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RESEARCH NOTES

ON THE MOMENTS OF RANDOM VARIABLES UNIFORMLY DISTRIBUTED OVER A POLYTOPE

S. PARAMASAMY
Department of Mathematics and Computer Science
The University of the West Indies
TRINIDAD

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ABSTRACT. Suppose $X = (X_1, X_2, ..., X_n)$ is a random vector uniformly distributed over a polytope. In this note, the author derives a formula for $E(X_i^r X_j^s ...)$, (the expected value of $X_i^r X_j^s ...$), in terms of the extreme points of the polytope.

KEY WORDS AND PHRASES: Uniform distribution, r^{th} moment, polytope.
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1. INTRODUCTION

Von Hohenbalken [2] presents an algorithm for decomposing a polytope $V = \{x | Ax \geq b, x \in R^n\}$ where A is an $m \times n$ matrix, into n-simplexes and states a formula for the center of gravity of V. In this note we explain how these results can be generalized to find formulas for $E(X_i^r X_j^s ...)$, where $X = (X_1, X_2, .., X_n)$ is a random vector (r.v.) uniformly distributed over V. Let $\int_Q f(x)dx$ be the Lebesque integral of a continuous function over a compact set $Q \in R^n$ In order to motivate the approach we have chosen, consider the problem of finding the r^{th} moment $E(X_i^r)$ using the definition formulating the integral

$$E(X_i^r) = \int_v x_i^r dx$$

as a set of n-fold iterated integrals and evaluating them. But how does one find the range of integration in a systematic way, given an arbitrary matrix A? It appears that no algorithm has yet been developed.

2. MAIN RESULTS

In order to simplify the arguments, we assume that $L(V)$, the Lebesque measure of V is positive In the following, we denote vectors by lower case letters with or without superscript. The i^{th} element the vector x is x_i and $L(Q)$ is the Lebesque measure of a compact set $Q \subset R^n$

Since V is a polytope, it is the convex hull of its extreme points, say $x^1, x^2, ..., x^e$ Let E be the set of these extreme points. Then there exists a set $S = \{S_1, S_2, ..., S_g\}$ of n simplexes such that the $n+1$ extreme points of each S_i are in E, $i = 1, 2, ..., g$ and that

i) $V = U_i S_i$
ii) $L(V) = \sum_i L(S_i)$.

The set E can be found using the algorithm in Dyer et al [1] while the set S can be found by decomposing E using the results given in [2]. Now i) and ii) imply that

$$E(X_i^a X_j^b ...) = \sum_{i=1}^{g} L(S_i) E(X_s^a X_j^b ... | X \subset S_i)/L(V), \quad a, b = 0, 1, ...$$

Let x^{ij}, $j = 1, 2, ..., n+1$, be the extreme points of S_i, $i = 1, 2, ..., g$ and let A^i be the matrix whose j^{th} column is $x^{ij} - x^{i,n+1}$, $j = 1, 2, ..., n$, and B^i the matrix whose j^{th} column is x^{ij}, $j = 1, 2, ..., n+1$. Then A^i is non-singular, $L(S_i) = |\det A^i|/n!$ and

$$S_i = \left\{ x | x = B^i y, y_j \geq 0, j = 1, 2, ..., n+1, \sum_i y_i = 1 \right\}. \tag{2.1}$$

THEOREM. Let b^{ij} be the j^{th} row of B^i, $a_1, a_2, ..., a_p$ a sequence of positive integers such that $a_1 + a_2 + ... + a_p = m$ and c_s the coefficient of $y_1^{s_1} y_2^{s_2} ... y_{n+1}^{s_{n+1}}$ in the expansion of $(b^{ij}y)^{a_1}(b^{ik}y)^{a_2}...(b^{ir}y)^{a_p}$, where $s = (s_1, s_2, ..., s_{n+1})$ such that $\sum_i s_i = m$ and $y = (y_1, y_2, ..., y_{n+1})$

