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ON COMPLETE CONVERGENCE FOR RANDOMLY INDEXED SUMS FOR A CASE WITHOUT IDENTICAL DISTRIBUTIONS

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ABSTRACT. In this note we extend the complete convergence for randomly indexed sums given by Klesov (1989) to nonidentical distributed random variables.

KEY WORDS AND PHRASES: complete convergence, random indexed sums, regular cover, array of rowwise independent random variables.
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1. INTRODUCTION AND PRELIMINARIES

The following concept of complete convergence was given by Hsu and Robbins [1].

DEFINITION 1.1. A sequence $\{X_n,\ n \geq 1\}$ of random variables converges completly to the constant C if

$$\sum_{n=1}^{\infty} P\big[|X_n - C| \geq \varepsilon\big] < \infty, \quad \forall \varepsilon > 0.$$

The main result of Hsu and Robbins [1] states that for a sequence $\{X_n,\ n \geq 1\}$ of i.i.d. random variables with zero expectation and $EX_1^2 < \infty$, we have

$$\sum_{n=1}^{\infty} P\big[|S_n| \geq n\varepsilon\big] < \infty, \quad \forall \varepsilon > 0, \tag{1.1}$$

where $S_n = \sum_{k=1}^{n} X_k$, i.e. the sequence of arithmetic means S_n/n, $n \geq 1$, completly convergence to 0. Erdös [2] proved the converse statement.

Extensions and generalizations of those results were summarized by A. Gut [3]. Extensions of (1.1) to randomly indexed sums of i.i.d. random variables one can find in Szynal [4], Gut

[5], Zhidong and Chun [6], Adler [7] and Klesov [8]. Some results concerning complete convergence for randomly indexed sums of nonidenticaly distributed random variables were given by Kuczmaszewska and Szynal [9], [10].

In this note we extend results on the complete convergence for randomly indexed sums in spirit of Gut [5] and Klesov [8] to nonidentical distributed random variables.

We use the following concept of regular cover of (the distribution of) a random variable.

DEFINITION 1.2. (See Pruss [11]). Let $X_1, \dots, X_n$ be random variables and let ξ be a random variable possible defined on a different probability space. Then $X_1, \dots, X_n$ are said to be a regular cover of (the distribution of) ξ provided we have

$$E\big[G(\xi)\big] = \frac{1}{n}\sum_{k=1}^{n} E\big[G(X_k)\big], \tag{1.2}$$

for any measurable function G for which both sides make sense. If $X_1, \dots, X_n$ are in addition independent, then we say they form an independent regular cover of ξ.

2. RESULTS.

The following theorem contains as a particular case the main result of Klesov [8].

THEOREM 2.1. Let $\{X_{nk},\ n \geq 1,\ k \geq 1\}$ be an array of rowwise independent random variables with $EX_{nk} = 0, E|X_{nk}|^r < \infty$, for some $r \geq 1$, and $n \geq 1,\ k \geq 1$, such that $X_{n1}, X_{n2}, \dots, X_{nk}, n \geq 1,\ k \geq 1$, form an independent regular cover of a random variable ξ with $E\xi = 0, E|\xi|^r < \infty$, for some $r \geq 1$. Suppose that $\{\nu_k,\ k \geq 1\}$ is a sequence of positive integer-valued random variables. Then for $S_{\nu_n} = \sum_{k=1}^{\nu_n} X_{nk}$ we have:

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\big[|S_{\nu_n}| \geq \varepsilon \nu_n^{\alpha}\big] < \infty, \quad \forall \varepsilon > 0, \tag{2.1}$$

for $\alpha > 1/2$, $\alpha r > 1$ and $\beta \geq 1$, whenever

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\big[\nu_n < n^{\beta}\big] < \infty, \tag{2.2}$$

and (2.1) holds true for $\alpha > 1/2$, $\alpha r > 1$, and $0 < \beta < 1$, whenever additionally with (2.2) the condition

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\big[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon \nu_n^{\alpha}\big] < \infty, \quad \forall \varepsilon > 0, \tag{2.3}$$

is satisfied.

