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A STUDY OF SOME NEW ABSOLUTE SUMMABILITY METHODS

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ABSTRACT. In this note we introduce a new method of absolute summability. A general theorem is given. Several results are also deduced.

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1. INTRODUCTION.

Let $\sum a_n$ be an infinite series with partial sums s_n. Let σ_n^δ and η_n^δ denotes the nth Cesaro mean of order $\delta(\delta > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series $\sum a_n$ is said to be summable $|C,\delta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}\left|\sigma_n^\delta - \sigma_{n-1}^\delta\right|^k < \infty ,$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-1}\left|\eta_n^\delta\right|^k < \infty .$$

Let $\{p_n\}$ be a sequence of real or complex constants with

$$P_n = p_0 + p_1 + \dots + p_n , \quad P_{-1} = p_{-1} = 0 .$$

The series $\sum a_n$ is said to be summable $|N,p_n|$, if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty , \tag{1}$$

where

$$t_n = P_n^{-1} \sum_{v=0}^{n} p_{n-v} s_v \; {\scriptstyle (t_{-1}=0)} .$$

We write $p = \{p_n\}$ and

$$M = \{p : p_n > 0 \;\&\; p_{n+1}/p_n \leq p_{n+2}/p_{n+1}, n = 0, 1, \dots\}$$

It is known that for $p \in M$, (1) holds if and only if (Das [4])

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} \left|\sum_{v=1}^{n} p_{n-v} v a_v\right| < \infty .$$

DEFINITION 1 (Sulaiman [5]). For $p \in M$, we say that $\sum a_n$ is summable $|N, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k} \left| \sum_{v=1}^{n} p_{n-v} v a_v \right|^k < \infty .$$

In the special case in which $p_n = A_n^{r-1}$, $r > -1$, where A_n^r is the coefficient of x^n in the power series expansion of $(1-x)^{-r-1}$ for $|x| < 1$, $|N, p_n|k$ summability reduces to $|C, r|_k$ summability.

The series $\sum a_n$ is said to be summable $|R, p_n|k$, $|\overline{N}, p_n|_k$, $k \geq 1$ (Bor [2] & [1]), if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty , \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty ,$$

respectively, where

$$T_n = P_n^{-1} \sum_{v=0}^{n} p_v s_v .$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $|R, p_n|_k$, $|\overline{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|R, p_n|$) summability.

We set

$$\begin{aligned} Q_n &= q_0 + q_1 + \dots + q_n , \quad q_{-1} = Q_{-1} = 0 . \\ U_n &= u_0 + u_1 + \dots + u_n, \quad u_{-1} = U_{-1} = 0 . \\ R_n &= p_0 q_n + p_1 q_{n-1} + \dots p_n q_0 \\ \triangle f_n &= f_n - f_{n+1} \end{aligned}$$

We assume $\{\phi_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive real constants. Here we give the following new definition.

DEFINITION 2. Let $\{p_n\}$, $\{q_n\}$ be sequences of positive real constants such that $q \in M$ We say that $\sum a_n$ is summable $|N, R_n, \phi_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \phi_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \sum_{v=1}^{n} P_{v-1} q_{n-v} a_v \right|^k < \infty .$$

DEFINITION 3 (Sulaiman [6]). The series $\sum a_n$ is said to be summable $|\overline{N}, p_n, \phi_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \phi_n^{k-1} |T_n - T_{n-1}|^k < \infty .$$

2. LEMMAS

LEMMA 1. Let $\{p_n\}$, $\{q_n\}$, and $\{u_n\}$ be sequences of positive real constants such that $q \in M$, $\{\alpha_n^{1-1/k} p_n / P_n R_{n-1}\}$ nonincreasing for $q_n \neq c$. Let T_n denote the $(\overline{N}, u_n)$-mean of the series $\sum a_n$ Let $[\epsilon_n\}$ be a sequence of constants and write $\beta_n^{1-1/k} \triangle T_{n-1} = \triangle_n$ If