Then

$$E(X_j^{a_1} X_k^{a_2} ... X_r^{a_p} | X \subset S_i) = \frac{n!}{(m+n)!} \sum_s s_1! s_2! ... s_{n+1}! c_s, \quad (j \neq k \neq ... \neq r).$$

PROOF. Let $U = (U_1, U_2, ..., U_n)$ be an $r.v$ uniformly distributed over

$$U = \left\{ u | u_i \geq 0, i = 1, 2, ..., n; \sum_i u_i \leq 1 \right\}.$$

We first show that $E(U_i^a U_j^b ... U_k^q) = \frac{n!a!b!...q!}{(v+n)!}$, where $v = a + b + ... + q$ and $i \neq j \neq ... \neq k$.

Now U is an n-simplex whose extreme points are 0, e^1, e^2, ..., e^n, where e^i is the unit vector whose i^{th} element is 1. Therefore $L(U) = \det I/n! = 1/n!$ (where I is the identity matrix). Let

$$g(i,1) = g_{i,1}(x_{i+1}, x_{i+2}, ..., x_x), \quad g(i,2) = g_{i,2}(x_{i+1}, x_{i+2}, ..., x_n), \quad i = 1, 2, ..., n-1,$$

where $g(i,1)$ and $g(i,2)$ are continuous functions in R^{n-i}. Suppose $g(i,1) \leq g(i,2)$ for all $(x_{i+1}, x_{i+2}, ..., x_n)$ in some compact set $Q_i \subset R^{n-i}$, $i = 1, 2, ..., n-1$ and that $f(x)$ is continuous over a compact set

$$G = \{x | a \leq x_n \leq b, g(i,1) \leq x_i \leq g(i,2), (x_{i+1}, x_{i+2}, ..., x_n) \in Q_i, i = 1, 2, ..., n-1\}.$$

Then

$$\int_G f(x)dx = \int_a^b dx_n \int_{g(n-1,1)}^{g(n-1,2)} dx_{n-1} \int_{g(n-2,1)}^{g(n-2,2)} dx_{n-2} \cdots \int_{g(1,1)}^{g(1,2)} f(x) dx_1$$

where the r.h.s is an n-fold iterated integral. Note that

$$U = \{u | 0 \leq u_n \leq 1, 0 \leq u_i \leq 1 - (u_{i+1} + u_{i+2} + ... + u_n), i = 1, 2, ..., n-1\}.$$

Consequently,

$$E(U_i^a U_j^b) = \int_0^{c(n)} dx_n \cdots \int_0^{c(2)} dx_2 \int_0^{c(1)} u_i^a u_j^b \ldots du_1 / L(U) \tag{2.2}$$

where $c(n) = 1$ and $c(i) = 1 - (u_{i+1} + u_{i+2} + ... + u_n)$ for $i = 1, 2, ..., n-1$.

Evaluating (2) for the case of $E(U_1^a)$, it is easily seen that $E(U_1^a) = E(U_i^a) = n!a!/(a+n)!$. For the case of $E(U_i^a U_j^b)$, it is easier to evaluate for $i = 1$ and $j = 2$. The integrand after integrating with respect to $u_1$is $f_u = \frac{n!a!c(1)^{(a+1)}}{(a+1)!}$. Now

$$\int_0^{c(2)} f_u du_2 = \frac{n!a!b!c(2)^{a+b+2}}{(a+b+2)!}$$

from which we derive easily that $E(U_i^a U_j^b) = \frac{n!a!b!}{(a+b+n)!}$ The proof can be completed by repeating this argument

