Internat. J. Math. & Math. Sci.
VOL. 21 NO. 1 (1998) 125-132

FIXED POINT THEOREMS FOR SEMI-GROUPS OF SELF MAPS OF SEMI-METRIC SPACES

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(Received May 2, 1996 and in revised form January 2, 1997)

ABSTRACT. We use selected semi-groups of self maps of a semi-metric space to obtain fixed point theorems for single maps and for families of maps – theorems which generalize results by Browder, Jachymski, Rhoades and Walters, and others. A basic tool in our approach is the concept of commuting maps.

KEY WORDS AND PHRASES: Semi-metric, semi-group, commuting maps, fixed points.
1991 AMS SUBJECT CLASSIFICATION CODES: Primary 54E25, 54H25.

1. INTRODUCTION. By a semi-group of maps we shall mean a family H of self maps of a set X which is closed with respect to composition of maps. Thus, if $f,g \in H$, then $f \circ g \in H$. Since composition of maps is associative, H is indeed a semi-group with respect to composition. We shall write fg for $f \circ g$ and fx for f(x) when convenient and confusion is not likely.

We shall utilize the following semi-groups of maps in subsequent sections.

1.1. Let $g: X \to X$, and $H = O_g = \{ g^n: n \in N \cup \{0\} \}$, where N is the set of positive integers, $g^0 = i_d$ – the identity map, $g^1 = g$ and $g^{n+1} = g \circ g^n$.

1.2. Let $g: X \to X$, and $H = C_g = \{ f: X \to X \mid fg = gf \}$. C_g is a semi-group. For if $f,h \in C_g$, then $(fh)g = f(hg) = f(gh) = (fg)h = (gf)h = g(fh)$, and thus, $fh \in C_g$.

1.3. $H = \{i_d\}$, and

1.4 If $f,g: X \to X$ and $fg = gf$, we can let $H = \{ f^n g^m: n,m \in N \cup \{0\} \}$.

If H is a semi-group of self maps of a set X and $a \in X$, then $H(a) = \{h(a): h \in H\}$. Consequently, if $g: X \to X$ and $H = O_g$, $O_g(a) = \{g^n(a) : n \in N \cup \{0\}\}$, and is called the *orbit of g at a.*

Just as we use semi-groups of maps to generalize the concept of orbits, we shall use semi-metric spaces to generalize results pertaining to metric spaces. We need the following definitions.

DEFINITION 1.1 A symmetric on a set X is a function $d: X x X \to [0, \infty)$ such that $d(x, y) = 0$ iff $x = y$ and $d(x, y) = d(y, x)$ for all $x,y \in X$.

Given a symmetric d on a set X, we generate an induced topology t(d) for X as follows. For $x \in X$ and $\epsilon > 0$, we let $S_\epsilon(x) = \{ y \in X: d(x, y) < \epsilon \}$. Then t(d) consists of all subsets U of X such that for each $p \in U$, $S_\epsilon(p) \subseteq U$ for some $\epsilon > 0$. Just as in the case of a metric, t(d) is a topology on X. However, if d is a symmetric, the sets $S_\epsilon(x)$ need not be neighborhoods of x. A semi-metric is a symmetric d such that all sets $S_\epsilon(x)$ are neighborhoods of x; i.e., $\exists\, U \in t(d)$ such that $x \in U \subseteq S_\epsilon(x)$. It is easy to verify that if d is a semi-metric, then a sequence $\{x_n\}$ in X converges to $x \in X$ in the topology t(d) iff $d(x_n, x) \to 0$. This is the property we desire. Hence, the following terminology.

DEFINITION 1.2 A semi-metric space is a topological space, denoted by (X; d), with topology t(d) induced on the set X by a semi-metric d.

For further discussion of symmetrics and semi-metrics refer to [1] or [2]. In this context, we note that a semi-metric need not be Hausdorff (or T_2) Example 2.2 in [2] gives an instance of such a semi-metric. Since we desire uniqueness of limits, we shall in most instances require that a semi-metric space (X; d) be Hausdorff. Note also that – as in metric spaces – we shall say a semi-metric space (X; d) is *complete* iff every Cauchy sequence in X converges to a point in X. If $g:X \to X$, then (X; d) is *g-orbitally complete* iff every Cauchy sequence in $O_g(x)$ converges to a point in X for all $x \in X$. A function $F.X \to [0, \infty)$ is lower semicontinuous iff $F(x) \leq \liminf_{n\to\infty} F(x_n)$ when $\{x_n\}$ is a sequence in X converging to x.