$$\sum_{n=v+1}^{m+1} \frac{\alpha_n^{k-1} p_n^k}{P_n^k R_{n-1}} q_{n-v-1} = 0\left\{ \frac{\alpha_v^{k-1} p_v^{k-1}}{P_v^k} \right\} ,$$

$$\sum_{n=1}^{\infty} \left(\frac{\alpha_n}{\beta_n}\right)^k \left(\frac{p_n}{P_n}\right)^k \left(\frac{P_{n-1}}{R_{n-1}}\right)^k \left(\frac{U_n}{u_n}\right)^k |\epsilon_n|^k |\triangle_n|^k < \infty ,$$

$$\sum_{n=1}^{\infty} \left(\frac{\alpha_n}{\beta_n}\right)^{k-1} |\epsilon_n|^k |\triangle_n|^k < \infty ,$$

and

$$\sum_{n=1}^{\infty}\left(\frac{\alpha_n}{\beta_n}\right)^{k-1}\left(\frac{U_{n-1}}{u_n}\right)^k|\Delta\epsilon_n|^k\,|\Delta_n|^k<\infty$$

then the series $\sum a_n\epsilon_n$ is summable $|N,R_n,\alpha_n|_k,\ \ k\geq 1.$

LEMMA 2 (Sulaiman [7]) Let $q\in M$. Then for $0<r\leq 1$,

$$\sum_{n=v+1}^{\infty}\frac{q_{n-v-1}}{n^r Q_{n-1}}=0(v^{-r})\ .$$

LEMMA 3 (Bor [2]). Let $k>1$ and $A=(a_{nv})$ be an infinite matrix. In order that $A\in(l^k;l^k)$, it is necessary that

$$a_{nv}=0(1)\quad (\text{all } n,v) \tag{3}$$

Proof of Lemma 1. Write

$$\tau_n=\sum_{v=1}^{n}P_{v-1}q_{n-v}a_v\epsilon_v\ .$$

Since

$$T_n=U_n^{-1}\sum_{v=0}^{n}u_v\sum_{r=0}^{v}a_r=U_n^{-1}\sum_{v=0}^{n}(U_n-U_{v-1})a_v\ ,$$

then

$$-\Delta T_{n-1}=\frac{u_n}{U_nU_{n-1}}\sum_{v=1}^{n}U_{v-1}a_v\ .$$

By Abel's transformation,

$$\begin{aligned}
\tau_n&=\sum_{v=1}^{n}U_{v-1}a_v\big(P_{v-1}q_{n-v}U_{v-1}^{-1}\epsilon_v\big)\\
&=\sum_{v=1}^{n-1}\left(\sum_{r=1}^{v}U_{r-1}a_r\right)\Delta\big(P_{v-1}q_{n-v}U_{v-1}^{-1}\epsilon_v\big)+\left(\sum_{r=1}^{n}U_{r-1}a_r\right)P_{n-1}q_0U_{n-1}^{-1}\epsilon_n\\
&=\sum_{v=1}^{n-1}\left\{-\frac{U_{v-1}U_v}{u_v}\Delta T_{v-1}\right\}\Big\{P_{v-1}\Delta_vq_{n-v}U_{v-1}^{-1}\epsilon_v+P_{v-1}q_{n-v-1}\frac{u_v}{U_{v-1}U_v}\epsilon_v\\
&\qquad -p_vq_{n-v-1}U_v^{-1}\epsilon_v+P_vq_{n-v-1}U_v^{-1}\Delta\epsilon_v\Big\}-P_{n-1}q_0U_nu_n^{-1}\epsilon_n\Delta T_{n-1}\\
&=\sum_{v=1}^{n-1}\Big\{-P_{v-1}\Delta q_{n-v}\frac{U_v}{u_v}\epsilon_v\Delta T_{v-1}-P_{v-1}q_{n-v-1}\epsilon_v\Delta T_{v-1}+p_vq_{n-v-1}\frac{U_{v-1}}{u_v}\\
&\qquad \times\epsilon_v\Delta T_{v-1}-P_vq_{n-v-1}\frac{U_{v-1}}{u_v}\Delta\epsilon_v\Delta T_{v-1}\Big\}-P_{n-1}q_0\frac{U_n}{u_n}\epsilon_n\Delta T_{n-1}
\end{aligned} \tag{2}$$

$$=\tau_{n,1}+\tau_{n,2}+\tau_{n,3}+\tau_{n,4}+\tau_{n,5}\ ,\quad \text{say}$$

In order to prove the lemma, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\alpha_n^{k-1}\left|\frac{p_n}{P_nR_{n-1}}\tau_{n,r}\right|^k<\infty,\quad r=1,2,3,4,5\,.$$

