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STEČKIN INEQUALITIES FOR SUMMABILITY METHODS

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ABSTRACT. Stečkin proved an inequality on Fejér means of Fourier series He said, "Proving similar inequality for other summability method is an interesting problem." We generalize Stečkin's inequality and give various applications to summability methods.

KEY WORDS AND PHRASES. Stečkin inequality, M. F Timan inequality, Zygmund typical means, various summability methods.
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1. INTRODUCTION

Let $C_{2\pi}$ be space of 2π-periodic continuous functions, $\|f\| := \max_{0 \le x \le 2\pi} |f(x)|$. Let $f \in C_{2\pi}$, its Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{i=1}^{\infty} (a_i \cos ix + b_i \sin ix).$$

Denote $\sqcap_n$ to be the set of trigonometric polynomials of order at most n, and

$$E_n(f) := E_n(f)_{C_{2\pi}} := \min_{t_n \in \sqcap_n} \|f - t_n\|.$$

For a triangular matrix $\wedge := \{\lambda_{i,m(n)}\}$ with $\lambda_{0,m(n)} = 1 (n = 0, 1, \ldots)$ we consider the linear summability method

$$U_{m(n)}(f,x) := \frac{a_0}{2} \lambda_{0,m(n)} + \sum_{i=1}^{m(n)} \lambda_{i,m(n)} (a_i \cos ix + b_i \sin ix),$$
$$=: \frac{a_0}{2} + \sum_{i=1}^{m(n)} \lambda_{i,m(n)} A_i(x). \tag{1 1}$$

If $\lambda_{i,n} = 1 - \frac{i}{n+1} (0 \le i \le n)$ we obtain Fejér means σ_n.

By M_i and C_i we denote positive constants independent of n, and f.

S. B. Stečkin proved in [1].

THEOREM A. Let $f \in C_{2\pi}$, then we have

$$\|f - \sigma_n(f)\| \le \frac{M_1}{n+1} \sum_{i=0}^{n} E_i(f). \tag{1 2}$$

Let $\mathbb{N}$ be the set of natural numbers.

If $k \in \mathbb{N}$ and $\lambda_{i,n} = 1 - \left(\frac{i}{n+1}\right)^k$, $(1 \le i \le n)$, we obtain Zygmund typical means Z_n^k

The following generalization is obtained by M. F Timan (see [2]).

THEOREM B. Let $f \in C_{2\pi}$, then we have

$$\|f - Z_n^k(f)\| \leq \frac{M_2}{(n+1)^k} \sum_{i=0}^{n} (i+1)^{k-1} E_i(f) . \tag{1 3}$$

In §2 we establish lemmas of comparison of summability methods with Zygmund typical means Z_n^k The generalization of Stečkin's inequality is proved in §3 Using the results of §2 and §3 some applications on various summability methods are given in §4-§5

2. LEMMA OF COMPARISON

Favard and Trigub [3] investigated comparison of linear summability methods of Fourier series. Butzer, Nessel, and Trebels investigated comparison of summability methods in Banach spaces

Let A be a linear operator mapping $C_{2\pi}$ to $C_{2\pi}$ and $\|A\|$ be its norm

LEMMA 1. Suppose that A_n is a sequence of linear operators mapping $C_{2\pi}$ to $C_{2\pi}$ with $\|A_n\| = O(1)$, and B_n is a sequence of linear operators mapping $C_{2\pi}$ to $\sqcap_n$ with $\|B_n\| = O(1)$ In order that for any $f \in C_{2\pi}$

$$\|f - A_n f\| \leq M_3 \bullet \|f - B_n f\| , \tag{2.1}$$

it is sufficient and necessary that for any $t \in \sqcap_n$

$$\|t_n - A_n(t_n)\| \leq M_4 \bullet \|t_n - B_n(t_n)\| . \tag{2.2}$$

PROOF. Necessity Obviously from (2.1) we obtain (2.2).

