Internat. J. Math. & Math. Sci.
VOL. 20 NO. 2 (1997) 409-411

TWO ELEMENTARY COMMUTATIVITY THEOREMS FOR GENERALIZED BOOLEAN RINGS

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(Received September 9, 1991 and in revised form April 17, 1992)

ABSTRACT. In this paper we prove that if R is a ring with 1 as an identity element in which $x^m - x^n \in Z(R)$ for all $x \in R$ and fixed relatively prime positive integers m and n, one of which is even, then R is commutative. Also we prove that if R is a 2-torsion free ring with 1 in which $(x^{2^k})^{n+1} - (x^{2^k})^n \in Z(R)$ for all $x \in R$ and fixed positive integer n and non-negative integer k, then R is commutative.

KEY WORDS AND PHRASES. Commutator, 2-torsion free ring.
1991 AMS SUBJECT CLASSIFICATION CODE. 16A70.

1. INTRODUCTION.

Throughout this paper, R is an associative ring with 1 as an identity element. We denote the centre of R by $Z(R)$ and the commutator $xy - yx$ by $[x,y]$. Recently, Quadri and Ashraf [1] proved that if R is a ring in which $x^{n+1} - x^n \in Z(R)$ for all $x \in R$ and fixed positive integer n, then R is commutative. In this paper, we generalize this result.

2. MAIN RESULTS.

We start with the following lemma of Bell [2].

LEMMA 2.1. Let $w \in R$. If for each $x \in R$ there exist relatively prime positive integers $n = n(x)$ and $m = m(x)$ such that

$$[w, x^n] = [w, x^m] = 0, \text{ then } w \in Z(R).$$

THEOREM 2.1. If R is a ring with $x^m - x^n \in Z(R)$ for all $x \in R$ and fixed relatively prime positive integers m and n, one of which is even, then R is commutative.

PROOF. Let

$$x^m - x^n \in Z(R) \qquad \text{for all } x \in R. \tag{2.1}$$

Assume m is even and n is odd. Using both x and $-x$ in (2.1) and then adding and subtracting, we get $2x^m \in Z(R)$ and $2x^n \in Z(R)$. Thus $[x^m, 2y] = [x^n, 2y] = 0$ for all $x, y \in R$; and by Lemma 2.1 $2y \in Z(R)$ for all $y \in R$. Now we replace x by $x+1$ to obtain

$$\left[\sum_{r=1}^{m-1} \binom{m}{r} x^{m-r}, y\right] = \left[\sum_{r=1}^{n-1} \binom{n}{r} x^{n-r}, y\right]$$

and since m is even and n is odd and $[2x, y] = 0$, we get $[x^2 p(x) - x, y] = 0$ for some $p(x) \in Z[x]$. Now the theorem follows from Herstein's result [3].

In Theorem 2.1, all the hypotheses are essential. If both m and n are odd or if one of m and n is even and the other odd, but they are not relatively prime; or if both m and n are even; or if R is a ring without the identity element 1 in the hypotheses of the theorem, then R need not be commutative.

EXAMPLE 2.1. It can be shown easily that

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a,b,c,d \in GF(3) \right\}$$

is a ring with identity element, in which

(i) $x^3 - x^9 \in Z(R)$

(ii) $x^3 - x^6 \in Z(R)$

(iii) $x^4 - x^{10} \in Z(R)$

for all $x \in R$, but R is not commutative.

EXAMPLE 2.2.

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a,b,c,d \in GF(2) \right\}$$

is a ring with identity element in which

(i) $x^5 - x^9 \in Z(R)$

(ii) $x^4 - x^8 \in Z(R)$

for all $x \in R$, but R is not commutative.

EXAMPLE 2.3.

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a,b,c, \in GF(3) \right\}$$

is a ring without identity element with $x^3 - x^4 \in Z(R)$ for all $x \in R$, but R is not commutative.

We state the following lemma which can be proved easily.

LEMMA 2.2. If $t = 2^k n$ where k and n are positive integers, then

$$\left\{ \binom{t+2^k}{(2r-1)+2^k} - \binom{t}{2r-1} \right\} \text{ and } \binom{t+2^k}{2r-1} \text{ are multiples of } 2^k \qquad \text{for } r = 1,2,3,\cdots \tfrac{t}{2}.$$

Now we give the following theorem which generalizes the theorem of Quadri and Ashraf [1] for 2-torsion free rings.

THEOREM 2.2. If R is a 2-torsion free ring with $(x^{2^k})^{n+1} - (x^{2^k})^n \in Z(R)$ for all $x \in R$ and fixed non-negative integer k and positive integer n, then R is commutative.

PROOF. If $k = 0$ then result follows from Theorem 2.1. Let $k > 0$ and

$$[x^{t+2^k}, y] = [x^t, y] \text{ for all } x, y \in R \text{ where } t = 2^k n$$

Now we replace x by $x + 1$ to obtain

$$\left[\sum_{r=1}^{t+2^k-1}\binom{t+2^k}{r}x^{t+2^k-r},y\right]=\left[\sum_{r=1}^{t-1}\binom{t}{r}x^{t-r},y\right] \tag{2.2}$$

Next we replace x by $-x$ in (2.2) and subtract the result from (2.2) and use the fact that R is 2-torsion free to get

$$\left[\sum_{r=1}^{2^{k-1}}\binom{t+2^k}{2r-1}x^{t+2^k-(2r-1)}+\sum_{r=1}^{t/2}\left\{\binom{t+2^k}{(2r-1)+2^k}-\binom{t}{2r-1}\right\}x^{t-(2r-1)},y\right]=0$$

By Lemma 2.2 and the fact that R is 2-torsion free, we get $[x^2p(x)-x,y]=0$ for all $x,y\in R$ and some $p(x)\in Z[x]$. Now R is commutative.

All the hypotheses of Theorem 2.2 are essential. In Example 2.1, R is a 2-torsion free ring with identity element in which $(x^{2^k})^m-(x^{2^k})^n\in Z(R)$ $(k=2, m=4, n=7)$ for all $x,y\in R$ and m and n are relatively prime positive integers and one of them is even, but R is not commutative. In Example 2.2, R is a 2-torsion ring with identity element in which $(x^{2^k})^{n+1}-(x^{2^k})^n\in Z(R)$ $(k=2,$ $n=1)$ for all $x\in R$, but R is not commutative. In Example 2.3, R is a 2-torsion free ring without identity element in which $(x^{2^k})^{n+1}-(x^{2^k})^n\in Z(R)$ $(k=2$ and $n=1)$ for all $x\in R$, but R is not commutative.

ACKNOWLEDGEMENT. The author expresses his sincere thanks to the referee for many helpful suggestions to modify the proof of Theorem 2.1.

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