PROOF. Firstly we prove that (2.2) and (2.3) with $\alpha > \frac{1}{2}$, $\alpha r > 1$, and $\beta > 0$ imply (2.1). Taking into account

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\big[|S_{\nu_n}| \geq \varepsilon \nu_n^{\alpha}\big] \leq \sum_{n=1}^{\infty} n^{\alpha r-2} P\big[|S_{\nu_n}| \geq \varepsilon \nu_n^{\alpha}, \nu_n \geq n^{\beta}\big] + \sum_{n=1}^{\infty} n^{\alpha r-2} P\big[\nu_n < n^{\beta}\big]$$

we see that we need only to show that

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\big[|S_{\nu_n}| \geq \varepsilon \nu_n^{\alpha}, \nu_n \geq n^{\beta}\big] < \infty. \tag{2.4}$$

Let $\delta > \frac{(\alpha r-1)}{\beta}$, $\frac{1}{(\alpha r)} < \gamma < 1$ and q be a positive integer such that $q > \frac{(1+\delta)}{(\alpha r-1)}$. Define the sets (cf. Klesov [8]):

$$B_n^{(1)} = \big[\exists k \le \nu_n : |X_{nk}| \ge \frac{\varepsilon \nu_n^{\alpha}}{q}\big],$$
$$B_n^{(2)} = \big[\exists \text{ at least } q \text{ indices } k \le \nu_n : |X_{nk}| \ge \nu_n^{\gamma\alpha}\big],$$
$$B_n^{(3)} = \big[\big|\sum_{k\le\nu_n} X_{nk} I[|X_{nk}| < \nu_n^{\gamma\alpha}]\big| \ge \frac{\varepsilon \nu_n^{\alpha}}{q}\big],$$

where $I[A]$ is the indicator function of an event A. Taking into account that

$$\big[|S_{\nu_n}| \ge \varepsilon \nu_n^{\alpha}\big] \subseteq B_n^{(1)} \cup B_n^{(2)} \cup B_n^{(3)}$$

we note that (2.4) will be proved if we show that

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\big[B_n^{(i)} \cap [\nu_n \ge n^{\beta}]\big] < \infty, \quad i = 1,2,3. \tag{2.5}$$

For $i = 1$ we have

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\big[B_n^{(1)} \cap [\nu_n \ge n^{\beta}]\big] \le \sum_{n=1}^{\infty} n^{\alpha r-2} P\big[\exists k \le \nu_n : |X_{nk}| \ge (\varepsilon \nu_n^{\alpha})/q\big]$$

$$\le \sum_{n=1}^{\infty} n^{\alpha r-2} P\big[\max_{k\le\nu_n} |X_{nk}| \ge \varepsilon' \nu_n^{\alpha}\big], \quad \varepsilon' = \varepsilon/q.$$

In the case $i = 2$ we state that

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\big[B_n^{(2)} \cap [\nu_n \ge n^{\beta}]\big]$$

$$\le \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j=1}^{\infty} P\big[B_n^{(2)} \cap [\nu_n = j], \nu_n \ge n^{\beta}\big]$$

$$\le \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j=1}^{\infty} \sum_{1\le k_1\le k_2\le\ldots\le k_q\le j} P\big[\nu_n = j, |X_{n1}| \ge j^{\gamma\alpha}, \ldots, |X_{nk_q}| \ge j^{\gamma\alpha}, \nu_n \ge n^{\beta}\big]$$

$$\le \sum_{n=1}^{\infty} n^{\alpha r-2-\beta\delta} \sum_{j=1}^{\infty} j^{\delta-qr\gamma\alpha} \sum_{1\le k_1\le k_2\le\ldots\le k_q\le j} E|X_{n1}|^r \ldots E|X_{nk_q}|^r I[\nu_n = j, \nu_n \ge n^{\beta}]$$

$$\le \sum_{n=1}^{\infty} n^{\alpha r-2-\beta\delta} \sum_{j=1}^{\infty} j^{\delta-qr\gamma\alpha} \sum_{1\le k_1\le k_2\le\ldots\le k_q\le j} E|X_{n1}|^r \ldots E|X_{nk_q}|^r$$

$$= \sum_{n=1}^{\infty} n^{\alpha r-2-\beta\delta} \sum_{j=1}^{\infty} j^{\delta-qr\gamma\alpha} \sum_{k_q=q}^{j} E|X_{nk_q}|^r \sum_{k_{q-1}=q-1}^{k_q-1} E|X_{nk_{q-1}}|^r \ldots \sum_{k_1=1}^{k_2-1} E|X_{nk_1}|^r.$$