Applying Hölder's inequality,

$$\sum_{m=2}^{m+1}\alpha_n^{k-1}\left|\frac{p_n}{P_nR_{n-1}}\tau_{n,1}\right|^k = \sum_{n=2}^{m+1}\alpha_n^{k-1}\left|\frac{p_n}{P_nR_{n-1}}\sum_{v=1}^{n-1}P_{v-1}\triangle_v q_{n-v}\frac{U_v}{u_v}\epsilon_v\triangle T_{v-1}\right|^k$$
$$\leq \sum_{n=2}^{m+1}\alpha_n^{k-1}\left(\frac{p_n}{P_nR_{n-1}}\right)^k\sum_{v=1}^{n-1}P_{v-1}^k|\triangle_v q_{n-v}|\left(\frac{U_v}{u_v}\right)^k|\epsilon_v|^k|\triangle T_{v-1}|^k$$
$$\times\left\{\sum_{v=1}^{n-1}|\triangle_v q_{n-v}|\right\}^{k-1}$$
$$= 0(1)\sum_{v=1}^{m}P_{v-1}^k\left(\frac{U_v}{u_v}\right)^k|\epsilon_v|^k|\triangle T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{\alpha_n^{k-1}p_n^k}{P_n^kR_{n-1}^k}|\triangle_v q_{n-v}|$$
$$= 0(1)\sum_{v=1}^{m}\left(\frac{\alpha_v}{\beta_v}\right)^{k-1}\left(\frac{p_v}{P_v}\right)^k\left(\frac{P_{v-1}}{R_{v-1}}\right)^k\left(\frac{U_v}{u_v}\right)^k|\epsilon_v|^k|\triangle_v|^k$$

$$\sum_{n=2}^{m+1}\alpha_n^{k-1}\left|\frac{p_n}{P_nR_{n-1}}\tau_{n,2}\right|^k = \sum_{n=2}^{m+1}\alpha_n^{k-1}\left|\frac{p_n}{P_nR_{n-1}}\sum_{v=1}^{n-1}\frac{P_{v-1}}{p_v}p_vq_{n-v-1}\epsilon_v\triangle T_{v-1}\right|^k$$
$$\leq \sum_{n=2}^{m+1}\frac{\alpha_n^{k-1}p_n^k}{P_n^kR_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_{v-1}}{p_v}\right)^k p_vq_{n-v-1}|\epsilon_v|^k|\triangle T_{v-1}|^k\left\{\sum_{v=1}^{n-1}\frac{p_vq_{n-v-1}}{R_{n-1}}\right\}^{k-1}$$
$$= 0(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v|\epsilon_v|^k|\triangle T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{\alpha_n^{k-1}p_n^k}{P_n^kR_{n-1}}q_{n-v-1}$$
$$= 0(1)\sum_{v=1}^{m}\left(\frac{\alpha_v}{\beta_v}\right)^{k-1}|\epsilon_v|^k|\triangle_v|^k$$