Sufficiency. Let $f \in C_{2\pi}$ and t_n^* be polynomial of nth best approximation, i.e., $\|f - t_n^*\| = E_n(f)$. Then by the boundness of $\|A_n\|$ and $\|B_n\|$, one gets

$$\begin{aligned}
\|f - A_n f\| &\leq \|f - t_n^*\| + \|t_n^* - A_n(t_n^*)\| + \|A_n(t_n^* - f)\| \\
&\leq E_n(f) + M_4 \bullet \|t_n^* - B_n(t_n^*)\| + M_5 \bullet E_n(f) \\
&\leq (1 + M_5) \bullet E_n(f) + M_4 \bullet \|t_n^* - f\| + M_4 \bullet \|f - B_n f\| + M_4 \bullet \|B_n(f - t_n^*)\| \\
&\leq (1 + M_5) \bullet E_n(f) + M_4 \bullet E_n(f) + M_4 \bullet \|f - B_n f\| + M_4 \bullet M_6 \bullet E_n(f) \\
&\leq (1 + M_5 + M_4 + M_4 \bullet M_6) \bullet E_n(f) + M_4 \bullet \|f - B_n f\| .
\end{aligned}$$

It is clear that if A_n are also mapping $C_{2\pi}$ to $\sqcap_n$, then converse inequality holds, this is

COROLLARY 1. In Lemma 1 if in addition: A_n is a sequence of linear operators mapping $C_{2\pi}$ to $\sqcap_n$, then for any $f \in C_{2\pi}$

$$M_7 \bullet \|f - B_n f\| \leq \|f - A_n f\| \leq M_8 \bullet \|f - B_n f\| .$$

Corollary 1 (case in (2.2): $A_n(t_n) = B_n(t_n), \forall t_n \in \sqcap_n$) is also obtained by Berman in [4]

LEMMA 2. Let $k \in \mathbb{N}$ and A_n be a sequence of linear operators mapping $C_{2\pi}$ to $C_{2\pi}$, in order that for every $f \in C_{2\pi}$

$$\|f - A_n f\| \leq M_9 \bullet \|f - Z_n^k(f)\| , \tag{2.3}$$

it is sufficient and necessary that

(i) $\|A_n\| = O(1)$,

(ii) A_n satisfies (b_k) (if k is even) and $(\tilde{b}_k)$ (if k is odd), here

Condition (b_k)· for some $k \in \mathbb{N}$

$$\|f - A_n f\| \le \frac{M_{10}}{(n+1)^k}\|f^{(k)}\|, \quad \forall f, \quad f^{(k)} \in C_{2\pi}.$$

Condition $(\tilde{b}_k)$ for some $k \in \mathbb{N}$

$$\|f - A_n f\| \le \frac{M_{11}}{(n+1)^k}\left\|\tilde{f}^{(k)}\right\|, \quad \forall f, \quad \tilde{f}^{(k)} \in C_{2\pi}.$$

Here $\tilde{f}(x)$ is a conjugate function of $f(x) \in C_{2\pi}$

Necessity It is evident (see [5]), $\|Z_n^k\| = O(1)$, hence by (2.3) we have $\|A_n\| = O(1)$ The statement (ii) follows from the following Zygmund's inequalities (see [5]) in Chap. VIII, § 8.7, problem 27·

(iii) For $f \in C_{2\pi}$ and $f^{(k)} \in C_{2\pi}$

$$\|f - Z_n^k(f)\| \le \frac{M_{12}}{(n+1)^k}\cdot\|f^{(k)}\|, \quad \text{if } k \text{ is even}, \tag{2.4}$$

(iv) For $f \in C_{2\pi}$ and $\tilde{f}^{(k)} \in C_{2\pi}$

$$\|f - Z_n^k(f)\| \le \frac{M_{13}}{(n+1)^k}\cdot\left\|\tilde{f}^{(k)}\right\|, \quad \text{if } k \text{ is odd}, \tag{2.5}$$

Sufficiency We note that for $t_n \in \sqcap_n$ we have

$$\frac{\left\|t_n^{(k)}\right\|}{(n+1)^k} = \|t_n - Z_n^k(t_n)\|, \quad (k \text{ even}) \tag{2.6}$$

$$\frac{\left\|\tilde{t}_n^{(k)}\right\|}{(n+1)^k} = \|t_n - Z_n^k(t_n)\|, \quad (k \text{ odd}) \tag{2.7}$$

