 5) we introduce the transformation

$$\cos(n\theta) = \sin^2(\frac{n\alpha}{2})\cos x + \cos^2(\frac{n\alpha}{2}) \ , \qquad 0<x<\pi \tag{2 7a}$$

$$\cos(n\phi) = \sin^2(\frac{n\alpha}{2})\cos y + \cos^2(\frac{n\alpha}{2}) \ , \qquad 0<y<\pi \tag{2 7b}$$

which transforms the equation into the form

$$\int_{0}^{\pi} H(y)\left[R+\log(2|\cos x-\cos y|\right]dy = h(x), \qquad 0<x<\pi \tag{2 8}$$

where

$$R = Q + \log\left[\frac{1}{2}\sin^2(n\alpha/2)\right] \tag{2 9}$$

$$h(x) = F_e'(\theta) \tag{2 10}$$

$$\text{and } H(y) = \frac{\sin^2(\frac{n\alpha}{2})\sin y}{n\sin(n\phi)} I(\phi) \tag{2 11}$$

Expanding $h(x)$ as Fourier cosine series

$$h(x) = \frac{1}{2}a_o + \sum_{m=1}^{\infty} a_m \cos(mx) \ , \qquad 0< x < \pi \tag{2 12}$$

where
$$a_m = \frac{2}{\pi}\int_0^\pi h(x)\cos(mx)dx \, , \qquad m \geq 0 \tag{2 13}$$

and making use of the bilinear form [10]

$$\log(2|\cos x - \cos y|) = -2\sum_{m=1}^{\infty}\frac{\cos(mx)\cos(my)}{m}, \qquad 0<x \, , y < \pi \tag{2 14}$$

, we find that the solution of (2 8) can be written in the form

$$H(y) = A_0 + \sum_{m=1}^{\infty} A_m \cos(my), \tag{2 15}$$

where

$$A_0 = \frac{a_0}{2\pi R} \quad , \ A_m = -\frac{1}{\pi}a_m \quad , \quad m \geq 1 \tag{2 16}$$

thus using (2 11) we find that the solution of (2 5) is given by

$$I_e(\phi) = \frac{\sqrt{2}n\cos(n\phi/2)}{\sqrt{\cos(n\phi) - \cos(n\alpha)}}\left[A_0 + \sum_{m=1}^{\infty} A_m \cos(my)\right], \qquad 0< \ \phi < \alpha \tag{2 17}$$

where we have used the relation

$$\sin(y) = \sqrt{2}\csc^2(\frac{n\alpha}{2})\sin(\frac{n\phi}{2})\sqrt{\cos(n\phi) - \cos(n\alpha)} \quad , \tag{2 18}$$

Now to solve eq (2 6) we first differentiate both sides with respect to θ

$$\int_0^\alpha \frac{nI_o(\phi)\sin(n\phi)}{\cos(n\phi) - \cos(n\theta)}d\phi = F'(\theta) \quad , \tag{2 19}$$

Applying the transformation (2 7) into (2 19) reduces it to

$$\int_0^\pi \frac{G(y)}{\cos y - \cos x}dy = g(x), \qquad 0< x < \pi \tag{2 20}$$

where

$$g(x) = F'(\theta) \tag{2 21}$$

$$G(y) = I_o(\phi)\sin y \tag{2 22}$$

Expanding $G(y)$ in Fourier cosine series

$$G(y) = B_0 + \sum_{m=1}^{\infty} B_m \cos(my), \qquad 0< y < \ \pi \tag{2 23}$$

and noting the integral

$$\int_0^\pi \frac{\cos(my)}{\cos y - \cos x} dy = \pi \sin(mx)\csc(x) \ , \qquad 0< x < \pi \tag{2 24}$$

which follows by putting $w = e^{iy}$ and integrating around the circle $|w| = 1$ Then

$$\int_0^\pi \frac{G(y)}{\cos y - \cos x} dy = \pi \sum_{m=1}^{\infty} B_m \sin(mx)\csc(x), \tag{2 25}$$

Thus if we expand

$$g(x)\sin x = \sum_{m=1}^{\infty} C_m \sin(mx) \tag{2 26}$$

$$C_m = \frac{2}{\pi} \int_0^\pi g(x)\sin x \sin(mx)dx \tag{2.27}$$

it follows that

$$B_m = \frac{1}{\pi} C_m \ , \qquad m \geq 1 \tag{2 28}$$

and hence

$$I_o(\phi) = \frac{1}{\sin y}\left[B_0 + \frac{1}{\pi}\sum_{m=1}^{\infty} C_m \cos(my)\right] \tag{2 29}$$

To evaluate the constant B_0 we use the finiteness condition that $G(y) \to 0$ as $y \to 0$ this gives

$$B_0 = -\frac{1}{\pi}\sum_{m=1}^{\infty} C_m \tag{2 30}$$

Substituting (2 30) into (2 29) we find from (2 22) and the relation (2 18) that

$$I_o(\phi) = \frac{1}{\pi \sin y}\sum_{m=1}^{\infty} C_m (1 - \cos my) \qquad -\alpha < \phi < \alpha \tag{2 31}$$