To produce fixed points we use a contractive function $P:[0, \infty) \to [0, \infty)$ which is nondecreasing and which satisfies: $\lim_{n\to\infty} P^n(t) = 0$ for each $t \in [0, \infty)$. Throughout this paper, P will denote such a map, and $\mathcal{P}$ will denote the family of all such maps P.

2. FIXED POINT THEOREMS.

The major results in this paper evolve from the following lemma.

LEMMA 2.1. Let X be a set, $g:X \to X$, and let $d:XxX \to [0, \infty)$. Let H be a semi-group of maps $h\, X \to X$ such that $H \subset C_g$ Suppose that for each pair $x,y \in X$ there is a choice of $r = r(x,y)$, $s=s(x,y) \in H$ and $u,v \in \{x,y\}$ for which

(i) $d(gx, gy) \leq P(d(ru, sv))$.

Then, if $n \in N$, for each pair $x,y \in X$ $\exists\, r_n,s_n \in H$ and $u_n,v_n \in \{x, y\}$ such that

(ii) $d(g^nx, g^ny) \leq P^n(d(r_nu_n, s_nv_n))$.

PROOF. (ii) holds for n=1 by (i), so suppose $n \in N$ for which (ii) is true. Then, if $x,y \in X$,

$$d(g^{n+1}x, g^{n+1}y) = d(g(g^nx), g(g^ny)) \leq P(d(ru, sv)), \tag{2.1}$$
where $r,s \in H$ and $u,v \in \{g^nx,g^ny\}$, by (i).

Specifically, $u = g^nu_o$, where $u_o \in \{x, y\}$ and $v = g^nv_o$ with $v_o \in \{x, y\}$. Thus

$d(ru, sv) = d(r(g^nu_o), s(g^nv_o))$ where $u_o, v_o \in \{x, y\}$. And since $r,s \in H \subset C_g$,

$$d(ru, sv) = d(g^n(ru_o), g^n(sv_o)) \leq P^n(d(r_nu_n, s_nv_n)), \text{ by (ii)}, \tag{2.2}$$
where $r_n, s_n \in H$ and $u_n, v_n \in \{ru_o, sv_o\}$.

Then $r_nu_n \in \{(r_nr)u_o, (r_ns)v_o\}$, where $r_nr, r_ns \in H$ (a semi-group). So, $r_nu_n= r_{n+1}u_{n+1}$, where $r_{n+1} \in H$ (i.e., $r_{n+1} \in \{r_nr, r_ns\}$) and $u_{n+1} \in \{u_o, v_o\} \subset \{x, y\}$. Similary, $s_nv_n = s_{n+1}v_{n+1}$, where $s_{n+1} \in H$ and $v_{n+1} \in \{x, y\}$. Thus (2.2) implies

$$d(ru, sv) \leq P^n(d(r_{n+1}u_{n+1}, s_{n+1}v_{n+1})), \text{ with } r_{n+1}, s_{n+1} \in H \text{ and } u_{n+1},v_{n+1} \in \{x, y\}. \tag{2.3}$$

But P is nondecreasing; therefore, (2.1) and (2.3) imply

$$d(g^{n+1}x, g^{n+1}y) \leq P(P^n(d(r_{n+1}u_{n+1}, s_{n+1}v_{n+1})) = P^{n+1}(d(r_{n+1}u_{n+1}, s_{n+1}v_{n+1}),$$

with $r_{n+1},s_{n+1} \in H$ and $u_{n+1}, v_{n+1} \in \{x, y\}$. Thus, (ii) is true for all n, by induction. □

THEOREM 2.1 Let (X; d) be a T_2 semi-metric space. Let $g:X \to X$ and let (X; d) be g-orbitally complete. Suppose H is a semi-group of self maps of X such that $H \subset C_g$, and there is an $a \in X$ for which H(a) is bounded and $g(H(a)) \subset H(a)$. If for each $x,y \in X$ $\exists$ a choice of $r,s \in H$ and $u,v \in \{x,y\}$ such that

(*) $d(gx, gy) \leq P(d(ru, sv))$,

then $g^n(a) \to c$ for some $c \in X$. If g is continuous at c, then $g(c) = c$. If d is lower semicontinuous , then c is a fixed point for all $h(\in H)$ continuous at c. Moreover, if g and each $h \in H$ are continuous at c, then c is the unique common fixed point of g and the family H.