Combining these with (ii) we get

$$\|t_n - A_n(t_n)\| \le M_{14}\cdot\|t_n - Z_n^k(t_n)\|, \quad (k \in \mathbb{N}),$$

the inequality (2.3) follows from this estimate and Lemma 1

3. STEČKIN'S PROBLEM

THEOREM 1. Suppose that A_n is a sequence of linear operators mapping $C_{2\pi}$ to $C_{2\pi}$, A_n satisfies $\|A_n\| = O(1)$ and condition $(\tilde{b}_k)$ for some $k \in \mathbb{N}$, then for any $f \in C_{2\pi}$ we have

$$\|f - A_n f\| \le \frac{M_{15}}{(n+1)^k}\sum_{i=0}^{n}(i+1)^{k-1}\cdot E_i(f). \tag{3.1}$$

PROOF. If k is odd, then (3.1) follows from Lemma 2 and Theorem B. If k is even, and we choose $T_n \in \sqcap_n$, such that $\|f - T_n\| = E_n(f)$, since $(\tilde{b}_k)$ and (2.7) $(Z_n^1 = \sigma_n)$ we have

$$\begin{aligned}\|T_n - A_n(T_n)\| &\le \frac{M_{16}}{(n+1)^k}\left\|\tilde{T}_n^{(k)}\right\| = \frac{M_{16}}{(n+1)^k}\left\|(\tilde{T}_n^{(k-1)})'\right\| \\ &= \frac{M_{16}}{(n+1)^k}\left\|\widetilde{(T_n^{(k-1)})'}\right\| = \frac{M_{16}}{(n+1)^{k-1}}\|T_n^{(k-1)} - \sigma_n(T_n^{(k-1)})\|,\end{aligned} \tag{3.2}$$

we have by Theorem A

$$\|T_n^{(k-1)} - \sigma_n(T_n^{(k-1)})\| \le \frac{M_1}{(n+1)} \sum_{\nu=0}^{n} E_\nu(T_n^{(k-1)}), \tag{3 3}$$

to estimate the sum of (3.3), we apply the inequality (see [5]) in Chap V §5 6 36 If $r \in \mathbb{N}$ and $g \in C_{2\pi}$ and $\sum_{i=1}^{\infty} i^{r-1} \bullet E_i(g) < \infty$, then $g^{(r)} \in C_{2\pi}$ and

$$E_\nu(g^{(r)}) \le M_{17}\{(\nu+1)^r \bullet E_\nu(g) + \sum_{i=\nu+1}^{\infty} i^{r-1} \bullet E_i(g)\}, \tag{3 4}$$

taking $r = k-1 (k \ge 2)$, $g = T_n$, since $E_i(T_n) = 0 (i \ge n)$ we obtain for $0 \le \nu \le n$

$$E_\nu(T_n^{(k-1)}) \le M_{17}\{(\nu+1)^{k-1} \bullet E_\nu(T_n) + \sum_{i=\nu}^{n} i^{k-2} \bullet E_i(T_n)\}, \tag{3.5}$$

for $0 \le i \le n$ from definition we have

$$E_i(g) \le \|g - 0\| = \|g\|, \quad \forall g \in C_{2\pi},$$

and we have (see [5]) in Chap II, §2.5

(1) for g and $h \in C_{2\pi}, E_i(g+h) \le E_i(g) + E_i(h)$;

(2) $E_n(f) \le E_i(f)$

hence

$$E_i(T_n) \le E_i(T_n - f) + E_i(f) \le \|T_n - f\| + E_i(f) = E_n(f) + E_i(f) \le 2E_i(f),$$

from (3.5) we have

$$E_\nu(T_n^{(k-1)}) \le 2M_{17} \bullet \{(\nu+1)^{k-1} \bullet E_\nu(f) + \sum_{i=\nu}^{n} i^{k-2} \bullet E_i(f)\},$$