Finally summing up the results we write the solution of eq (1 1) as

$$I(\phi) = \frac{\sqrt{2}n\cos(n\phi/2)}{\sqrt{\cos(n\phi) - \cos(n\alpha)}}\left[A_o + \sum_{m=1}^{\infty} A_m \cos(my)\right]$$

$$-\frac{1}{\pi \sin y}\sum_{m=1}^{\infty} C_m (1 - \cos my) \ , \qquad -\alpha < \phi < \alpha \tag{2 32}$$

where

$$A_0 = \frac{a_0}{4\pi\left[qn + \log(2^{-n}|\sin(n\alpha/2|)\right]} \tag{2 33 }$$

$$A_m = -\frac{1}{\pi} a_m \ , \qquad m \geq 1 \tag{2 34}$$

$$a_m = \frac{2}{\pi}\int_0^\pi h(x)\cos(mx)dx \ , \qquad m \geq 0 \tag{2 35}$$

$$C_m = \frac{2}{\pi}\int_0^\pi g(x)\sin x \sin(mx)dx \tag{2 36}$$

$$h(x) = F_e'(\theta) \ , \qquad g(x) = F_o'(\theta) \tag{2 37}$$

3.APPLICATION IN POTENTIAL THEORY

The singular integral eq (1 1) governs the solutions of various Dirichlit and Neumann problems for two-dimensional Laplace equation in the domain D consisting of all the points (r,θ) lying ooutside the n-circular infinite strips

$$C_k \; r = a \;, \left|\theta - \frac{2k\pi}{n}\right| < \alpha \,, k \;\; 0,..,n\text{-}1 \,, \tag{3 1}$$

These types of problems appear in different physical fields such as electrostatics, magnetostatics,steady state heat flow , and others

A Dirichlit problem We seek a function $\Phi(r,\theta)$ that satisfies the follwing boundary value problem

$$\nabla^2\Phi = 0 \tag{3 2}$$

$$\Phi(a,\theta) = f(\theta) \quad , \qquad \left|\theta - \frac{2k\pi}{n}\right| < \alpha \;, \quad k \;\; 0,..,n\text{-}1 \tag{3 3}$$

$$\Phi(r,\theta)| \sim -\frac{\Gamma}{2\pi}\log r \qquad r \to \infty \tag{3 4}$$

and are continuous across the arcs

$$C'_k \; r = a \;, \qquad \frac{2k\pi}{n} + \alpha \quad < \theta < 2(k+1)\frac{\pi}{n} - \alpha \tag{3 5}$$

The function $f(\theta)$ satisfies the symmetry condition

$$f\left(\theta + \frac{2k\pi}{n}\right) = f(\theta) \quad , \qquad -\alpha < \theta < \alpha \tag{3 6}$$

$$\text{and} \quad \Gamma = \sum_{k=0}^{n-1} \int_{C_k} \left[\frac{\partial\Phi}{\partial r}\right]_{r=a} ds \tag{3 7}$$

Using Green's function approach and the symmetry in the problem the solution can be represented as

$$\Phi(r,\theta) = -\frac{a}{2\pi}\sum_{k=0}^{n-1}\int_{-\alpha}^{\alpha} I(\phi)\log\sqrt{r^2 + a^2 - 2ra\cos(\theta - \phi - \frac{2k\pi}{n})}d\phi \tag{3 8}$$

where the density function $I(\phi)$ is defined as

$$I(\phi) = \left[\frac{\partial\Phi}{\partial r}\right]_{r=a}^{r=a'} \tag{3 9}$$

The boundary condition (3 3) leads to the following integral equation

(3 10)

which is eq (1 1) with $q = \log(2a)$, $F(\theta) = -\frac{2\pi}{a}f(\theta)$, and hence it has the solution (2 32) with the substitution of these values

B. Neumann problem We seek a function $\Psi(r,\theta)$ satisfying the boundary value problem

$$\nabla^2\Psi = 0 \tag{3 11}$$

$$\Psi(a,\theta) = p(\theta), \qquad \left|\theta - \frac{2k\pi}{n}\right| < \alpha \;, k \;\; 0,..,n\text{-}1 \tag{3 12}$$

$$\Psi(r,\theta) \to 0 \text{ , as } r \to \infty \tag{3.13}$$

Ψ, and $\dfrac{\partial \Psi}{\partial r}$ are continuous across the arcs C_k'

