COMMON FIXED POINT THEOREMS AND APPLICATIONS

HEMANT KUMAR PATHAK

Department of Mathematics, Kalyan Mahavidyalaya
Bhilai Nagar (M.P.) 490 006, INDIA

SHIH SEN CHANG

Department of Mathematics, Sichuan University
Chengdu, Sichuan 610064, PEOPLE'S REPUBLIC OF CHINA

YEOL JE CHO

Department of Mathematics, Gyeongsang National University
Jinju 660-701, KOREA

JONG SOO JUNG

Department of Mathematics, Dong-A University
Pusan 604-714, KOREA

(Received November 22, 1994 and in revised form January 18, 1995)

ABSTRACT. The purpose of this paper is to discuss the existence of common fixed points for mappings in general quasi-metric spaces. As applications, some common fixed point theorems for mappings in probabilistic quasi-metric spaces are given. The results presented in this paper generalize some recent results.

KEY WORDS AND PHRASES. General quasi-metric space, probabilistic quasi-metric space, fixed point, periodic point, periodic index.
1991 AMS SUBJECT CLASSIFICATION CODES. 54H25, 54A40, 54C60.

1. INTRODUCTION

In this paper, we show the existence of common fixed points for commuting mappings in general quasi-metric spaces. As applications, we give some fixed point theorems for commuting mappings in probabilistic quasi-metric spaces. Our main results generalize and improve some recent results in [1], [4], [5] and [6].

Let $(G, \leq, <)$ be a partial order set satisfying the following conditions:

(G-1) 0 is the minimal element in G, i.e., $0 \leq u$ for all $u \in G$,

(G-2) for any $u, v \in G$, $\sup\{u, v\}$ exists and belongs to G,

(G-3) for any $u \in G$, $u \not< u$,

(G-4) for any $u, v, w \in G, u < w$ and $v < w \Rightarrow \sup\{u, v\} < w$, and $u < v$, $v \leq w \Rightarrow u < w$.

DEFINITION 1.1 Let X be a nonempty set. (X, r) is called a *general quasi-metric space* if $r : X \times X \to G = (G, \leq, <)$ satisfies the following conditions:

(QM-1) $r(x, y) = 0$ if and only if $x = y$,

(QM-2) $r(x, y) = r(y, x)$.

It follows from the definition that every general quasi-metric space includes a metric space as its special case.

DEFINITION 1.2 Let X be a nonempty set and let T be a self-mapping of X. A point $x \in X$ is called a *periodic point* of T if there exists a positive integer k such that $T^k x = x$. The least positive integer satisfying this condition is called the *periodic index* of x.

DEFINITION 1.3 A mapping $F : (-\infty, \infty) \to [0, \infty)$ is called a *distribution function* if it is nondecreasing and left-continuous with $\inf F(t) = 0$ and $\sup F(t) = 1$.

In what follows we always denote by $F(T)$ the set of all fixed points of T, $P(T)$ the set of all periodic points of T and $\mathcal{D}$ the set of all distribution functions, respectively, and let $\mathcal{D}^+ = \{F \in \mathcal{D} : F(t) = 0 \text{ for all } t < 0\}$.

DEFINITION 1.4 $(X, \mathcal{F})$ is called a *probabilistic quasi-metric space* if X is a nonempty set, $\mathcal{F}$ is a mapping from $X \times X$ into $\mathcal{D}^+$ (we shall denote the distribution function $\mathcal{F}(x, y)$ by $F_{x,y}(t)$ which represent the value of $F_{x,y}$ at $t \in (-\infty, \infty)$) satisfying the following conditions:

(PQM-1) $F_{x,y}(0) = 0$,

(PQM-2) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,

(PQM-3) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in (-\infty, \infty)$.

DEFINITION 1.5 $(X, \mathcal{F})$ is called a *probabilistic metric space* if $(X, \mathcal{F})$ is a probabilistic quasi-metric space and the following condition is satisfied:

(PQM-4) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$, then $F_{x,z}(t_1 + t_2) = 1$.