hence

$$\sum_{\nu=0}^{n} E_\nu(T_n^{(k-1)}) \le 2M_{17} \bullet \left\{ \sum_{\nu=0}^{n} (\nu+1)^{k-1} \bullet E_\nu(f) + \sum_{\nu=0}^{n} \sum_{i=\nu}^{n} i^{k-2} \bullet E_i(f) \right\}, \tag{3.6}$$

we have

$$\sum_{\nu=0}^{n} \sum_{i=\nu}^{n} i^{k-2} \bullet E_i(f) = \sum_{i=0}^{n} \sum_{\nu=0}^{i} i^{k-2} \bullet E_i(f) = \sum_{i=0}^{n} i^{k-2} \bullet (i+1) \bullet E_i(f) \le \sum_{i=0}^{n} (i+1)^{k-1} \bullet E_i(f), \tag{3.7}$$

combining (3.2), (3.3), (3.6), (3.7) we have

$$\|T_n - A_n(T_n)\| \le \frac{4M_{16} \bullet M_1 \bullet M_{17}}{(n+1)^k} \sum_{i=0}^{n} (i+1)^{k-1} \bullet E_i(f), \tag{3.8}$$

since $\|A_n\| = O(1)$ ($\|A_n\| \le M_{18}$) we have

$$\begin{aligned} \|f - A_n f\| &\le \|f - T_n\| + \|T_n - A_n(T_n)\| + \|A_n(T_n - f)\| \\ &\le E_n(f) + \frac{4M_{16} \bullet M_1 \bullet M_{17}}{(n+1)^k} \sum_{i=0}^{n} (i+1)^{k-1} \bullet E_i(f) + M_{18} \bullet E_n(f). \end{aligned} \tag{3.9}$$

since $E_n(f) \le E_i(f) (0 \le i \le n)$ we have

$$E_n(f) \le \frac{c_1}{(n+1)^k}\sum_{i=0}^{n}(i+1)^{k-1}\bullet E_n(f)$$
$$\le \frac{c_1}{(n+1)^k}\sum_{i=0}^{n}(i+1)^{k-1}\bullet E_i(f), \tag{3.10}$$

combining (3 9) and (3.10) we get

$$\|f - A_n f\| \le \frac{M_{15}}{(n+1)^k}\sum_{i=0}^{n}(i+1)^{k-1}\bullet E_i(f).$$

REMARK 1. Let $k \in \mathbb{N}$ and A_n be a sequence of linear operators mapping $C_{2\pi}$ to $C_{2\pi}$ In order that for any $f \in C_{2\pi}$

$$\|f - A_n f\| \le M_{19}\bullet\omega_k\left(f, \frac{1}{n}\right),$$

it is sufficient and necessary that A_n satisfies conditions $\|A_n\| = O(1)$ and (b_k) (see [6]) on page 182

We have (see [5]) in Chap. VI, 6.11, for $f \in C_{2\pi}$

$$\omega_k\left(f, \frac{1}{n}\right) \le \frac{M_{20}}{n^k}\sum_{i=0}^{n}(i+1)^{k-1}\bullet E_i(f),$$

from Remark 1 we also obtain

$$\|f - A_n f\| \le \frac{M_{21}}{(n+1)^k}\sum_{i=0}^{n}(i+1)^{k-1}\bullet E_i(f).$$

Let $f \in C_{2\pi}$ and $\omega_1(f,\delta)$ be the modulus of continuity of f. Classes of functions $\mathrm{Lip}(1,M):=\{f|\omega_1(f,\delta) \le M\delta\}$, and $\mathrm{Lip}\,1 := \left\{\bigcup_{M>0}\mathrm{Lip}\,(1,M)\right\}$

LEMMA 3. Let A_n be a sequence of linear operators mapping $C_{2\pi}$ to $C_{2\pi}$, $A_n(1,x) = 1$, if for $\tilde{f}^{(k-1)} \in \mathrm{Lip}\,1$

$$\|f - A_n f\| = O\left(\frac{1}{n^k}\right), \tag{3.11}$$

then for $f \in C_{2\pi}$ and $\tilde{f}^{(k)} \in C_{2\pi}$ we have

$$\|f - A_n f\| \le \frac{M_{22}}{(n+1)^k}\bullet\left\|\tilde{f}^{(k)}\right\|. \tag{3 12}$$