An immediate consequence of the above result is that if $Y = (Y_1, Y_2, ..., Y_{n+1})$ is an r v uniformly distributed over

$$Y = \left\{ y | y_i \geq 0, i = 1, 2, .., n+1, \sum_i y_i = 1 \right\}$$

then $E(Y_{i_1}^a Y_{i_2}^b \dots Y_{i_r}^q) = \frac{n!a!b!\dots q!}{(m+n)!}$ for all $i_1, i_2, ..., i_r \in \{1, 2, ..., n+1\}$ and for all $r \leq n$. We now show that the result is true for $r = n+1$ so that $E(Y_1^a Y_2^b ... Y_n^q Y_{n+1}^t) = \frac{n!a!b!\dots q!t!}{(w+n)!}$, where $w = a + b + ... + q + t$. The proof is by induction. It is easily proved, using the substitution $Y_{n+1} = (1 - Y_1 - Y_2 - ... - Y_n)$ that the result is true for $t = 1$. So assume that it is true for all positive integers, $a, b, ..., p$ and for some $t \geq 2$. Then

$$E(Y_1^a Y_2^b ... Y_n^q Y_{n+1}^t (1 - Y_1 - Y_2 - ... - Y_n)$$

simplifies, by the induction hypothesis, to the required result, proving that

$$E(Y_j^a Y_k^b ... Y_r^q) = \frac{n!a!b!...q!}{(m+n)!} \tag{2 3}$$

for all $\{j, k, ..., r\} \subset \{1, 2, .., n+1\}$ such that $j \neq k \neq ... \neq r$. Now using (1) and the properties of the matrix B^i, it can be easily proved that

$$E(X_j^{a_1} X_k^{a_2} ... X_r^{a_p} | X \subset S_i) = E(b^{ij} Y)^{a_1} (b^{ik} Y)^{a_2} \dots (b^{ir} Y)^{a_p} .$$

The theorem now follows immediately from (2.3).

COROLLARY 1. Let the j^{th} row of B^i be $b^{ij} = (b_{j1}, b_{j2}, ..., b_{j,n+1})$. Then

$$E(X_j^m | X \subset S_i) = \frac{n!m!}{(m+n)!} \sum b_{j1}^p b_{j2}^q \dots b_{j,n+1}^r$$

where the summation is over all non-negative integers such that $p + q + ... r = m$.

PROOF. The result follows from (2.3) and the equation

$$E(b_{j1} Y_1 + b_{j2} Y_2 + ... + b_{j,n+1} Y_{n+1})^m = \sum \frac{m!}{p!q!...r!} b_{j1}^p b_{j2}^q ... b_{j,n+1}^r E(Y_1^p Y_2^q ... Y_{n+1}^r) .$$

COROLLARY 2. Let p_k be the sum of the elements of the k^{th} row of B^i and J be the $(n+1) \times (n+1)$ matrix whose $(k, j)^{\text{th}}$ element is $p_k p_j$, $k, j = 1, 2, ..., n$. Then

$$E(XX^t | X \subset S_i) = \frac{n!(B^i (B^i)^t + j)}{(2+n)!} .$$

PROOF. Now

$$\begin{aligned} E(X_k^2) &= \sum_j b_{k,j}^2 E(Y_j^2) + 2 \sum_{s>j} b_{k,s} b_{k,j} E(Y_s Y_j) \\ &= \frac{n!}{(n+2)!} \left(2(b^{ik})^t b^{ik} + 2 \sum_{s>j} b_{k,s} b_{k,j} \right) \\ &= \frac{n!}{(n+2)!} (p_k^2 + (b^{ik})^t b^{ik}) \end{aligned}$$

(where $(b^{ik})^t$ is the transpose of b_{ik}). Similarly,

$$E(X_k X_r) = \sum_j b_{k,j} b_{r,j} E(Y_j^2) + \sum_{j=s} b_{k,s} b_{k,j} E(Y_s Y_j)$$
$$= \frac{n!}{(n+2)!} \left(p_k p_r + (b^{ik})^t b^{ir}\right).$$

Hence the corollary

An Application that requires $u_i = E(X_i)$ and $E(X_i - u_i)^2$ is studied in [3].

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