Now using the assumption (1.2) we get

$$\sum_{n=1}^{\infty} n^{\alpha r-2-\beta\delta} \sum_{j=1}^{\infty} j^{\delta-qr\gamma\alpha} \sum_{k_q=q}^{j} E|X_{nk_q}|^r \sum_{k_{q-1}=q-1}^{k_q-1} E|X_{nk_{q-1}}|^r \ldots \sum_{k_1=1}^{k_2-1} E|X_{nk_1}|^r$$

$$\le \sum_{n=1}^{\infty} n^{\alpha r-2-\beta\delta} E|\xi|^r \sum_{j=1}^{\infty} j^{\delta-qr\gamma\alpha+1} \sum_{k_q=q}^{j} E|X_{nk_q}|^r \sum_{k_{q-1}=q-1}^{k_q-1} E|X_{nk_{q-1}}|^r \ldots \sum_{k_1=2}^{k_3-1} E|X_{nk_2}|^r$$

$$\leq \sum_{n=1}^{\infty} n^{\alpha r-2-\beta\delta}\big(E|\xi|^r\big)^q \sum_{j=1}^{\infty} j^{\delta+q(1-r\gamma\alpha)} < \infty$$

as $\delta > \frac{\alpha r-1}{\beta}$, $\gamma > \frac{1}{\alpha r}$ and $q > \frac{1+\delta}{\gamma\alpha r-1}$.

To prove (2.5) for $i = 3$ we write

$$Y_{kj}^n = X_{nk}I[|X_{nk}| < j^{\gamma\alpha}] - EX_{nk}I[|X_{nk}| < j^{\gamma\alpha}],$$

$1 \leq k \leq j, j \geq 1$ and $n \geq 1$.

Then we see that

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\big[B_n^{(3)} \cap [\nu_n \geq n^\beta]\big] \tag{2.6}$$

$$\leq \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j\geq[n^\beta]} P\Big[\Big|\sum_{k\leq j} X_{nk}I[|X_{nk}| < j^{\gamma\alpha}]\Big| \geq \frac{\varepsilon j^\alpha}{q}, \nu_n = j\Big]$$

$$\leq \text{const} \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j\geq[n^\beta]} j^{-\alpha s} E\Big|\sum_{k\leq j} X_{nk}I[|X_{nk}| < j^{\gamma\alpha}]\Big|^s$$

$$\leq \text{const}\Big[\sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j\geq[n^\beta]} j^{-\alpha s} E\Big|\sum_{k\leq j} Y_{kj}^n\Big|^s + \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j\geq[n^\beta]} j^{-\alpha s}\Big|\sum_{k\leq j} EX_{nk}I[|X_{nk}| < j^{\gamma\alpha}]\Big|^s\Big]$$

for every $s > 0$ and a positive constant c.

We note that the second term in the last inequality is finite as

$$\sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j\geq[n^\beta]} j^{-\alpha s}\Big|\sum_{k\leq j} EX_{nk}I[|X_{nk}| < j^{\gamma\alpha}]\Big|^s \tag{2.7}$$

$$\leq \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j\geq[n^\beta]} j^{-\alpha s}\Big(\sum_{k\leq j} \frac{E|X_{nk}|^r I[|X_{nk}| \geq j^{\gamma\alpha}]}{(j^{\gamma\alpha})^{r-1}}\Big)^s$$

$$\leq \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j\geq[n^\beta]} j^{-\alpha s-\gamma\alpha(r-1)s}\Big(\sum_{k\leq j} E|X_{nk}|^r\Big)^s$$

$$\leq \text{const} \sum_{j=1}^{\infty} j^{-\alpha s+c-\gamma\alpha(r-1)s+s}\big(E|\xi|^r\big)^s = \text{const}\big(E|\xi|^r\big)^s \sum_{j=1}^{\infty} j^{c-s(\alpha+\alpha r\gamma-\gamma\alpha-1)} < \infty$$

for $s > \frac{c}{\alpha(1-\gamma)+\gamma\alpha r-1}$.

Now we can write

$$E\Big|\sum_{k\leq j} Y_{kj}^n\Big|^s = \int_0^\infty z^{s-1} P\Big[\Big|\sum_{k\leq j} Y_{kj}^n\Big| \geq z\Big]dz \tag{2.8}$$

$$= \int_0^{j^{\gamma\alpha}} z^{s-1} P\Big[\Big|\sum_{k\leq j} Y_{kj}^n\Big| \geq z\Big]dz + \int_{j^{\gamma\alpha}}^\infty z^{s-1} P\Big[\Big|\sum_{k\leq j} Y_{kj}^n\Big| \geq z\Big]dz$$

$$\leq j^{\gamma\alpha s} + \int_{j^{\gamma\alpha}}^\infty z^{s-1} P\Big[\Big|\sum_{k\leq j} Y_{kj}^n\Big| \geq z\Big]dz.$$