PROOF. If $f \in C_{2\pi}$ and $\tilde{f}^{(k)} \in C_{2\pi}$ we have

$$\left|\tilde{f}^{(k-1)}(x+t) - \tilde{f}^{(k-1)}(x)\right| \le \left\|\tilde{f}^{(k)}\right\|t, \quad (t \ge 0),$$

if $D := \left\|\tilde{f}^{(k)}\right\| > 0$ then $\tilde{f}^{(k-1)}/D \in \mathrm{Lip}\,(1,1)$, from (3.11) we obtain $\left\|\frac{f}{D} - A_n\left(\frac{f}{D}\right)\right\| = O\left(\frac{1}{n^k}\right)$, hence

$$\|f - A_n f\| \le \frac{M_{22}}{(n+1)^k}\bullet D = \frac{M_{22}}{(n+1)^k}\left\|\tilde{f}^{(k)}\right\|, \tag{3.13}$$

if $\left\|\tilde{f}^{(k)}\right\| = 0$, then $f \equiv \mathrm{const}$ (see [5]) in §5.9 1, since $A_n(1,x) = 1$, obviously (3.12) holds

Sequence of Fejér mean σ_n is saturated with order (n^{-1}) and saturation class $S(L_n) := \{f|\tilde{f} \in \text{Lip } 1\}$, using Lema 3 we obtain that σ_n satisfies $(\tilde{b}_1)$, since $\|\sigma_n\| = 1$ and Theorem 1, we obtain Theorem A

PROBLEM 1. Let A_n be a sequence of linear operators mapping $C_{2\pi}$ to $C_{2\pi}$, finding sufficient and necessary conditions on A_n such that Timan type inequality (1 3) holds

4. APPLICATIONS

We give applications on linear summability method $\bigcup_{m(n)}$ of Fourier series Firstly we have $\bigcup_{m(n)}(1, x) = 1$

EXAMPLE 1. (C, α) means $\sigma_n^\alpha(\alpha > 0)$. $\lambda_{i,n} = \frac{A_{n-i}^\alpha}{A_n^\alpha}(0 \leq i \leq n)$, $A_n^\alpha : = \frac{(\alpha+1)(\alpha+2) \quad (\alpha+n)}{n!}$

Trigub proved [3].

LEMMA 4. Let $\alpha > 0$ and $f \in C_{2\pi}$, then

$$C_3 \cdot \|f - \sigma_n(f)\| \leq \|f - \sigma_n^\alpha(f)\| \leq C_2 \cdot \|f - \sigma_n(f)\| .$$

THEOREM 2. Let $\alpha > 0$ and $f \in C_{2\pi}$, then we have

$$\|f - \sigma_n^\alpha(f)\| \leq \frac{M_{23}}{n+1} \sum_{i=0}^{n} E_i(f) . \tag{4.1}$$

PROOF. Obviously from Theorem A and Lemma 4 we obtain Theorem 2.

Let $\omega(\delta)$ be a modulus of continuity and $\omega(\delta) > 0 \, (0 < \delta \leq \pi)$. Class of functions $H_\omega : = \{f|\omega_1(f, \delta) \leq \omega(\delta), 0 \leq \delta \leq \pi\}$.

Let $\omega_1^{**}(\delta)$ be a modified function of first order of $\omega(\delta)$ (see [1])

$$\omega_1^{**}(\delta) : = \delta \cdot \inf_{0 < \eta \leq \delta} \left\{ \eta^{-1} \cdot \inf_{\eta \leq \xi \leq \pi} \omega(\xi) \right\}, \text{ we have } \omega_1^{**}(\delta) \leq \omega(\delta).$$

Let $a_n > 0$, $b_n > 0$, $a_n \approx b_n$ means that there are $C_4 > 0$, $C_5 > 0$ such that $C_4 a_n \leq b_n \leq C_5 a_n$.