The function $P(\theta)$ satisfies the symmetry condition

$$p\left(\theta + \frac{2k\pi}{n}\right) = p(\theta) \text{ , } \quad -\alpha < \theta < \alpha \tag{3.14}$$

and the consistency condition

$$\int_{-\alpha}^{\alpha} p(\theta) d\theta = 0 \tag{3.15}$$

Green's function approach leads to the follwing representation

$$\Psi(r,\theta) = \frac{-a}{2\pi}\sum_{k=0}^{n-1}\int_{-\alpha}^{\alpha} J(\phi)\left[\frac{\partial}{\partial\rho}\log\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\phi-\frac{2k\pi}{n})}\right]_{\rho=a} d\phi \tag{3.16}$$

$$\text{where } J(\phi) = \left[\Psi(r,\phi)\right]_{r=a^-}^{r=a^+} \text{ , } \quad -\alpha < \phi < \alpha . \tag{3.17}$$

Furthermore $J(\theta)$ satisfies the edge condition

$$J(\pm\alpha) = 0 \tag{3.18}$$

Using the boundary condition (3.12) we obtain

$$\sum_{k=0}^{n-1}\int_{-\alpha}^{\alpha} J(\phi)\csc^2\frac{1}{2}(\theta-\phi-\frac{2k\pi}{n})d\phi = 8\pi a p(\theta) \tag{3.19}$$

Integrating by parts using the edge condition (3.18) we obtain

$$\sum_{k=0}^{n-1}\int_{-\alpha}^{\alpha} J'(\phi)\cot\frac{1}{2}(\theta-\phi-\frac{2k\pi}{n})d\phi = -4\pi a p(\theta) \tag{3.20}$$

Upon integration with respect to θ we get

$$\sum_{k=0}^{n-1}\int_{-\alpha}^{\alpha} J'(\phi)\log\left|\sin\frac{1}{2}(\theta-\phi-\frac{2k\pi}{n})\right| d\phi = -2\pi a P(\theta) + C \tag{3.21}$$

where $P(\theta) = \int p(\theta)d\theta$ and C is an arbitrary constant Eq (3.21)is special case of eq (1.1) with $q = 0$ $F(\theta) = C - 2\pi a p(\theta)$, and $I(\phi) = J'(\phi)$ To determine the constant C we use the edge condition (3.18) Finally the density $J(\theta)$ is given by

$$J(\theta) = \int_{-\alpha}^{\theta} J'(\phi)d\phi \text{ , } \quad -\alpha < \theta < \alpha \tag{3.22}$$

4.ILLUSTRATIVE EXAMPLE

As an example to illustrate our results we consider the electostatics potential problem of the n-equal infinite co-axial perfectly conducting strips in free space charged so that thetotal charge per height on each strip is unity.In cylindrical coordinates (r,θ,z) let the stips be defined by

$$C_k \;\; r = a, \;\; \left|\theta - \frac{2k\pi}{n}\right| < \alpha \text{ , } k = 0,\ldots,n-1 \text{ , , } \quad -\infty \le z \le \infty \tag{4.1}$$

then the electrostatics potential satisfies the Dirichlit problem A , where on the boundary we assume that Φ has the constant potential K Thus

$$\Phi(r,\theta) = -a\sum_{k=0}^{n-1}\int_{-\alpha}^{\alpha} I(\phi)\log\sqrt{r^2+a^2-2ra\cos(\theta-\phi-\frac{2k\pi}{n})}\,d\phi \tag{4 2}$$

and $I(\phi)$ satisfy the integral equation (3 10) with $f(\theta)=K$,and hence

$$h(x) = F_e(\theta) = \frac{-K}{a}\ ,\quad g(x) = F_o(\theta) = 0 \tag{4 3}$$

Substituting these valus in (2 32) we find that

$$I(\phi) = \frac{-Kn\cos(n\phi/2)}{\sqrt{2}\pi a\log(a^n|\sin(n\alpha/2)|)\sqrt{\cos(n\phi)-\cos(n\alpha)}},\quad -\alpha<\phi<\alpha \tag{4 4}$$

To find the value of K we use the condition that the total charge per unit height on each strip is unity,that is,

$$a\sum_{k=0}^{n-1}\int_{-\alpha}^{\alpha} I(\phi)d\phi = n\ ,\ \text{or}\quad \int_{-\alpha}^{\alpha} I(\phi)d\phi = \frac{1}{2a} \tag{4 5}$$

Inserting (4 4) into (4 5) and evaluating the resulting integral wefind that

$$K = -\log(a^n|\sin n\alpha/2|) \tag{4 7}$$

and the charge density in this case is

$$I(\phi) = \frac{n\cos(n\phi/2)}{\sqrt{2}\pi a\sqrt{\cos(n\phi)-\cos(n\alpha)}}\ ,\ -\alpha<\phi<\alpha \tag{4 8}$$

while the charge concentration factor is given by

$$N = \lim_{\phi\to\alpha}(\alpha-\phi)^{1/2} I(\phi) = \frac{1}{2a\pi}\sqrt{n\cot(n\alpha/2)} \tag{4 9}$$

All the above formulae are believed to be new For n=4 the formulae (4 6),(4 7),and (4 8) agree with those in [5],

and when $\alpha \to \pi/n$ we recover the known results for a charged infinite cylinder when the total charge per unit height is n

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