PROOF. We first prove that $\{g^n(a)\}$ is Cauchy. By Lemma 2.1., for each pair $n,k \in N$, there is a choice of $r_n, s_n \in H$ and $u_n, v_n \in \{a, g^k a\}$ such that

$$d(g^n(a), g^{n+k}(a)) = d(g^n(a), g^n(g^k(a)) \leq P^n(d(r_n u_n, s_n v_n)). \tag{2.4}$$

Now $r_n \in H$ implies $r_n a \in H(a)$ and $r_n g^k(a) = g^k(r_n a) \in H(a)$, since $g(H(a)) \subseteq H(a)$ implies that $g^k(H(a)) \subseteq H(a)$. Thus $r_n u_n \in H(a)$. Similarly, $s_n v_n \in H(a)$. But H(a) is bounded, so $\exists\ M \geq 0$ such that $d(x, y) \leq M$ for $x,y \in H(a)$. Thus, $d(r_n u_n, s_n v_n) \leq M$ for $n \in N$. Then (2.4) implies

$$d(g^n(a), g^{n+k}(a)) \leq P^n(M), \text{ for } n,k \in N \tag{2.5}$$

since P is nondecreasing. But $P^n(M) \to 0$ as $n \to \infty$. So given $\epsilon>0$, $\exists\ n_o \in N$ such that for any $m> n \geq n_o$, (2,5) implies $d(g^n(a), g^m(a)) \leq P^n(M) < \epsilon$, with $m=n+k$. Consequently, $\{g^n(a)\}$ is Cauchy.

Since (X; d) is g-orbitally complete, $g^n(a), g^{n+1}(a) \to c$ for some $c \in X$. If g is continuous, $g(g^n(a)) = g^{n+1}(a) \to g(c)$; thus, $c = g(c)$ since X is Hausdorff.

Now suppose that d is lower semicontinuous and that $h(\in H)$ is continuous at c. Then, since $H \subseteq C_g$ and $g^n(a) \to c$,

$$g^n(h(a)) = h(g^n(a)) \to h(c). \tag{2 6}$$

But (*) and Lemma 2.1 tell us that $\exists\ r_n, s_n \in H$ and $u_n, v_n \in \{a, h(a)\}$ such that

$$d(g^n(a), g^n(h(a))) \leq P^n(d(r_n u_n, s_n v_n)). \tag{2.7}$$

Then $r_n u_n = r_n a \in H(a)$ or $r_n u_n = r_n h(a) \in H(a)$ ($r_n h \in H$, since H is a semi-group.). Thus, in either event, $r_n u_n \in H(a)$. In like manner, we conclude that $s_n v_n \in H(a)$. Then, as above, $d(r_n u_n, s_n v_n) \leq M$ for $n \in N$, which implies by (2.7)

$$d(g^n(a), g^n(h(a))) \leq P^n(M) \to 0. \tag{2.8}$$

But since $g^n(a) \to c$, (2.6) implies that $(g^n(a), g^n(h(a))) \to (c, h(c))$ in XxX. Since d is lower semicontinuous, $d(c, h(c)) \leq \lim_{n\to\infty} d(g^n(a), g^n(h(a))) = 0$ by (2.8), so $h(c) = c$.

To complete the proof we have yet to show that if c is a common fixed point for g and every $h \in H$, then c is the only such point. So suppose that $z \in X$ and that $z = g(z) = h(z)$ for all $h \in H$. Then by (*) and Lemma 3.1, we can write:

$$d(c, z) = d(g^n(c), g^n(z)) \leq P^n(d(r_n u_n, s_n v_n)) \tag{2.9}$$

where $r_n, s_n \in H$ and $u_n, v_n \in \{c, z\}$. But then $r_n u_n \in \{c, z\}$. Similarly, $s_n v_n \in \{c, z\}$. Therefore, $d(r_n u_n, s_n v_n) = 0$ or $d(c, z)$. Thus (2.9) says that $d(c, z) \leq P^n(d(c, z))$. Since $P^n(d(c,z)) \to 0$, $c = z$; i.e., c is unique. □

The following example shows that the family H in Theorem 2.1 can have fixed points other than the unique common fixed point of g and H.