$$\sum_{m=2}^{m+1}\alpha_n^{k-1}\left|\frac{p_n}{P_nR_{n-1}}\tau_{n,3}\right|^k = \sum_{n=2}^{m+1}\alpha_n^{k-1}\left|\frac{p_n}{P_nR_{n-1}}\sum_{v=1}^{n-1}p_vq_{n-v-1}\frac{U_{v-1}}{u_v}v\epsilon_v\triangle T_{v-1}\right|^k$$
$$\leq \sum_{n=2}^{m+1}\frac{\alpha_n^{k-1}p_n^k}{P_n^kR_{n-1}}\sum_{v=1}^{n-1}p_vq_{n-v-1}\left(\frac{U_{v-1}}{u_v}\right)^k|\epsilon_v|^k|\triangle T_{v-1}|^k$$
$$\times\left\{\sum_{v=1}^{n-1}\frac{p_vq_{n-v-1}}{R_{n-1}}\right\}^{k-1}$$
$$= 0(1)\sum_{v=1}^{m}p_v\left(\frac{U_v}{u_v}\right)^k|\epsilon_v|^k|\triangle T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{\alpha_n^{k-1}p_n^k}{P_n^kR_{n-1}^k}q_{n-v-1}$$
$$= 0(1)\sum_{v=1}^{m}\left(\frac{\alpha_v^{k-1}}{\beta_v}\right)\left(\frac{p_v}{P_v}\right)^k\left(\frac{P_{v-1}}{R_{v-1}}\right)^k\left(\frac{U_v}{u_v}\right)^k|\epsilon_v|^k|\triangle_v|^k$$

$$\sum_{m=2}^{m+1} \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \tau_{n,4} \right|^k = \sum_{n=2}^{m+1} \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} P_v q_{n-v-1} \frac{U_{v-1}}{u_v} \Delta\epsilon_v \Delta T_{v-1} \right|^k$$

$$\leq \sum_{n=2}^{m+1} \frac{\alpha_n^{k-1} p_n^k}{P_n^k R_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k p_v q_{n-v} \left(\frac{U_{v-1}}{u_v}\right)^k |\Delta\epsilon_v|^k |\Delta T_{v-1}|^k \times \left\{ \sum_{v=1}^{n-1} \frac{p_v q_{n-v-1}}{R_{n-1}} \right\}^{k-1}$$

$$= 0(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k p_v \left(\frac{U_{v-1}}{u_v}\right)^k |\Delta\epsilon_v|^k |\Delta T_{v-1}|^k \cdot \sum_{n=v+1}^{m+1} \frac{\alpha_n^{k-1} p_n}{P_n^k R_{n-1}} q_{n-v-1}$$

$$= 0(1) \sum_{v=1}^{m} \left(\frac{\alpha_v}{\beta_v}\right)^{k-1} \left(\frac{\alpha_v^{k-1}}{\beta_v}\right) \left(\frac{U_v}{u_v}\right)^k |\Delta\epsilon_v|^k |\Delta_v|^k$$

$$\sum_{n=1}^{m} \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \tau_{n,5} \right|^k = \sum_{n=1}^{m} \alpha_n^{k-1} \left| \frac{p_n}{P_{nR_{n-1}}} P_{n-1} q_0 \frac{U_n}{u_n} \epsilon_n \Delta T_{n-1} \right|^k$$

$$= 0(1) \sum_{n=1}^{m} \left(\frac{\alpha_n}{\beta_n}\right)^{k-1} \left(\frac{p_n}{P_n}\right)^k \left(\frac{P_{n-1}}{R_{n-1}}\right)^k \left(\frac{U_n}{u_n}\right)^k |\epsilon_n|^k |\Delta_n|^k .$$

This completes the proof of Lemma 1.

3. MAIN RESULT

THEOREM. Let $q \in M$ such that $\{\alpha_n^{1-1/k} p_n / P_n R_{n-1}\}$ nonincreasing for $q_n \neq c$. Let $P_n R_{n-1} u_n = 0(p_n P_{n-1} U_n)$, $\sum \alpha_n^{k-1} (p_n/P_n)^k$ divergent, and

$$\sum_{n=v}^{\infty} \frac{\alpha_n^{k-1} p_n^k}{P_n^k R_{n-1}} q_{n-v-1} = 0\left\{ \frac{\alpha_v^{k-1} p_v^{k-1}}{P_v^k} \right\} .$$