For more details on probabilistic metric spaces, refer to [3] and [7].

2. COMMON FIXED POINT THEOREMS

Now, we give our main theorems.

THEOREM 2.1 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. If for any $x \in X$ and any two positive integers $n, q \geq 2$ with

$$\begin{aligned} T^i x \neq T^j x, \quad 0 \leq i < j \leq n-1, \\ S^{i'} x \neq S^{j'} x, \quad 0 \leq i' < j' \leq q-1, \end{aligned} \tag{2.1}$$

$$r(T^n S^i x, S^q T^i x)$$

$$< \max\left\{ \sup_{1 \leq j \leq n, 1 \leq j' \leq q} r(T^j x, S^{j'} x), \sup_{1 \leq j \leq n} r(T^j x, x), \sup_{1 \leq j' \leq q} r(S^{j'} x, x) \right\} \tag{2.2}$$

for $i = 1, 2, ..., n-1$ and $j = 1, 2, ..., q-1$, then S and T have a common fixed point in X if and only if there exist integers m, n, p, q, $m > n \geq 0$, $p > q \geq 0$, and a point $x \in X$ such that

$$T^m x = S^p x = T^n x = S^q x \tag{2.3}$$

and either $F(S) \neq \emptyset$ or $F(T) \neq \emptyset$. If this condition is satisfied, then either $T^n x$ or $S^q x$ is a common fixed point of S and T.

PROOF. Let $x^* \in X$ be a common fixed point of T, i.e., $x^* = Sx^* = Tx^*$. Then (2.3) is true in case $m = p = 1$ and $n = q = 0$.

Conversely, suppose that there exist a point $x \in X$ and four integers m, n, p, q, $m > n \geq 0$, $p > q \geq 0$, such that (2.3) is satisfied. Without loss of generality, we can assume that $x \in F(S)$ and m is the minimal integer satisfying $T^k x = T^n x$, $k > n$. Putting $y = T^n x$ and $p_1 = m - n$, we have $T^{p_1} y = y$ and p_1 is the minimal integer satisfying $T^k y = y$, $k \geq 1$.

By (2.2), it follows that

$$r(T^n x, T^i x) < \max\left\{ \sup_{1 \leq j \leq n} \{r(T^j x, x)\}, \sup_{1 \leq j \leq n} \{r(T^j x, x\}, 0 \right\},$$

i.e.,

$$r(T^n x, T^i x) < \sup_{1 \leq j \leq n} \{r(T^j x, x)\}. \tag{2 4}$$

Next, we prove that y is a common fixed point of S and T. Suppose the contrary. Then y is not a fixed point of T. Also, $p \geq 2$ and

$$T^i y \neq T^j y, \quad 0 \leq i < j \leq p_1 - 1.$$

By (2.4), it follows that, for $i = 1, 2, ..., p_1 - 1$,

$$r(y, T^i y) = r(T^{p_1} y, T^i y) < \sup_{1 \leq j \leq p_1} \{r(T^j y, y)\} \leq \sup_{1 \leq j \leq p_1 - 1} \{r(T^j y, y)\}.$$

It follows from (G-4) that

$$\sup_{1 \leq j \leq p-1} \{r(y, T^j y)\} < \sup_{1 \leq j \leq p-1} \{r(T^j y, y)\},$$

which is a contradiction. Therefore, $y = T^m x$ is a fixed point of T. Further, since S and T are commuting, we have

$$y = T^m x = T^m S x = S T^m x = S y,$$

i.e., $y = T^m x$ is a common fixed point of S and T. In this case, when $x \in F(T)$, we have, by interchanging the role of S and T, that $y = S^q x$ is a common fixed point of S and T. This completes the proof.