EXAMPLE 2.1. Let $X=\{0, 1\}$, $g(x) = 0$ for $x \in X$ $h(x) = x$ for $x \in X$. Let $d(x,y) = |x - y|$ and $H = \{h^n\colon n \in N\}$. Since $h^n(x) = x$ for $n \in N$, $H = \{i_d\}$. Since $d(gx, gy) = 0$ for all $x,y \in X$, it is immediate that g and H satisfy the hypothesis of Theorem 3.1 with $a = 0$, and 1 is a fixed point of H but not of g.

The first corollary provides conditions necessary and sufficient to ensure that a family H of continuous self maps of a semi-metric space has a fixed point.

COROLLARY 2.1. Let (X; d) be a complete Hausdorff semi-metric space with d lower semicontinuous. A semi-group H of continuous self maps of X has a common fixed point iff H(a) is bounded for some $a \in X$, and $\exists\, P \in \mathcal{P}$ and a continuous self map g of X which satisfies the following.

1. $H \subseteq C_g$ and $g(H(a)) \subseteq H(a)$
2. For any $x,y \in X$, $\exists$ a choice of $r, s \in H$ and $u, v \in \{x, y\}$ such that
   $d(gx, gy) \leq P(d(ru, sv))$.

PROOF. That the conditions are sufficient follows immediately from Theorem 2.1.. To prove necessity, suppose $a \in X$ and that $h(a) = a$ for $h \in H$. Then $H(a) = \{a\}$ and is thus bounded. Let $g(x) = a$ for $x \in X$. It is immediate that $gh = hg$ for all $h \in H$, so $H \subseteq C_g$. Moreover, $g(h(a)) = a$ for all $h \in H$, so that $g(H(a)) \subseteq H(a)$ and statement 1. of the Corollary holds. Statement 2. follows upon noting that $d(gx, gy) = d(a, a) = 0$ for all $x, y \in X$. (We can let $P(t) = t/2$, e.g..) □

NOTE 2.1. The next result and proof suggest that the function g of Theorem 2.1 may have an infinitude or unbounded set of fixed points, although H may have just one. Example 3.1 in the next section confirms this.

COROLLARY 2.2. Let (X; d) be a complete Hausdorff semi-metric space with d lower semicontinuous. A semi-group H of continuous self maps of X has a common fixed point provided $\exists$ $a \in X$ such that H(a) is bounded, and for any $x,y \in X$ $\exists$ $r, s \in H$ and $u,v \in \{x, y\}$ such that

$$d(x, y) \leq P(d(ru, sv)).$$

PROOF. Let $g = i_d$, the identity map. □

COROLLARY 2.3. Let g be a self map of a metric space (X, d) which is g-orbitally complete. If $\exists$ $a \in X$ such that $O_g(a)$ is bounded and $k \in N$ such that for each pair $x,y \in X$ there is a choice of $n=n(x, y)$, $m=m(x, y) \in N$ and $u,v \in \{x, y\}$, for which

$$(*') \qquad d(g^k x, g^k y) \leq P(d(g^n u, g^m v))$$

then $g^n(a) \to c$ for some $c \in X$. Moreover, if $x_o \in X$ and $O_g(x_o)$ is bounded, then $g^n(x_o) \to c$. If g is continuous at c, c is the unique fixed point of g.

PROOF. Let $H = O_g$. Note that $H \subseteq C_{g^k}$, $g(H(a)) = g(O_g(a)) \subseteq O_g(a)$, and that $g^k(H(a)) \subseteq H(a)$. Since d is a metric, d is actually uniformly continuous [3]. Thus, $g^{kn}(a) \to c$ as $n \to \infty$ for some $c \in X$, by Theorem 2.1 applied to g^k. If g is continuous at c, each $g^n \in H(= O_g)$ is continuous at c. So, as an element of H, $g(c) = c$ by Theorem 2.1..