Then the necessary and sufficient conditions that $\sum a_n \epsilon_n$ is summable $|N, R_n, \alpha_n|_k$ whenever $\sum a_n$ is summable $|\overline{N}, p_n, \beta_n|_k$, $k \geq 1$, are

$$\text{(i)} \qquad \epsilon_n = 0\left\{ \left(\frac{P_n R_{n-1} u_n}{p_n P_{n-1} U_n}\right) \left(\frac{\beta_n}{\alpha_n}\right)^{1-1/k} \right\},$$

$$\text{(ii)} \qquad \Delta\epsilon_n = 0\left\{ \left(\frac{u_n}{U_{n-1}}\right) \left(\frac{\beta_n}{\alpha_n}\right)^{1-1/k} \right\} .$$

PROOF. Sufficiency. Follows form Lemma 1.

Necessity of (i). Multiplying (2) by $\alpha_n^{1-1/k} p_n / P_n R_{n-1}$, the last term on the right becomes

$$\frac{\alpha_n^{1-1/k} p_n}{P_n R_{n-1}} \tau_{n,5} = - \frac{p_n P_{n-1} U_n}{P_n R_{n-1} u_n} \epsilon_n \Delta T_{n-1}$$

$$= - \left\{ \frac{p_n P_{n-1} U_n}{P_n R_{n-1} u_n} \left(\frac{\alpha_n}{\beta_n}\right)^{1-1/k} \epsilon_n \right\} (\beta_n^{1-1/k} \Delta T_{n-1})$$

Following Bor [2]. By (3), it is possible to write the matrix transforming $(\beta_n^{1-1/k} \Delta T_{n-1})$ into $((\alpha_n^{1-1/k} p_n / P_n R_{n-1}) \tau_n)$. Since $|\overline{N}, p_n, \beta_n|_k$ implies $|N, R_n, \alpha_n|_k$, the matrix $\epsilon(l^k; l^k)$. By Lemma 3, a necessary condition for this implication is that the elements (in particular the diagonal elements) of this matrix should be bounded. Hence (i)

Necessity of (ii). Suppose $|\overline{N}, p_n, \beta_n|_k$ of $\sum a_n$ implies $|N, R_n, \alpha_n|_k$ of $\sum a_n\epsilon_n$. From (2)

$$|\tau_{n,4}| \le \sum_r |\tau_{n,r}| + |\tau_n| , \quad r = 1, 2, 3, 5 .$$

By Minkowski's inequality, using (i), we have, via the proof of Lemma 1,

$$\begin{aligned}\sum_{n=1}^{m} \alpha_n^{k-1}\left(\frac{p_n}{P_nR_{n-1}}\right)^k |\tau_{n,4}|^k &\le 0(1)\left\{\sum_{n=1}^{m} \alpha_n^{k-1}\left(\frac{p_n}{P_nR_{n-1}}\right)^k |\tau_{n,r}|^k + \sum_{n=1}^{m} \alpha_n^{k-1}\left(\frac{p_n}{P_nR_{n-1}}\right)^k |\tau_n|^k\right\} \\ &\le 0(1)\sum_{n=1}^{m} |\triangle_n|^k \\ &= 0(1)\sum_{n=1}^{m} \beta_n^{k-1}|\triangle T_{n-1}|^k .\end{aligned}$$

Therefore

$$\sum_{n=1}^{m} \alpha_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \left|\frac{1}{R_{n-1}}\sum_{v=1}^{n-1} P_v q_{n-v-1}\frac{U_{v-1}}{u_v}\triangle\epsilon_v\triangle T_{v-1}\right|^k = 0(1)\sum_{n=1}^{m} \beta_n^{k-1}|\triangle T_{n-1}|^k$$

Now, put $\triangle T_{n-1} = \left(\frac{p_n}{P_n}\right)\left(\frac{\alpha_n}{\beta_n}\right)^{1-1/k}$, we obtain

$$\sum_{n=1}^{m} \alpha_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \left|\frac{1}{R_{n-1}}\sum_{v=1}^{n-1} p_v q_{n-v-1}\left\{\left(\frac{U_{v-1}}{u_v}\right)\left(\frac{\alpha_v}{\beta_v}\right)^{1-1/k}\triangle\epsilon_v\right\}\right|^k = 0(1)\sum_{n=1}^{m} \alpha_n^{k-1}\left(\frac{p_n}{P_n}\right)^k$$