On the other hand, by using Theorem 3 of [6], we have the following:

THEOREM 2.2 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. Assume that for any $x, y \in X$, $x \neq y$, there exists a positive integer $p(x, y)$ such that

$$r((ST)^{p(x,y)}x, (ST)^{p(x,y)}y) < \sup\{r(x,y), r(x, (ST)^{p(x,y)}x), r(y, (ST)^{p(x,y)}y), r(x, (ST)^{p(x,y)}y), r(y, (ST)^{p(x,y)}x)\}.$$

Then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k such that for any $u, v \in A = \{x, STx, ..., (ST)^{k-1}x\}$, $u \neq v$, there exist $x', y' \in A$, $x' \neq y'$, satisfying the following conditions:

$$(ST)^{p(x',y')}x' = u, \quad (ST)^{p(x',y')}y' = v$$

and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(St) \neq \emptyset$. If these conditions are satisfied, then the point x is the unique common fixed point of S and T.

PROOF. The necessity condition is obvious.

The sufficiency condition follows from Theorem 3 of [6] as follows: In Theorem 3 of [6], if we replace T by ST, we can conclude that ST has a unique fixed point x in X. Since S and T are commuting,

$$Sx = S(STx) = ST(Sx)$$

and so Sx is also a fixed point of ST. Uniqueness gives $Sx = x$. Similarly, $Tx = x$. This completes the proof.

The following is a special case of Theorem 2.2 by setting $p(x, y) = p(x)$:

COROLLARY 2.3 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. Assume that for any $x \in X$, there exists a positive integer $p(x)$ such that every $y \in X$, $x \neq y$,

$$r((ST)^{p(x)}x, (ST)^{p(x)}y) < \sup\{r(x,y), r(x, (ST)^{p(x)}x), r(y, (ST)^{p(x)}x), r(y, (ST)^{p(x)}y), r(x, (ST)^{p(x)}y), r(y, (ST)^{p(x)}x)\}.$$

Then S and T has a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k such that for any $u, v \in A = \{x, STx, ..., (ST)^{k-1}x\}$, $u \neq v$, there exist $x', y' \in A$, $x' \neq y'$, satisfying the following conditions:

$$(ST)^{p(x')}x' = u, \quad (ST)^{p(x')}y' = v$$

and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If these conditions are satisfied, then the point x is the unique common fixed point of S and T.

The following is obtained from Corollary 2.3 by setting $p(x) = p$:

COROLLARY 2.4 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. Assume that there exists a positive integer p such that for any $x, y \in X$, $x \neq y$,

$$r((ST)^p x, (ST)^p y) < \sup\{r(x,y), r(x,(ST)^p x), r(y,(ST)^p y), r(x,(ST)^p y), r(y,(ST)^p x)\}.$$

Then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If this condition is satisfied, then the point x is the unique common fixed point of S and T.

By setting $p = 1$ in Corollary 2.4, we have the following:

COROLLARY 2.5 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. Assume that for any $x, y \in X$, $x \neq y$,

$$r(STx, STy) < \sup\{r(x,y), r(x,STx), r(y,STy), r(x,STy), r(y,STx)\}.$$

Then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If this condition is satisfied, then the point x is the unique common fixed point of S and T.

By using Theorem 5 of [6], we have the following:

THEOREM 2.6 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. Assume that there exist positive integers p, q such that for any $x, y, \in X$, $x \neq y$,

$$r((ST)^p x, (ST)^q y) < \sup\{r(x,y), r(x,(ST)^p x), r(y,(ST)^q y), r(x,(ST)^q y), r(y,(ST)^p x)\}.$$

Then S and T have a fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k which satisfies the following condition:

$$k \neq 2|p_2 - q_2|,$$

where $p = p_1 k + p_2$, $q = q_1 k + q_2$, $0 \leq p_2, q_2 < k$ and p_1, q_1 are non-negative integers and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If this condition is satisfied, then the point x is the unique common fixed point of S and T.

PROOF. The necessity condition is obvious.

To prove converse, if we use Theorem 5 of [6] by replacing T with ST, then ST has a unique fixed point x in X. Therefore, employing the same argument as in the proof of Theorem 2.2, it follows that the point x is the unique fixed point of S and T. This completes the proof.