But the Fuk-Nagaev inequality (cf. Fuk and Nagaev [12]):

$$P\Big[\Big|\sum_{i=1}^{n} X_i\Big| \geq x\Big]$$

$$\leq 2\Big(\sum_{i=1}^{n} P\big[|X_i| \geq \eta x\big] + \frac{1}{(\eta x)^t}\sum_{i=1}^{n}\int_0^{\eta x}|u|^t dF_{X_i}(u) + exp\Big(-\frac{(1-\eta)^2 x^2}{2e^t\sum_{i=1}^{n} EX_i^2}\Big)\Big),$$

where $t \geq 2$, $\eta = \frac{t}{t+2}$, allows us to show that

$$\int_{j^{\gamma\alpha}}^{\infty} z^{s-1}P\big[|\sum_{k\leq j} Y_{kj}^n| \geq z\big]dz \tag{2.9}$$

$$\leq 2\Big(\sum_{k=1}^{j}\int_{j^{\gamma\alpha}}^{\infty} z^{s-1}P\big[|Y_{kj}^n| \geq \eta z\big]dz + \frac{2}{\eta^t}\sum_{k=1}^{j}\int_{j^{\gamma\alpha}}^{\infty} z^{s-t-1}\int_0^{\eta z}|u|^t dF_{Y_{kj}^n}(u)$$

$$+\int_{j^{\gamma\alpha}}^{\infty} z^{s-1}exp\Big(-\frac{(1-\eta)^2 z^2}{2e^t\sum_{k=1}^{j}E(Y_{kj}^n)^2}\Big)dz\Big).$$

Now we see that

$$\sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{j\geq[n^\beta]} j^{-\alpha s}\sum_{k\leq j}\int_{j^{\gamma\alpha}}^{\infty} z^{s-1}P\big[|Y_{kj}^n| \geq \eta z\big]dz \tag{2.10}$$

$$= \Big(\frac{1}{\eta}\Big)^{s-1}\sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{j\geq[n^\beta]} j^{-\alpha s}\sum_{k=1}^{j} E|Y_{kj}^n|^s \leq \text{const}\sum_{j=1}^{\infty} j^{-\alpha s+c} j(j^{\gamma\alpha})^s < \infty$$

for $s > \frac{c+2}{\alpha(1-\gamma)}$.

Moreover, using the assumption on a regular cover (cf. Definition 1.2), we have

$$\sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{j\geq[n^\beta]} j^{-\alpha s}\sum_{k=1}^{j}\int_{j^{\gamma\alpha}}^{\infty} z^{s-t-1}\Big(\int_0^{\eta z}|u|^t dF_{Y_{kj}^n}(u)\Big)dz \tag{2.11}$$

$$\sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{j\geq[n^\beta]} j^{-\alpha s}\sum_{k=1}^{j}\int_{j^{\gamma\alpha}}^{\infty} z^{s-t-1}\big(E|Y_{kj}^n|^t I[|Y_{kj}^n| < z]\big)dz$$

$$\leq \text{const}\sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{j\geq[n^\beta]} j^{-\alpha s}\sum_{k=1}^{j} E|X_{nk}|^t I[|X_{nk}| < j^{\gamma\alpha}]j^{\gamma\alpha(s-t)}$$

$$\leq \text{const}\sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{j\geq[n^\beta]} j^{-\alpha s+\gamma\alpha(s-r)}\sum_{k=1}^{j} E|X_{nk}|^r$$

$$\leq \text{const}E|\xi|^r\sum_{j=1}^{\infty} j^{-\alpha s+c+\gamma\alpha(s-r)+1} < \infty$$

for $s > \frac{c+2-\gamma\alpha r}{r(1-\alpha)}$.

Further on, we note that

$$\sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{j\geq[n^\beta]} j^{-\alpha s}\int_{j^{\gamma\alpha}}^{\infty} z^{s-1}exp\Big(-\frac{(1-\eta)^2 z^2}{2e^t\sum_{k=1}^{j}E(Y_{kj}^n)^2}\Big)dz \tag{2.12}$$

$$\leq \text{const}\sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{j\geq[n^\beta]} j^{-\alpha s}\Big(\sum_{k=1}^{j} E(Y_{kj}^n)^2\Big)^{s/2}\int_0^{\infty} y^{s/2-1}e^{-y}dy$$

$$\leq \text{const}\sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{j\geq[n^\beta]} j^{-\alpha s}\Big(\sum_{k=1}^{j} E(Y_{kj}^n)^2\Big)^{s/2}.$$

Assume now that $r \geq 2$. Then we have

$$\sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j\geq[n^\beta]} j^{-\alpha s}\Big(\sum_{k=1}^{j} E(Y_{kj}^n)^2\Big)^{s/2} \leq \text{const} \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j\geq[n^\beta]} j^{-\alpha s}\big(jE|\xi|^2\big)^{s/2}$$

$$\leq \text{const} \sum_{j=1}^{\infty} j^{-\alpha s+c+s/2} < \infty \tag{2.13}$$

for $s > \frac{c+1}{\alpha-1/2}$.