This should imply

$$\frac{1}{R_{n-1}}\sum_{v=1}^{n-1} p_v q_{n-v-1}\left\{\left(\frac{U_{v-1}}{u_v}\right)\left(\frac{\alpha_v}{\beta_v}\right)^{1-1/k}\triangle\epsilon_v\right\} = 0(1) .$$

But $\sum_{v=1}^{n-1} p_v q_{n-v-1} = R_{n-1}$, we get

$$\left(\frac{U_{v-1}}{u_v}\right)\left(\frac{\alpha_v}{\beta_v}\right)^{1-1/k}\triangle\epsilon_v = 0(1) .$$

This completes the proof of the theorem.

REMARK. It is clear that

$$|\overline{N}, p_n, P_n/p_n|_k = |\overline{N}, p_n|_k , \quad |\overline{N}, p_n, n|_k = |R, p_n|_k , \quad |\overline{N}, 1, n|_k = |C, 1|_k ,$$

and from our definition we may deduce that

$$q_n = 1 \Rightarrow |N, \quad P_n, \phi_n|_k = |\overline{N}, p_n, \phi_n|_k ,$$

which implies

$$|N, P_n, P_n/p_n|_k = |\overline{N}, p_n|_k$$

and

$$|N, P_n, n|_k = |R, p_n|_k \cdot p_n = 1 \Rightarrow |N, Q_n, n|_k = |N, q_n|_k .$$

4. APPLICATIONS

COROLLARY 1. Let $P_n u_n = 0(p_n U_n)$. Then the necessary and sufficient conditions that $\sum a_n\epsilon_n$ be summable $|\overline{N}, p_n|_k$ whenever $\sum a_n$ is summable $|\overline{N}, u_n|_k$, $k \ge 1$, are

$$\epsilon_n = 0\left\{\left(\frac{u_n P_n}{U_n p_n}\right)^{1/k}\right\}, \qquad \Delta\epsilon_n = 0\left\{\left(\frac{u_n}{U_{n-1}}\right)\left(\frac{p_n}{P_n}\right)^{1-1/k}\right\}.$$

PROOF. Follows from the theorem by putting $q_n = 1$, $\alpha_n = P_n/p_n$, and $\beta_n = U_n/u_n$

COROLLARY 2 (Bor and Thorpe [3]) Let $P_n u_n = 0(p_n U_n)$ and $p_n U_n = 0(P_n u_n)$ Then $\sum a_n$ is summable $|\overline{N}, p_n|_k$ iff it is summable $|\overline{N}, u_n|_k$, $k \geq 1$

PROOF. Follows from Corollary 1 by putting $\epsilon_n = 1$

COROLLARY 3. Let $Q_{n-1}u_n = 0(U_n)$ Then the necessary and sufficient conditions that $\sum a_n\epsilon_n$ be summable $|N, q_n|_k$ whenever $\sum a_n$ is summable $|\overline{N}, u_n|_k$, $k \geq 1$, are

$$\epsilon_n = 0\left\{\left(\frac{Q_{n-1}}{n}\right)\left(\frac{nu_n}{U_n}\right)^{1/k}\right\}, \qquad \Delta\epsilon_n = 0\left\{\left(\frac{u_n}{U_{n-1}}\right)\left(\frac{U_n}{nu_n}\right)^{1-1/k}\right\}.$$

PROOF. Follows from the theorem by putting $p_n = 1$, $\alpha_n = n$, $\beta_n = U_n/u_n$ and making use of Lemma 2.

COROLLARY 4. Let $Q_{n-1}u_n = 0(U_n)$. Then a necessary and sufficient condition that $\sum a_n$ be summable $|N, q_n|_k$ whenever it is summable $|\overline{N}, u_n|_k$, $k \geq 1$, is

$$n^{k-1}U_n = 0(Q_{n-1}^k u_n).$$

PROOF. Follows from Corollary 3 by putting $\epsilon_n = 1$.