COROLLARY 2. Let $\alpha > 0$, we have

$$\sup_{f \in H_\omega} \|f - \sigma_n^\alpha(f)\| \approx \frac{1}{n} \sum_{i=1}^{n} \omega^{**}\left(\frac{1}{i}\right), \tag{4.2}$$

$$\sup_{f \in H_\omega} \|f - \sigma_n^\alpha(f)\| \approx \frac{1}{n} \sum_{i=1}^{n} \omega\left(\frac{1}{i}\right), \tag{4.3}$$

$$\sup_{f \in H_\omega} \|f - \sigma_n^\alpha(f)\| \approx \frac{1}{n} \int_{\frac{1}{n}}^{\pi} \frac{\omega(u)}{u^2} \, du . \tag{4.4}$$

PROOF. For (4.2) $(\alpha \geq 1)$ see Sun [7]. For (4.3) $(\alpha = 1)$ see Devore [8] on page 227 For (4.4) $(\alpha \geq 1)$ see Mazhar and Totik [9]. Using Lemma 4 we have Corollary 2.

Stečkin also proved (see [1])

LEMMA 5. For f and $\tilde{f} \in C_{2\pi}$, we have

$$\|f - \sigma_n(f)\| = O(E_n(f)) + O\left(\omega_1\left(\tilde{f}, \frac{1}{n}\right)\right) .$$

Lemma 4 implies

COROLLARY 3. Let $\alpha > 0$, for f and $\tilde{f} \in C_{2\pi}$ we have

$$\|f - \sigma_n^\alpha(f)\| = O(E_n(f) + O\left(\omega_1\left(\tilde{f}, \frac{1}{n}\right)\right). \qquad (4\ 5)$$

(4 2) and (4 5) answer two problems of Sun [7] on $\sigma_n^\alpha (0 < \alpha < 1)$

EXAMPLE 2. M Riesz means $R_n^{(\lambda,\delta)}$ $\lambda_{i,n} = \lambda\left(\frac{i}{n+1}\right)(0 \le i \le n)$, $\lambda(u) = (1-u^\lambda)^\delta (\lambda \in \mathbb{N}, \delta > 0)$

B Nagy proved that (see [5]) in Chap VIII, §8 7, problem 13, $\left\|R_n^{(\lambda,\delta)}\right\| = O(1)$ G Sunouchi proved that [6] on page 72, $R_n^{(\lambda,\delta)}$ is saturated with order $(n^{-\lambda})$ and the saturation class is $S(R_n^{(\lambda,\delta)}) : = \{f \mid \tilde{f}^{(\lambda-1)} \in \mathrm{Lip}\, 1\ (\lambda \text{ odd}) \text{ and } f^{(\lambda-1)} \in \mathrm{Lip}\, 1\ (\lambda \text{ even})\}$, using Theorem 1 and Remark 1 we obtain that for any $f \in C_{2\pi}$

$$\|f - R_n^{(\lambda,\delta)}(f)\| \le \frac{M_{24}}{(n+1)^\lambda} \sum_{i=0}^{n} (i+1)^{\lambda-1} \bullet E_i(f), \quad (\lambda \in \mathbb{N}).$$

EXAMPLE 3. Operators L_n determined by convolution with kernels of Korovkin (see [8]) on page 107 L_n is saturated with order (n^{-1}) and saturation class $S(L_n) : = \{f | \tilde{f} \in \mathrm{Lip}\, 1\}$, hence we obtain Stečkin type inequality

EXAMPLE 4. Nishishiraho and Wang Si-Lei proved (see [10])

LEMMA 6. Suppose that there exists a sequence $\{\phi_n\}$ of positive real numbers converging to zero, which satisfies

$$\lim_{n\to\infty} \frac{(1-\lambda_{i,n})}{\phi_n} = K, \quad \text{and let} \quad \sum_{i=0}^{n} |\Delta^2 \lambda_{i,n}| = 0(\phi_n),$$

where $\Delta^2 \lambda_{i,n} = \lambda_{i,n} - 2\lambda_{i+1,n} + \lambda_{i+2,n}$, and $\lambda_{i,n} = 0 (i > n)$. If $\phi_n = \frac{6}{n}$, then $\bigcup_n$ is saturated with the order (n^{-1}) and saturation class $S(\bigcup_n) : = \{f | \tilde{f} \in \mathrm{Lip}\, 1\}$, using Theorem 1 and Lemma 3 we obtain Stečkin type inequality.