Similarly it can be proved that for $r < 2$

$$\sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j\geq[n^\beta]} j^{-\alpha s}\Big(\sum_{k=1}^{j} EY_{kj}^2\Big)^{s/2} \leq \text{const} \sum_{j=1}^{\infty} j^{-s[\alpha-1/2-\gamma\alpha(2-r)/2]+c} < \infty \tag{2.14}$$

whenever $s > \frac{c+1}{\alpha-1/2+\gamma\alpha(2-r)/2}$ and γ is such that $\gamma < \frac{2\alpha-1}{2-r}$.

Collecting the estimates (2.7) - (2.14) we see that the series in (2.6) converges which completes the proof of (2.1) for $\beta > 0$.

But for the stronger requirement $\beta \geq 1$ we note that the condition (2.3) is fulfilled under the assumption $E|X_{nk}|^r < \infty$, $r \geq 1$, $k \geq 1$, $n \geq 1$.

Indeed, we see that

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\big[\max_{k\leq\nu_n} |X_{nk}| \geq \varepsilon\nu_n^\alpha\big]$$

$$\leq \sum_{n=1}^{\infty} n^{\alpha r-2} P\big[\nu_n < n^\beta\big] + \sum_{n=1}^{\infty} n^{\alpha r-2} P\big[\max_{k\leq\nu_n} |X_{nk}| \geq \varepsilon\nu_n^\alpha, \nu_n \geq n^\beta\big],$$

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\big[\max_{k\leq\nu_n} |X_{nk}| \geq \varepsilon\nu_n^\alpha, \nu_n \geq n^\beta\big]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} P\big[\max_{k\leq\nu_{2^m}} |X_{2^m k}| \geq \varepsilon\nu_{2^m}^\alpha, \nu_{2^m} \geq (2^m)^\beta\big]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} \sum_{j=m}^{\infty} P\big[\max_{k\leq\nu_{2^m}} |X_{2^m k}| \geq \varepsilon\nu_{2^m}^\alpha, (2^j)^\beta \leq \nu_{2^m} < (2^{j+1})^\beta\big]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} \sum_{j=m}^{\infty} P\big[\max_{k\leq(2^{j+1})^\beta} |X_{2^m k}| \geq \varepsilon(2^j)^{\alpha\beta}\big]$$

$$\leq \text{const} \sum_{m=1}^{\infty} P\big[\max_{k\leq(2^{m+1})^\beta} |X_{2^m k}| \geq \varepsilon(2^m)^{\alpha\beta}\big] \sum_{k=1}^{m} (2^k)^{\alpha r-1}$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} P\big[\max_{k\leq(2^{m+1})^\beta} |X_{2^m k}| \geq \varepsilon(2^m)^{\alpha\beta}\big]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} \sum_{k\leq(2^{m+1})^\beta} P\big[|X_{2^m k}| \geq \varepsilon(2^m)^{\alpha\beta}\big]$$

$$\leq \text{const} \sum_{m=1}^{\infty} \sum_{k\leq(2^{m+1})^\beta} \frac{E|X_{2^m k}|^r}{(2^m)^{\alpha\beta r}}$$

$$= \text{const} E|\xi|^r \sum_{m=1}^{\infty} (2^m)^{\alpha r-1-\beta(\alpha r-1)} < \infty$$

for $\beta \geq 1$, which gives (2.3) and ends the proof of Theorem 2.1.

Now we note that the condition (2.3) ($0 < \beta < 1$) is fulfilled under a stronger moment condition than that of Theorem 2.1.

COROLLARY. Let $\{X_{nk},\ n \geq 1,\ k \geq 1\}$ be an array of rowwise independent random variables such that $X_{n1}, X_{n2}, \ldots, X_{nk},\ n \geq 1,\ k \geq 1$, form an independent regular cover of a random variable ξ, and assume that $EX_{nk} = 0$, $E|X_{nk}|^{\frac{\alpha r-1+\beta}{\alpha\beta}} < \infty$, $n \geq 1$, $k \geq 1$, $E\xi = 0$, and $E|\xi|^{\frac{\alpha r-1+\beta}{\alpha\beta}} < \infty$ for $r \geq 1$, $\alpha > 1/2$, $\alpha r > 1$, $0 < \beta < 1$.
If $\{\nu_n,\ n \geq 1\}$ is a sequence of positive integer-valued random variables such that