COROLLARY 5. Let $\{n^{1-1/k}p_n/P_nP_{n-1}\}$ nonincreasing, $P_n u_n = 0(p_n U_n)$, and

$$\sum_{n=v}^{\infty} \frac{n^{k-1}p_n^k}{P_n^k P_{n-1}} = 0\left(\frac{v^{k-1}p_v^{k-1}}{P_v^k}\right).$$

Then the necessary and sufficient conditions that $\sum a_n\epsilon_n$ be summable $|R, p_n|_k$ whenever $\sum a_n$ is summable $|\overline{N}, u_n|_k$, $k \geq 1$, are

$$\epsilon_n = 0\left\{\left(\frac{P_n}{np_n}\right)\left(\frac{nu_n}{U_n}\right)^{1/k}\right\}, \qquad \Delta\epsilon_n = 0\left\{\left(\frac{u_n}{U_{n-1}}\right)\left(\frac{U_n}{nu_n}\right)^{1-1/k}\right\}.$$

PROOF. Follows from the theorem by putting $q_n = 1$, $\alpha_n = n$ and $\beta_n = U_n/u_n$.

COROLLARY 6. Let $P_n u_n = 0(p_n U_n)$. Then the necessary and sufficient conditions that $\sum a_n\epsilon_n$ be summable $|\overline{N}, p_n|_k$ whenever $\sum a_n$ is summable $|R, u_n|_k$, $k \geq 1$, are

$$\epsilon_n = 0\left\{\left(\frac{nu_n}{U_n}\right)\left(\frac{P_n}{np_n}\right)^{1/k}\right\}, \qquad \Delta\epsilon_n = 0\left\{\left(\frac{u_n}{U_{n-1}}\right)\left(\frac{np_n}{P_n}\right)^{1-1/k}\right\}.$$

PROOF. Follows from the theorem by putting $q_n = 1$, $\alpha_n = P_n/p_n$ and $\beta_n = n$

The following four results follows from Corollary 3 and they are generalizations for the results of [8].

COROLLARY 7. The necessary and sufficient conditions that $\sum a_n\epsilon_n$ are summable $|C, \alpha|_k$, $0 \leq \alpha \leq 1$, whenever $\sum a_n$ is summable $|C, 1|_k$, $k \geq 1$, are

$$\epsilon_n = 0(n^{\alpha-1}), \qquad \Delta\epsilon_n = 0(n^{-1}).$$

PROOF. Follows by putting $q_n = A_n^{\alpha-1}$, $u_n = 1$.

COROLLARY 8. The necessary and sufficient conditions that $\sum a_n\epsilon_n$ be summable $|N, 1/(n+1)|_k$ whenever $\sum a_n$ is summable $|C, 1|_k$, $k \geq 1$, are

$$\epsilon_n = 0(\log n/n), \qquad \Delta\epsilon_n = 0(n^{-1}).$$

PROOF. Follows by putting $q_n = 1/(n+1)$, $u_n = 1$

COROLLARY 9. The necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|N, 1/(n+1)|_k$ whenever $\sum a_n$ is summable $|R, \log n, 1|_k$, $k \geq 1$, are

$$\epsilon_n = 0\left\{(\log n)^{1-1/k}/n\right\}, \qquad \triangle\epsilon_n = 0\left\{1/n(\log n)^{1/k}\right\}.$$

PROOF. Follows by putting $q_n = u_n = 1/(n+1)$.

COROLLARY 10. The necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|C, \alpha|_k$, $0 \leq \alpha \leq 1$, whenever $\sum a_n$ is summable $|R, \log n, 1|_k$, $k \geq 1$, are

$$\epsilon_n = 0\{n^{\alpha-1}/(\log n)^{1/k}\}, \qquad \triangle\epsilon_n = 0\{1/n(\log n)^{1/k}\}.$$

PROOF. Follows by putting $q_n = A_n^{\alpha-1}$, $u_n = 1/(n+1)$.

Lastly it may be mentioned that many other results could be obtained either from the theorem or from its corollaries.

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