We have yet to prove that $g^n(a) \to c$ and that $g^n(x_o) \to c$ for x_o with $O_g(x_o)$ bounded. To see that $g^n(a) \to c$, let $\epsilon > 0$. Since $(g^k)^m(a) \to c$ as $m \to \infty$, $\exists$ $m_1 \in N$ such that

$$d(g^{km}(a), c) < \epsilon/2, \text{ for } m > m_1 \tag{2.10}$$

By Lemma 2.1 and (*') of the hypothesis, if $m \in N$, for each pair $x, y \in X$ there exist $r_m, s_m \in H$ and $u_m, v_m \in \{x, y\}$ such that

$$d((g^k)^m(x), (g^k)^m(y)) \leq P^m(d(r_m u_m, s_m v_m)). \tag{2.11}$$

Since $O_g(a)$ is bounded, $\exists$ $M \geq 0$ such that

$$d(r_m u_m, s_m v_m) \leq M \text{ if } r_m u_m \text{ and } s_m v_m \text{ are in } O_g(a). \tag{2.12}$$

Now $P^m(M) \to 0$ as $m \to \infty$, so we can choose $m_o \in N$ such that

$$m_o > m_1 \text{ and } P^{m_o}(M) < \epsilon/2. \tag{2.13}$$

Let $n > km_o$. Then $n = km_o + t_n$ for some $t_n \in N$, and (2.11) implies

$$d(g^n(a), g^{km_o}(a)) = d(g^{km_o}(g^{t_n}(a)), g^{km_o}(a)) \leq P^{m_o}(d(r_{m_o}u_{m_o}, s_{m_o}v_{m_o})) \tag{2.14}$$

where $r_{m_o} \in H = O_g$, $u_{m_o} \in \{a, g^{t_n}(a)\}$, so that $r_{m_o}u_{m_o} \in O_g(a)$. Similarly, $s_{m_o}v_{m_o} \in O_g(a)$. Therefore, (2.12) implies that

$d(r_{m_o}u_{m_o}, s_{m_o}v_{m_o}) \leq M$, and since P is nondecreasing, we have

$P^{m_o}(d(r_{m_o}u_{m_o}, s_{m_o}v_{m_o})) \leq P^{m_o}(M) < \epsilon/2$, by (2.13). Thus (2.14) implies

$$d(g^n(a), g^{km_o}(a)) < \epsilon/2.$$

But then (2.10) with $m = m_o$ and the triangle inequality imply that $d(g^n(a), c) < \epsilon$, since $m_o > m_1$. We therefore conclude that $g^n(a) \to c$

If $x_o \in X$ such that $O_g(x_o)$ is bounded, the above argument shows us that $\exists\, p \in X$ such that $g^n(x_o) \to p$. To see that $p = c$, first observe that $S = O_g(x_o) \cup O_g(a)$ is also bounded since d is a metric; i.e., $\exists\, M_o \geq 0$ such that $d(x, y) \leq M_o$ if $x, y \in S$. We can therefore apply (*′) and Lemma 2.1 as before to conclude that

$$d(c, p) = \lim_{m \to \infty} d(g^{km}(a), g^{km}(x_o)) \leq \lim_{m \to \infty} P^m(M_o) = 0. \ \square$$

The following example shows that the hypothesis in Corollary 2.3 that the orbit $O_g(a)$ be bounded for at least one $a \in X$ is indeed necessary.

EXAMPLE 2.2. Let $X = [1, \infty)$, $P(t) = t/2$ for $t \in [0, \infty)$, $d(x, y) = |x - y|$ and $g(x) = 3x$ for $x,y \in X$. Then $P^n(t) = t/2^n \to 0$ as $n \to \infty$, $g:X \to X$ and $d(gx, gy) = |gx - gy| = 3|x - y| \leq 9/2|x - y| = 1/2\,|9x - 9y| = 1/2\,|g^2x - g^2y| = P(d(g^2x, g^2y))$. But since $g^n(x) = 3^nx \to \infty$ for each $x \in X$, $O_g(x)$ is bounded for no $x \in X$ and g has no fixed point.

In [4] Rhoades and Watson introduced the concept of a "generalized contraction".

DEFINITION 2.1 Let (X, d) be a metric space. A function $f:X \to X$ is a *generalized contraction* (with respect to Q) if $\exists\, p,q \in N$ such that for all $x,y \in X$

(i) $\quad d(f^px, f^qy) \leq Q(M(x, y))$,

where

$$M(x, y) = \max\{d(f^ix, f^jy), d(f^ix, f^{i'}x), d(f^jy, f^{j'}y) : 0 \leq i, i' \leq p,\ 0 \leq j, j' \leq q\}.$$

(Q is a nondecreasing function $Q: [0, \infty) \to [0, \infty)$ such that $Q(s) < s$ for $s > 0$.)