REMARK. Theorems 2.1 ~ 2.6 generalize some main results in [1], [2] and [4].

3. APPLICATIONS TO PROBABILISTIC QUASI-METRIC SPACES

First of all, we define partial orders " $\leq$ " and " $<$ " on $\mathcal{D}^+$ as follows, respectively: For any F_1, $F_2 \in \mathcal{D}^+$ and $t > 0$,

$$F_1 \leq F_2 \Rightarrow F_1(t) \geq F_2(t),$$
$$F_1 < F_2 \Rightarrow F_1(t) > F_2(t).$$

In the sequel, we denote $G = (\mathcal{D}^+, \leq, <)$. It is obvious that G satisfies the following conditions:

(G-1) there exists a minimal element $0 \overset{\text{def}}{=} H \in G$, where

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0, \end{cases}$$

(G-2) for any $F_1,\ F_2 \in G$,

$$\sup\{F_1, F_2\}(t) \overset{\text{def}}{=} \min\{F_1(t), F_2(t)\}\,,$$

(G-3) for any $F \in G,\ F \not< F$,

(G-4) for any $F_1,\ F_2,\ F_3 \in G$,

$$F_1 < F_3,\ F_2 < F_3 \Rightarrow \sup\{F_1, F_2\} < F_3\,,$$
$$F_1 < F_2,\ F_2 \leq F_3 \Rightarrow F_1 < F_3\,.$$

THEOREM 3.1 (Embedding Theorem) Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space. Then $(X, \mathcal{F})$ is a general quasi-metric space, where $G = (D^+,\ \leq\ ,\ <\ )$ is the partial order set induced by the way as above.

PROOF. Let $r(x, y) = F_{x,y}$ for all $x,\ y \in X$. It is easy to see that r satisfies the conditions (QM-2) and (QM-2) of Definition 1.1.

The following results are obtained from Theorems 2.1 ~ 2.6 and Theorem 3.1 immediately:

THEOREM 3.2 Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space and let S and T be two commuting self-mappings of X. If for any $x \in X$ and positive integer $n,\ q \geq 2$ with

$$T^i x \neq T^j x\,, \quad 0 \leq i < j \leq n-1\,,$$
$$S^i x \neq S^j x\,, \quad 0 \leq i < j \leq q-1\,,$$

$$F_{T^n S^i x, S^q T^i x}(T) > \min\left\{ \min_{1 \leq j \leq n, 1 \leq j' \leq q} F_{T^j x, S^{j'} x}(t),\ \min_{1 \leq j \leq n} F_{T^j x, x}(t),\ \min_{1 \leq j' \leq q} F_{S^{j'} x, x}(t) \right\}$$

for $i = 1, 2, ..., n-1$ and $j = 1, 2, ..., q-1$, then S and T have a common fixed point in X if and only if there exist integers $m,\ p,\ q,\ m > n \geq 0,\ p,\ q\ \geq 0$, and a point $x \in X$ such that

$$T^m x = S^p x = T^n x = S^q x$$

and either $F(S) \neq \emptyset$ or $F(T) \neq \emptyset$. If this condition is satisfied, then either $T^m x$ or $S^q x$ is a common fixed point of S and T.

THEOREM 3.3 Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space and let S and T be two commuting self-mappings of X. If there exist positive integers $p,\ q$ such that for any $x,\ y \in X,\ x \neq y$, and for all $t > 0$,

$$F_{(ST)^p x,(ST)^q y}(t) > \min\left\{F_{x,y}(t), F_{x,(ST)^p x}(t), F_{y,(ST)^q y}(t), F_{x,(ST)^q y}(t), F_{y,(ST)^p x}(t)\right\},$$

then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k which satisfies the condition (2.5) in Theorem 2.6 and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If this condition is satisfied, then the point x is the unique common fixed point of S and T.