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\left[\nu_n < n^\beta\right] < \infty,$$

then for any given $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\left[|S_{\nu_n}| \geq \varepsilon \nu_n^\alpha\right] < \infty.$$

PROOF. It is enough to see that under the considered case the condition (2.3) is satisfied. Since

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\left[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon \nu_n^\alpha\right]$$

$$\leq \sum_{n=1}^{\infty} n^{\alpha r-2} P\left[\nu_n < n^\beta\right] + \sum_{n=1}^{\infty} n^{\alpha r-2} P\left[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon \nu_n^\alpha, \nu_n \geq n^\beta\right],$$

then we need only to note that

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\left[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon \nu_n^\alpha, \nu_n \geq n^\beta\right]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} P\left[\max_{k \leq \nu_{2^m}} |X_{2^m k}| \geq \varepsilon \nu_{2^m}^\alpha, \nu_{2^m} \geq (2^m)^\beta\right]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} \sum_{j=m}^{\infty} P\left[\max_{k \leq \nu_{2^m}} |X_{2^m k}| \geq \varepsilon \nu_{2^m}^\alpha, (2^j)^\beta \leq \nu_{2^m} < (2^{j+1})^\beta\right]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} \sum_{j=m}^{\infty} P\left[\max_{k \leq (2^{j+1})^\beta} |X_{2^m k}| \geq \varepsilon (2^j)^{\alpha\beta}\right]$$

$$\leq \text{const} \sum_{m=1}^{\infty} P\left[\max_{k \leq (2^{m+1})^\beta} |X_{2^m k}| \geq \varepsilon (2^m)^{\alpha\beta}\right] \sum_{k=1}^{m} (2^k)^{\alpha r-1}$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} P\left[\max_{k \leq (2^{m+1})^\beta} |X_{2^m k}| \geq \varepsilon (2^m)^{\alpha\beta}\right]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} \sum_{k \leq (2^{m+1})^\beta} P\left[|X_{2^m k}| \geq \varepsilon (2^m)^{\alpha\beta}\right]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1+\beta} P\left[|\xi|^{\frac{1}{\alpha\beta}} \geq \varepsilon (2^m)\right] \leq \text{const} E|\xi|^{\frac{\alpha r-1+\beta}{\alpha\beta}} < \infty.$$

Note that the moment condition of Corollary is close to optimal which shows the following statement.

THEOREM 2.2. Let $\{X_{nk},\ n \geq 1,\ k \geq 1\}$ be an array of rowwise independent random variables such that $X_{n1}, X_{n2}, \ldots, X_{nk},\ n \geq 1,\ k \geq 1$, form an independent regular cover of a random variable ξ, and assume that $EX_{nk} = 0$.
Then for $r \geq 1,\ \alpha > 1/2,\ \alpha r > 1,\ \beta > 0$, the convergence of the series

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\left[|S_{[n^\beta]}| \geq \varepsilon n^{\alpha\beta}\right] < \infty \tag{2.15}$$

implies $E|\xi|^{\frac{\alpha r-1+\beta}{\alpha\beta}} < \infty$.
PROOF. Let μ_n be a median of S_n, i.e. $\mu_n = \{t : P[S_n < t] \geq 1/2\}$. By the standard symmetrization inequalities (cf. Loève [13]) we have

$$P\left[|S_{[n^\beta]}| \geq \varepsilon n^{\alpha\beta}\right]$$

$$\geq \frac{1}{2} P\left[|S^s_{[n^\beta]}| \geq 2\varepsilon n^{\alpha\beta}\right] \geq \frac{1}{4} P\left[|S_{[n^\beta]} - \mu_{[n^\beta]}| \geq 2\varepsilon n^{\alpha\beta}\right]$$

$$\geq \frac{1}{4} P\left[S_{[n^\beta]} - \mu_{[n^\beta]} \geq 2\varepsilon n^{\alpha\beta}\right],$$

which by (2.15) gives

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\left[S_{[n^\beta]} - \mu_{[n^\beta]} \geq 2\varepsilon n^{\alpha\beta}\right] < \infty. \tag{2.16}$$

We note that $\tau_n = \sup\{\tau : P[\xi \geq \tau] \geq \frac{1}{4n^\beta}\}$. We note that $\tau_n \geq \tau_{n-1}$, and

$$P\left[\xi \geq \tau_n\right] \geq \frac{1}{4n^\beta},\ P\left[\xi \leq \tau_n\right] \geq 1 - \frac{1}{4n^\beta}. \tag{2.17}$$