NOTE 2.2. Jachymski [5] studied the relation (i) and observed that it satisfies

(ii) $\quad d(f^rx, f^ry) \leq Q(\max\{ d(f^iu,f^jv) : 0 \leq i,j \leq r \text{ and } u,v \in \{x, y\}\})$

where $r = \max\{p, q\}$, since Q is nondecreasing. But (ii), and hence (i), satisfy the relation (*′) in Corollary 2.3. In fact, the following theorem by Jachymski – except the last sentence therein – is a consequence of Corollary 2.3. This last sentence refers to (9) which is essentially (i) above with the restriction that either $i, i' \in \{0, p\}$ or $j, j' \in \{0, q\}$.

THEOREM 4. ([5]) Let f be a generalized contraction and let (X, d) be f-orbitally complete. If $\lim_{n \to \infty} Q^n(s) = 0$ for $s \in [0, \infty)$ and there exists a point $x_o \in X$ with a bounded orbit, then the sequence $\{f^nx_o\}$ converges to some $z \in X$. Moreover, for any $x \in X$ with a bounded orbit, $f^nx \to z$. Furthermore, if f satisfies (9) z is the unique fixed point of f.

The following example shows that if we use the more general contractive property (*′) of Corollary 2.3, continuity at c or restrictions of the ilk found in (9) of Theorem 4 are needed to ensure that the point c (or z) is a fixed point.

EXAMPLE 2.3. Let $X = [0, 1]$ and let $d(x, y) = |x - y|$. Define $g: X \to X$ by $g(x) = \frac{1}{2}(x+1)$ for $x \in [0, 1)$ and $g(1) = \frac{1}{2}$. Then it is easy to see that $g^n(x) = (x - 1)2^{-n}+1$ $(x \neq 1)$, and $g^n(1) = 1 - 2^{-n}$ for $n \in N$. Thus, $g^n(a) \to 1$ for any $a \in X$. Since X is bounded, $O_g(a)$ is bounded for each $a \in X$. Thus, to see that the hypothesis of Corollary 2.3 is satisfied, we need only to verify that (*′) holds. A check shows that

$$d(g^2x, g^2y) = \tfrac{1}{2}\, d(gx, gy) \quad \text{for all } x,y \in X. \tag{2.15}$$

Thus, (*′) holds trivially with $k=2$, $n = m = 1$, and $u = x$, $v= y$ for all x,y in $[0, 1]$. However, $g^n(a) \to 1$ for any a, but g is not continuous at 1 and 1 is not a fixed point of g. Note that property (9) of Jachymski's Theorem 4. does not hold since in this instance $p = q = 2$, and g^2, g^0 do not appear in the right member of (2.15).

3. THE BOUNDED CASE. The following is an example of a function g and a family H which satisfy the hypothesis of Theorem 2.1 and for which the set F_g – the set of fixed points of g – is not bounded We then consider the significance of this phenomenon.

EXAMPLE 3.1. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ for $x,y \in X$. Let $g(x) = x$ for $x \in X$; i.e., g is the identity map. So $F_g = [0, \infty)$, and is unbounded. Let $P(t) = t/2$ for $t \in [0, \infty)$ and define $h_n(x) = nx$ for $x \in X$ and $n \in N$. If $H = \{h_n: n \in N\}$, and $h_n, h_m \in H$, then $h_nh_m(x) = h_n(h_m(x)) = h_n(mx) = (nm)x = h_{nm}(x)$. Thus $h_nh_m = h_{nm} \in H$, so that H is indeed a semi-group. Since g is the identity, the conditions $H \subseteq C_g$ and $g(H(a)) \subseteq H(a)$ (for any $a \in X$) are satisfied trivially. Moreover, $d(gx, gy) = |x - y| \leq \frac{3}{2}|x - y| = \frac{1}{2}|3x - 3y| = P(d(h_3x, h_3y))$, so that (*) and hence the hypothesis of Theorem 2.1 is satisfied. a=0 is the unique fixed point for g and H, but g has an infinitude of other fixed points.