The following is a special case of Theorem 3.3 obtained by setting $p = q = 1$:

COROLLARY 3.4 Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space and let S and T be two commuting self-mappings of X. If for any $x,\ y \in X,\ x \neq y$, and $t > 0$,

$$F_{STx,STy}(t) > \min\{F_{x,y}(t), F_{x,STx}(t), F_{y,STy}(t), F_{x,STy}(t), F_{y,STx}(t)\}\,,$$

then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If this condition is satisfied, then the point x is the unique common fixed point of S and T.

THEOREM 3.5 Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space and let S and T be two commuting self-mappings of X. If for any $x \in X$, $x \neq y$, there exists a positive integer $p(x,y)$ such that for all $t > 0$,

$$F_{(ST)^{p(x,y)}x,(ST)^{p(x,y)}y}(t) > \min\{F_{x,y}(t), F_{x,(ST)^{p(x,y)}x}(t), F_{y,(ST)^{p(x,y)}y}(t), F_{x,(ST)^{p(x,y)}y}(t), F_{y,(ST)^{p(x,y)}x}(t)\},$$

then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k such that for any $u, v \in A = \{x, (ST)x, ..., (ST)^{k-1}\}$, $u \neq v$, there exist x', $y' \in A$, $x' \neq y'$, satisfying the following conditions:

$$(St)^{p(x',y')}x' = u, \quad (ST)^{p(x',y')}y' = v$$

and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If these conditions are satisfied, then the point x is the unique common fixed point of S and T.

By setting $p(x,y) = p(x)$ in Theorem 3.5, we have the following:

COROLLARY 3.6 Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space and let S and T be two commuting self-mappings of X. If there exists a positive integer $p(x)$ such that for every $y \in X$, $x \neq y$, and for all $t > 0$,

$$F_{(ST)^{p(x)}x,(ST)^{p(x)}y}(t) > \min\{F_{x,y}(t), F_{x,(ST)^{p(x)}x}(t), F_{y,(ST)^{p(x)}y}(t), F_{x,(ST)^{p(x)}y}(t), F_{y,(ST)^{p(x)}x}(t)\},$$

then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k such that for any u, $v \in A = \{x, (ST)x, ..., (ST)^{k-1}x\}, u \neq v$, there exist x', $y' \in A$, $x' \neq y'$, satisfying the following conditions:

$$(ST)^{p(x')}x' = u, \quad (ST)^{p(x')}y' = v$$

and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If these conditions are satisfied, then the point x is the unique common fixed point of T.

REMARK. Theorems 3.3 ~ 3.6 include Theorems 1 ~ 5 in [4] and Theorems 3.3 ~ 3.6 in [5] as special cases.

ACKNOWLEDGMENT. The authors wish to express the deepest appreciation to the referee for his helpful comments on this paper. The Present Studies were supported in part by the Basic Science Research Institute Program, Ministry of Education, Korea, 1994, Project No. BSRI-94-1405.

REFERENCES

[1] CHANG, S.S., On Rhoades' open questions and some fixed point theorems for a class of mappings, *Proc. Amer. Math. Soc.* **97** (1986), 343-346.

[2] CHANG, S.S., On the generalized 2-metric space and probabilistic 1-metric spaces with applications to fixed point theory, *Math. Japonica* **34** (1989), 885-900.

[3] CHANG, S.S., CHO, Y.J. and KANG, S.M., *Probabilistic Metric Spaces and Nonlinear Operator Theory*, Sichuan University Press, P.R. China, 1994.

[4] CHANG, S.S. and HUANG, N.J., Fixed point theorems for some mappings in probabilistic metric spaces, *J. Natural Sci.* **12** (1989), 474-475.

[5] CHANG, S.S., HUANG, N.J. and CHO, Y.J., Fixed point theorems in general quasi-metric spaces and applications, to appear in *Math. Japonica.*

[6] CHANG, S.S. and CHANG, Q.C., On Rhoades' open questions, *Proc. Amer. Math. Soc.* **109** (1990), 269-274.

[7] SCHWEIZER, B. and SKLAR, A., *Probabilistic Metric Spaces*, North-Holland, New York, 1983.