5. POLYNOMIALS OF INTERPOLATION AND CAO-GONSKA OPERATORS

Let $f(x) \in C_{2\pi}$ and $\bigcup_n^*(f, x)$ be linear summability (with $\wedge = \{\lambda_{i,n}\}$) of trigonometric polynomial of interpolation on nodes $y_i = \frac{2i\pi}{2n+1}$ $(i = 0, 1, ..., 2n)$ [4] [5]. Berman proved [4] and [5] in 8.7, problem 7

LEMMA 7. Let $K_n(v) : = \frac{1}{2} + \sum_{i=1}^{n} \lambda_{i,n} \cos iv$, $\int_0^\pi |K_n(v)| dv = O(1)$, then for $f \in C_{2\pi}$

$$M_{25} \bullet \|f - U_n(f)\| \le \|f - U_n^*(f)\| \le M_{26} \bullet \|f - U_n(f)\|.$$

THEOREM 3. Let $k \in \mathbb{N}$ and $\int_0^\pi |K_n(v)| dv = O(1)$, and $\wedge = \{\lambda_{i,n}\}$ satisfies $(\tilde{b}_k)$, then, for any $f \in C_{2\pi}$

$$\|f - U_n^*(f)\| \le \frac{M_{27}}{(n+1)^k} \sum_{i=0}^{n} (i+1)^{k-1} E_i(f).$$

PROOF. From Lemma 7 and Theorem 1 we obtain Theorem 3.

Let $f \in C[-1, 1]$, the Pičugov-Lehnhoff operators are defined by $(\theta = \arccos x, x \in [-1, 1]$, $K_{m(n)}(v) : = \frac{1}{2} + \sum_{i=1}^{m(n)} \lambda_{i,m(n)} \cos iv)$

$$G_{m(n)}(f(t), x) : = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\cos(v + \arccos x)) K_{m(n)}(v) dv. \qquad (5\ 1)$$

Let $T_i(x):=\cos(i\arccos x)$ be the i-th Cebyšev polynomial, and $x_{\gamma,N_0}:=\cos\frac{2\gamma-1}{2N_0}\pi$, $1\le\gamma\le N_0$, the Cao and Gonska polynomials are defined by (see [11])

$$\wedge_{m(n),N_0}(f,x):=\frac{1}{N_0}\sum_{\gamma=1}^{N_0}f(x_{\gamma,N_0})\left\{1+2\sum_{i=1}^{m(n)}\lambda_{i,m(n)}\cdot T_i(x_{\gamma,N_0})\cdot T_i(x)\right\},\tag{5 2}$$

specifically $\wedge_{n-1,n}$ are the Varma-Mills operators (see [11])

LEMMA 8. Let $N_0\ge m(n)+1$ and $\int_0^\pi|K_{m(n)}(v)|dv=O(1)$, then for any $f\in C[-1,1]$

$$M_{28}\cdot\|f-G_{m(n)}(f)\|_{C[-1,1]}\le\|f-\wedge_{m(n),N_0}(f)\|_{C[-1,1]}\le M_{29}\cdot\|f-G_{m(n)}(f)\|_{C[-1,1]}.$$

PROOF. (see [12]).

THEOREM 4. Let $k\in\mathbb{N}$, $N_0\ge m(n)+1$, and $\wedge=\{\lambda_{i,m(n)}\}$ satisfies $\int_0^\pi|K_{m(n)}(v)|dv=O(1)$ and $(\tilde{b}_k)$, then for any $f\in C[-1,1]$

$$\|f-\wedge_{m(n),N_0}(f)\|_{C[-1,1]}\le\frac{M_{30}}{(n+1)_K}\sum_{i=0}^{n}(i+1)^{k-1}\cdot E_i(f)_{C[-1,1]}.$$

PROOF. Letting $\phi(t)=f(\cos t)$, using Lemma 8 and Theorem 1 we obtain Theorem 4.

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