In the remainder of the paper, if (X; d) is a T_2 semi-metric space and $g:X \to X$, we shall say that *g has property P relative to a semi-group H of self maps of X* iff for each pair $x, y \in X$ $\exists\, r, s \in H$ and $u, v \in \{x, y\}$ such that

(*) $d(gx, gy) \leq P(d(ru, sv))$.

NOTE 3.1. If a function $g:X \to X$ has property P relative to a semi-group H of self maps of X for which $H \subseteq C_g$, Lemma 2.1 implies that if $n \in N$, for any pair $x,y \in X$ there exist $r_n, s_n \in H$ and $u_n, v_n \in \{x, y\}$ such that

(**) $d(g^nx, g^ny) \leq P^n(d(r_nu_n, s_nv_n))$

PROPOSITION 3.1. Let (X; d) be a T_2 semi-metric space and let $g:X \to X$. Suppose H is a semigroup of self maps of X such that $H \subseteq C_g$. If g has property P relative to H and F_g is nonempty and bounded, then

(i) F_g is a singleton $\{c\}$, and (ii) $c = g(c) = h(c)$ for all $h \in H$.

PROOF. To prove (i), we first note that $h(F_g) \subseteq F_g$ for all $h \in H$. For if $h \in H$ and $a = g(a)$, then $g(h(a)) = h(g(a)) = h(a)$, so that $h(a) \in F_g$. Moreover, since F_g is bounded, $\exists\, M \geq 0$ such that $d(a, c) \leq M$ for $a,c \in F_g$.

Now by hypothesis, $\exists\, c \in F_g$. We assert that c is unique. For suppose $a \in F_g$. Then Note 3.1 says that we can choose $u_nv_n \in \{a, c\}$ and $r_n,s_n \in H$ such that

$$d(a, c) = d(g^n(a), g^n(c)) \leq P^n(d(r_nu_n, s_nv_n)). \tag{3.1}$$

But since $h(F_g) \subseteq F_g$ for $h \in H$, and since $a, c \in F_g$, $r_nu_n, s_nv_n \in F_g$. So by the above,

$$d(r_nu_n, s_nv_n) \leq M.$$

Therefore, since P and hence P^n is nondecreasing, (3.1) yields:

$$d(a, c) \leq P^n(M) \quad \text{for } n \in N. \tag{3.2}$$

Since $P^n(M) \to 0$ as $n \to \infty$, (3.2) implies that $a = c$.

(ii) is an immediate consequence of (i), since $(h \in H) \Rightarrow h(F_g) \subseteq F_g$. Therefore, if $h \in H$, $h(c) \in \{c\}$; i.e, $h(c) = c$. □

COROLLARY 3.1. Let $(X; d)$ be a bounded and complete T_2 semi-metric space. Let $g: X \to X$ and let H be a semi-group of self maps of X such that $H \subset C_g$ and $g(H(a)) \subset H(a)$ for $a \in X$ If g has property P relative to H, then for each $x \in X$ $\exists$ a point $c_x \in X$ such that $g^n(x) \to c_x$. If g is continuous at one such c_x, then F_g is a singleton, $\{c\}$, and $c_x = c$ for all x. Moreover, $c = h(c)$ for all $h \in H$.

PROOF. Since X is bounded, $H(x)$ is bounded for each $x \in X$. Therefore, Theorem 2.1 implies that $g^n(x) \to c_x$ for some $c_x \in X$. If g is continuous at one such c_x, then $g(c_x) = c_x$. But then F_g is bounded and nonempty, so that Proposition 3.1 implies that $F_g = \{c\}$, a singleton, and that $c = h(c)$ for all $h \in H$. □

Corollary 3.1 has Theorem 1.[6] by Browder and a result by Zitarosa [7] on contractive self maps of a bounded complete metric space as special cases with $H = \{i_d\}$.

The proof of our next theorem, as did the proof of Corollary 2.3, requires that the union of two bounded sets be bounded. So we again need a metric. Also, observe that in Example 3.1 the set $H(a) = \{ na \cdot n \in N \}$ is unbounded for $a \neq 0$.