If the τ_n are all negative then $P\left[\xi < 0\right] = 1$ so $E\left(\xi^+\right)^{\frac{\alpha r-1+\beta}{\alpha\beta}} = 0 < \infty$. Thus, assume that for n sufficiently large we have $\tau_n \geq 0$. Moreover, we note that by (2.17)

$$P\left[X_{nk} > \tau_n\right] \leq P\left[X_{n1} > \tau_1\right] + \ldots + P\left[X_{n[n^\beta]} > \tau_n\right] \tag{2.18}$$

$$\leq n^\beta P\left[\xi > \tau_n\right] = n^\beta\left(1 - P\left[\xi \leq \tau_n\right]\right) \leq \frac{1}{4}.$$

Furthemore, for $k \in \{1, \ldots, [n^\beta]\}$ define $\{\rho_{nk}, 1 \leq k \leq [n^\beta]\}$ with

$$\rho_{nk} = \sup\{\rho : P[S_{[n^\beta]} - X_{nk} \geq \rho] \geq \frac{1}{3}\}.$$

Then we have

$$P\left[S_{[n^\beta]} - X_{nk} \geq \rho_{nk}\right] \geq \frac{1}{3},\ P\left[S_{[n^\beta]} - X_{nk} \leq \rho_{nk}\right] \geq \frac{2}{3}. \tag{2.19}$$

Using the independence $S_{[n^\beta]} - X_{nk}$ and X_{nk}, (2.18) and (2.19) we get

$$P\left[S_{[n^\beta]} \leq \tau_n + \rho_{nk}\right] \geq P\left[X_{nk} \leq \tau_n,\ S_{[n^\beta]} - X_{nk} \leq \rho_{nk}\right]$$

$$= P\left[X_{nk} \leq \tau_n\right] P\left[S_{[n^\beta]} - X_{nk} \leq \rho_{nk}\right]$$

$$= \left(1 - P\left[X_{nk} > \tau_n\right]\right) P\left[S_{[n^\beta]} - X_{nk} \leq \rho_{nk}\right] \geq \frac{1}{2}.$$

Now using

$$T_{nk} := [X_{nk} > 2\varepsilon n^{\alpha\beta} + \tau_n],\ R_{nk} := [S_{[n^\beta]} - X_{nk} \geq \rho_{nk}]$$

we see that

$$P\big[S_{[n^\beta]} \geq 2\varepsilon n^{\alpha\beta} + \mu_{[n^\beta]}\big] \tag{2.20}$$

$$\geq P\big[S_{[n^\beta]} > 2\varepsilon n^{\alpha\beta} + \tau_n + \rho_{nk}\big] \geq P\Big[\bigcup_{k=1}^{[n^\beta]} (T_{nk} \cap R_{nk})\Big]$$

$$= \sum_{k=1}^{[n^\beta]} P\big[(T_{n1} \cap R_{n1})^c \cap \ldots \cap (T_{nk-1} \cap R_{nk-1})^c \cap (T_{nk} \cap R_{nk})\big]$$

$$\geq \sum_{k=1}^{[n^\beta]} P\big[T_{n1}^c \cap \ldots \cap T_{nk-1}^c \cap T_{nk} \cap R_{nk}\big]$$

$$\geq \sum_{k=1}^{[n^\beta]} \big\{P\big[T_{nk} \cap R_{nk}\big] - P\big[(T_{n1} \cup \ldots \cup T_{nk-1}) \cap R_{nk}\big]\big\}$$

$$\geq \sum_{k=1}^{[n^\beta]} P\big[T_{nk}\big]\Big\{P\big[R_{nk}\big] - \sum_{k=1}^{[n^\beta]} P\big[T_{nk}\big]\Big\}$$

Having $\tau_n \geq 0$ for sufficiently large n we get

$$\sum_{k=1}^{[n^\beta]} P\big[T_{nk}\big] = \sum_{k=1}^{[n^\beta]} P\big[X_{nk} \geq 2\varepsilon n^{\alpha\beta} + \tau_n\big]$$

$$< n^\beta P\big[\xi > 2\varepsilon n^{\alpha\beta} + \tau_n\big] = n^\beta\big(1 - P\big[\xi \leq 2\varepsilon n^{\alpha\beta} + \tau_n\big]\big) \leq \frac{1}{4},$$

where we have used the covering identity (1.1) as well as (2.17).
Thus, (2.20) implies that

$$P\big[S_{[n^\beta]} \geq 2\varepsilon n^{\alpha\beta} + \mu_{[n^\beta]}\big] \geq \frac{1}{12}[n^\beta]P\big[\xi > 2\varepsilon n^{\alpha\beta} + \tau_n\big]$$

for n sufficiently large.
Hence, by (2.16) we conclude that

$$\sum_{n=1}^{\infty} n^{\alpha r - 2 + \beta} P\big[\xi > 2\varepsilon n^{\alpha\beta} + \tau_n\big] < \infty$$

which is equivalent to

$$\sum_{m=1}^{\infty} (2^m)^{\alpha r - 1 + \beta} P\big[\xi > 2\varepsilon(2^m)^{\alpha\beta} + \tau_{2^m}\big] < \infty. \tag{2.21}$$

