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ON RADII OF STARLIKENESS AND CONVEXITY FOR CONVOLUTIONS OF STARLIKE FUNCTIONS

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ABSTRACT. In this paper, we obtain the radiuses of univalence, starlikeness and convexity for convolutions of starlike functions.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in the unit disc $D = \{z : |z| < 1\}$, and let S denote the subclass of $\mathcal{A}$ consisting of univalent functions. Let S^* and K be the usual subclasses of S consisting of starlike and convex functions, respectively, that is, $S^* = \{f : Re(zf'(z)/f(z)) > 0\}$ and $K = \{f : Re(1 + zf''(z)/f'(z)) > 0\}$. The convolution or Hadamard product of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the following power series $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. Hadamard products have many interesting properties and important applications, see [3] and [4]. It is well known that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*$, then $z + \sum_{n=0}^{\infty} \frac{a_n}{n} z^n = \int_0^z \frac{f(t)}{t} dt \in K$.

Theorem A (see [1]). If $f \in K$ and $g \in K (g \in S^*)$, then $f * g \in K (f * g \in S^*)$.

However, it is also known that if $f \in S^*$ and $g \in S^*$, $f * g$ need not be in S^*. Furthermore, Sheil-Small in [2] showed that $f * g$ need not be in S for $f \in S^*$ and $g \in S^*$.

2. MAIN RESULTS

Lemma 1. Let $F(z) = z + \sum_{n=2}^{\infty} n^2 z^n$. Then $F(z)$ is starlike in $|z| < 2 - \sqrt{3} \approx 0.268$. The result is sharp.

Proof. Noting that

$$F(z) = \frac{(z+1)z}{(1-z)^3} \tag{1}$$

and differentiating logarithmically both sides of (1), we have

$$\frac{zF'(z)}{F(z)} = \frac{z^2 + 4z + 1}{(1+z)(1-z)} = \frac{1+z}{1-z} - \frac{1}{1+z} + \frac{1}{1-z}. \tag{2}$$

It follows from (2) that

$$Re\left(\frac{zF'(z)}{F(z)}\right) \geq \frac{1-r}{1+r} - \frac{1}{1-r} + \frac{1}{1+r} = \frac{r^2 - 4r + 1}{(1+r)(1-r)},$$

where $r = |z|$. Thus, if $|z| < 2 - \sqrt{3}$, then $Re(zF'(z)/F(z)) > 0$. So $F(z)$ is starlike for $|z| < 2 - \sqrt{3}$. Since $F'(-2 + \sqrt{3}) = 0$, we know that the result is sharp.

Lemma 2. Let $F(z) = z + \sum_{n=2}^{\infty} n^2 z^n$, then $F(z)$ is convex in $|z| < 5 - 2\sqrt{6} \approx 0.101$. The result is sharp.

Proof. Using (1), we have

$$1 + \frac{zF''(z)}{F'(z)} = \frac{(1+z)(z^2 + 10z + 1)}{(1-z)(z^2 + 4z + 1)} = \frac{1+z}{1-z} + \frac{2}{1-z} - \frac{2+\sqrt{3}}{z + 2 + \sqrt{3}} - \frac{2-\sqrt{3}}{z + 2 - \sqrt{3}}, \tag{3}$$

$$Re\left(1+\frac{zF''(z)}{F'(z)}\right) \geq \frac{1-r}{1+r}+\frac{2}{1+r}-\frac{2+\sqrt{3}}{2+\sqrt{3}-r}-\frac{2-\sqrt{3}}{2-\sqrt{3}-r}=\frac{(1-r)(r^2-10r+1)}{(1+r)(r^2-4r+1)}$$

for $r=|z|<2-\sqrt{3}$. Thus, we have $Re(1+zF''(z)/F'(z))>0$ for $|z|<5-2\sqrt{6}$. Hence $F(z)$ is convex for $|z|<5-2\sqrt{6}$. It is clear that the result is sharp.

Theorem 1. Let $f\in S^*$ and $g\in S^*$, then $f*g$ is univalent and starlike for $|z|<r_0=2-\sqrt{3}$ and the constant r_0 cannot be replaced by any larger number.

Proof. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in S^*$, $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in S^*$ and $G(z)=f(z)*g(z)$. Then

$$G(z)=(z+\sum_{n=2}^{\infty}n^2z^n)*(z+\sum_{n=2}^{\infty}\frac{a_n}{n}z^n)*(z+\sum_{n=2}^{\infty}\frac{b_n}{n}z^n).$$

We know that $z+\sum_{n=2}^{\infty}\frac{a_n}{n}z^n\in K$ and $z+\sum_{n=2}^{\infty}\frac{b_n}{n}z^n\in K$. By Theorem A, we get

$$(z+\sum_{n=2}^{\infty}\frac{a_n}{n}z^n)*(z+\sum_{n=2}^{\infty}\frac{b_n}{n}z^n)\in K.$$

Now, let $H(z)=(z+\sum_{n=2}^{\infty}\frac{a_n}{n}z^n)*(z+\sum_{n=2}^{\infty}\frac{b_n}{n}z^n)$, then $H(z)=z+\sum_{n=2}^{\infty}\frac{a_nb_n}{n^2}z^n$. So that

$$G(z)=(z+\sum_{n=2}^{\infty}n^2z^n)*H(z)=F(z)*H(z),$$

where $F(z)=z+\sum_{n=2}^{\infty}n^2z^n$. By Lemma 1, we know that $F(z)$ is starlike for $|z|<r_0=2-\sqrt{3}$. Hence $F(r_0z)/r_0\in S^*$. Since $H(z)\in K$, by Theorem A we have

$$B(z)=(F(r_0z)/r_0)*H(z)=z+\sum_{n=2}^{\infty}a_nb_nr_0^{n-1}z^n\in S^*.$$

Therefore, $G(z)=r_0B(z/r_0)$ is starlike for $|z|<r_0=2-\sqrt{3}$.

Finally, we show that r_0 cannot be replaced by any larger number. Taking $\frac{z}{(1-z)^2}\in S^*$, for $G(z)=\frac{z}{(1-z)^2}*\frac{z}{(1-z)^2}=z+\sum_{n=2}^{\infty}n^2z^n$, we have $G'(-r_0)=0$. Thus, for any $r>r_0$, $G(z)$ is not univalent for $|z|<r$. This completes the proof of our theorem.

Theorem 2. Let $f\in S^*$ and $g\in S^*$, then $f*g$ is convex for $|z|<r_1=5-2\sqrt{6}$ and the constant r_1 cannot be replaced by any larger number.

Proof. By the method used in the proof of Theorem 1 and by using Lemma 2, we get Theorem 2 immediately and the sharpness of the result in Theorem 2 is obtained from (3).

Remark. The constant r_0 in Theorem 1 is usually refered to as the radius of univalence and starlikeness, while the constant r_1 in Theorem 2 is called the radius of convexity.

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