THEOREM 3.1. Let g be a self map of a metric space (X, d) which is g-orbitally complete. Suppose that H is a semi-group of self maps of X such that $H \subset C_g$ and that g has property P relative to H. If $g(H(a)) \subset H(a)$ and $H(a)$ is bounded for all $a \in X$, then g has a contractive point c; i.e., $g^n(x) \to c$ for all $x \in X$. Moreover, $c = g(c) = h(c)$ for all $h \in H$ if g is continuous at c

PROOF. Let $a \in X$. By Theorem 2.1, since $H(a)$ is bounded, $g^n(a) \to c$ for some $c \in X$ But $g^n(x) \to c_x \in X$ for any $x \in X$ since $H(x)$ is bounded. We show $c_x = c$ for any $x \in X$. To this end, let $x \in X$. Then $H(x) \cup H(a)$ is bounded. Since g has property P relative to H and $H \subset C_g$, Note 3.1 implies that for all $n \in N$ we have:

$$d(g^n(a), g^n(x)) \leq P^n(d(r_n u_n, s_n v_n)), \tag{3.3}$$

where $r_n, s_n \in H$ and $u_n v_n \in \{a, x\}$; hence, $r_n u_n, s_n v_n \in H(a) \cup H(x)$ for $n \in N$. But $H(a) \cup H(x)$ is bounded, and so $\exists\, M \geq 0$ such that $d(r_n u_n, s_n v_n) \leq M$ for all n. Thus, $P^n(d(r_n u_n, s_n v_n)) \leq P^n(M) \to 0$ as $n \to \infty$. Hence (3.3) and the above imply:

$$d(c, c_x) = \lim_{n\to\infty} d(g^n(a), g^n(x)) = 0.$$

Thus $c = c_x$. If g is continuous at c, then $c = g(c)$ But since $g^n(x) \to c$ for all x, c is the only fixed point of g. Therefore, $c = h(c)$ for all $h \in H$ by Proposition 3.1. □

COROLLARY 3.2. Let g be a self map of a metric space (X, d) which is g-orbitally complete. Suppose that $O_g(x)$ is bounded for all $x \in X$. If g has property P relative to O_g, then $\exists$ $z \in X$ such that $g^n(x) \to z$ for all $x \in X$. z is a unique fixed point of g iff the function $F(x) = d(x, g(x))$ is lower semicontinuous at z.

PROOF. Since trivially, $O_g \subset C_g$ and $g(O_g(x)) \subset O_g(x)$ for all $x \in X$, Corollary 3.2 follows immediately from Theorem 3.1 (with the observation that the last statememt is a well known consequence of "$g^n(x) \to z$") □

We conclude with a theorem (rephrased) by Jachymski [5] which generalizes theorems of Rhoades and Watson [4], and which is a consequence of our Corollary 2.3.

THEOREM 2. [5] Assume that f is a generalized contraction, and (X, d) is f-orbitally complete. If $\lim_{n\to\infty} Q^n(s) = 0$ for $s \in [0, \infty)$ and $\lim_{s\to\infty}(s - Q(s)) = \infty$, then there exists $z \in X$ such that $f^n x \to z$ for any $x \in X$. z is a unique fixed point of f if and only if the function $F(x) = d(x, f(x))$ is lower semicontinuous at z.

To see that Theorem 2. [5] does indeed follow from Corollary 2.3, first observe that (as noted before) a generalized contraction satisfies (*') of Corollary 2.3. Moreover, Lemma 3. [5] tells us that if

$\lim_{s\to\infty}(s - Q(s)) = \infty$, then the orbits $O_f(x)$ are bounded for all $x \in X$. Therefore, Corollary 2.3 assures us that $\exists\, z \in X$ such that $f^n(x) \to z$ for all $x \in X$. The assertion that z is the unique fixed point of f follows as in the proof of Corollary 3.2

4. **RETROSPECT.** In closing we emphasize the general nature and utility of the semi-groups H of self maps employed. For example, in Corollary 2 2 H is any family of continuous self maps closed under composition with H(a) bounded at some one point $a \in X$ – no commutativity requirements are imposed. Corollary 2.3 demonstrates the utility of options provided by H in letting $H = O_g$. And Example 1.4 indicates how, when given a map $g:X \to X$, we can generate semigroups H which satisfy $g(H(a)) \subset H(a)$.

Note also that Hausdorff semi-metric spaces (X; d) generalize metric spaces, even if the semi-metric d is lower semi-continuous. In fact, Cook [8] provides an example of a semi-metric space with a continuous semi-metric which is developable but not normal, and hence not a metric

A final comment. The semi-group C_g has been used to some extent in fixed point research See, e.g., [9, 10, 11].

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