Similarly as in Pruss [11] (cf. Lemma 4) we can show that for m sufficiently large we have

$$\tau_{2^{m+1}} \leq 2^m \varepsilon + \tau_{2^m}.$$

Assume that M is a positive integer number such that

$$\tau_{2^{m+1}} \leq 2\varepsilon(2^m)^{\alpha\beta} + \tau_{2^m} \quad \text{for} \quad m \geq M.$$

Iterating this inequality for $m \geq M$ we obtain

$$\tau_{2^m} < 2\varepsilon(2^m)^{\alpha\beta} + \tau_{2^M}$$

which gives $2\varepsilon(2^m)^{\alpha\beta} + \tau_{2^m} < 4\varepsilon(2^m)^{\alpha\beta} + \tau_{2^M}$.
Therefore, using (2.21), we have

$$\sum_{m=1}^{\infty}(2^m)^{\alpha r-1+\beta}P\left[\xi > 4\varepsilon(2^m)^{\alpha\beta} + \tau_{2^M}\right] < \infty$$

which proves that

$$\infty > \sum_{n=1}^{\infty} n^{\alpha r-2+\beta}P\left[\xi > 4\varepsilon n^{\alpha\beta} + \tau_{2^M}\right]$$

$$\geq \sum_{n=1}^{\infty} n^{\alpha r-2+\beta}P\left[\xi > (4\varepsilon + \tau_{2^M})n^{\alpha\beta}\right] \geq \text{const}E\left(\xi^+\right)^{\frac{\alpha r-1+\beta}{\alpha\beta}}.$$

Similarly one can show that $E\left(\xi^-\right)^{\frac{\alpha r-1+\beta}{\alpha\beta}} < \infty$, which completes the proof of Theorem 2.2.

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REFERENCES

1. HSU, P.L. and ROBBINS, H., "Complete Convergence and the Law of Large Numbers", *Proc. Nat. Acad. Sci. U.S.A.* **33**, (1947) 25-31.
2. ERDÖS, P., "On a Theorem of Hsu and Robbins", *Ann. Math. Statist.* **20**, (1949) 286-291.
3. GUT, A., "Complete Convergence", technical report.
4. SZYNAL, D., "On Almost Complete Convergence for the Sums of a Random Numbers of Independent Random Variables", *Bull.Acad. Polon. Math.* **20** (1972) 571-574.
5. GUT, A., "Complete Convergence and Convergence Rates for Randomly Indexed Partial Sums with an Application to Some First Passage Times", *Acta Math. Hung.* **42** (1983) 225-232.
6. ZHIDONG, B. and CHUN, S., "The Complete Convergence for Partial Sums of I.I.D. Random Variables", *Sci. Sinica Ser. A* **28** (1985) 1261-1277.
7. ADLER, A., "On Complete Convergence of the Sum of a Ramdom Number of Random Variables", *Calcutta Statist. Assoc. Bull.*, **37** (1988) 161-169.
8. KLESOV, O.I., "The Rate of Convergence of Sums of Random Number of Random Variables", *Theor. Probability and Math. Statist.* **39** (1989) 75-81.
9. KUCZMASZEWSKA, A. and SZYNAL, D., "On the Complete Convergence for Randomly Indexed Partial Sums and the Limit Behaviour of a Sequence of Quantiles", *J. Statist. Plann. Inference.* **30** (1992) 371-383.
10. KUCZMASZEWSKA, A. and SZYNAL, D., "On Complete Convergence in a Banach Space", *Internat, J. Math. & Math. Sci.* **17** (1994) 1-14.
11. PRUSS, A.R., "Randomly Sampled Riemann Sums and Complete Convergence in the Law of Large Numbers for a Case without Identical Distribution", *Proc. Amer. Math. Soc.* **124** (1996) 919-929
12. FUK, D. Kh. and NAGAEV, S. V., "Probabilistic Inequalities for Sums of Independent Random Variables", *Theory Probab. Appl.* **16** (1971) 660-675.
13. LOÈVE, M., *Probability Theory.* 4rd ed. Vol 1 van Nostrand Princeton, (1977).