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Research Article

Multiresolution Expansion and Approximation Order of Generalized Tempered Distributions

Byung Keun Sohn

Department of Mathematics, Inje University, Kimhae 621-749, Republic of Korea

Correspondence should be addressed to Byung Keun Sohn; mathsohn@inje.ac.kr

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Let $\mathscr{K}_M^r{}'(\mathbb{R})$ be the generalized tempered distributions of $e^{M(x)}$-growth with restricted order $r \in \mathbb{N}_0$, where the function $M(x)$ grows faster than any linear functions as $|x| \to \infty$. We show the convergence of multiresolution expansions of $\mathscr{K}_M^r{}'(\mathbb{R})$ in the test function space $\mathscr{K}_M^r(\mathbb{R})$ of $\mathscr{K}_M^r{}'(\mathbb{R})$. In addition, we show that the kernel of an integral operator $K : \mathscr{K}_M^r{}'(\mathbb{R}) \to \mathscr{K}_M^r{}'(\mathbb{R})$ provides approximation order in $\mathscr{K}_M^r{}'(\mathbb{R})$ in the context of shift-invariant spaces.

1. Introduction

Multiresolution analysis was shown to be very useful in extending the expansions in orthogonal wavelets from $L^2(\mathbb{R})$ to a certain class of tempered distributions. Some interactions between wavelets and tempered distributions have been presented by Walter's work in [1–3]. Walter has found the analytic representation of tempered distributions of polynomial growth with restricted order, $\mathscr{S}_r'(\mathbb{R})$, $r \in \mathbb{N}_0$, by wavelets [1] and the multiresolution expansions' pointwise convergence of $\mathscr{S}_r'(\mathbb{R})$ [3]. Pilipović and Teofanov have showed the uniform convergence on compact sets of the derivatives of multiresolution expansions of $\mathscr{S}_r'(\mathbb{R})$ and the convergence of multiresolution expansions of $\mathscr{S}_r'(\mathbb{R})$ in the test function space $\mathscr{S}_r(\mathbb{R})$ of $\mathscr{S}_r'(\mathbb{R})$. As an application, Pilipović and Teofanov have shown that the kernel of an integral operator $K : \mathscr{S}_r'(\mathbb{R}) \to \mathscr{S}_r'(\mathbb{R})$ provides approximation order in $\mathscr{S}_r'(\mathbb{R})$ in the context of shift-invariant spaces [4].

In the meantime, the tempered distributions of polynomial growth were extended to tempered distributions of $e^{|x|}$-growth, $\mathscr{K}_1'(\mathbb{R})$, in [5, 6] and $e^{|x|^p}$-growth, $\mathscr{K}_p'(\mathbb{R})$, in [7, 8] or $e^{M(x)}$-growth, $\mathscr{K}_M'(\mathbb{R})$, in [9, 10], where the function $M(x)$ grows faster than any linear functions as $|x| \to \infty$. We have considered the analytic representation of tempered distributions of $e^{M(x)}$-growth with restricted order, $\mathscr{K}_M^r{}'(\mathbb{R})$, by wavelets [11]. Also, we have shown that the multiresolution expansions of $\mathscr{K}_M^r{}'(\mathbb{R})$ converges pointwise to the value of the distribution where it exists [12].

In this paper, we will show the uniform convergence on compact sets of the derivatives of multiresolution expansions of $\mathscr{K}_M^r{}'(\mathbb{R})$ and convergence of multiresolution expansions of $\mathscr{K}_M^r{}'(\mathbb{R})$ in the test function space $\mathscr{K}_M^r(\mathbb{R})$ of $\mathscr{K}_M^r{}'(\mathbb{R})$. In addition, we will show that the kernel of an integral operator $K : \mathscr{K}_M^r{}'(\mathbb{R}) \to \mathscr{K}_M^r{}'(\mathbb{R})$ provides approximation order in $\mathscr{K}_M^r{}'(\mathbb{R})$. This is an extension of the works of Pilipović and Teofanov [4] in the context of generalized tempered distributions, $\mathscr{K}_M'(\mathbb{R})$.

2. The Generalized Tempered Distribution Spaces $\mathscr{K}_M'(\mathbb{R})$

Throughout this paper, we will use C or C_i to denote the positive constants, which are independent parameters and may be different at each occurrence.

Let $\mu(\xi)$ $(0 \le \xi \le \infty)$ denote a continuous increasing function such that $\mu(0) = 0$ and $\mu(\infty) = \infty$. For $x \ge 0$, we define

$$M(x) = \int_0^x \mu(\xi)\, d\xi. \tag{1}$$

The function $M(x)$ is an increasing, convex, and continuous function with $M(0) = 0$, $M(\infty) = \infty$ and satisfies the fundamental convexity inequality $M(x_1) + M(x_2) \leq M(x_1 + x_2)$. Further, we define $M(x)$ for negative x by means of the equality $M(x) = M(-x)$. Note that since the derivative $\mu(x)$ of $M(x)$ is unbounded in $\mathbb{R}$, the function $M(x)$ will grow faster than any linear function as $|x| \to \infty$. Now we list some properties of $M(x)$ which will be frequently used later. Consider the following:

$$\begin{aligned} &M(x) + M(y) \leq M(x+y) \quad \forall x, y \geq 0, \\ &M(x+y) \leq M(2x) + M(2y) \quad \forall x, y \geq 0. \end{aligned} \tag{2}$$

Using the function $M(x)$, we define the space $\mathscr{K}_M(\mathbb{R})$ as the space of all functions $\varphi \in C^\infty(\mathbb{R})$ such that

$$\|\varphi\|_{\mathscr{K}_M} = \sup_{x\in\mathbb{R},\alpha\leq k} e^{M(kx)} \left| \frac{d^\alpha}{dx^\alpha}\varphi(x)\right| < \infty, \quad k = 1, 2, \ldots. \tag{3}$$

The topology in $\mathscr{K}_M(\mathbb{R})$ is defined by the family of the seminorms $\|\cdot\|_{\mathscr{K}_M}$. Then $\mathscr{K}_M(\mathbb{R})$ become a Fréchet space and $\mathscr{D}(\mathbb{R}) \hookrightarrow \mathscr{K}_M(\mathbb{R}) \hookrightarrow \mathscr{S}(\mathbb{R}) \hookrightarrow \mathscr{E}(\mathbb{R})$ are continuous and dense inclusions; here $\mathscr{D}(\mathbb{R})$ denotes the spaces of all $C^\infty(\mathbb{R})$ functions with compact supports, $\mathscr{S}(\mathbb{R})$ the spaces of polynomially decreasing functions (Schwartz functions), and $\mathscr{E}(\mathbb{R})$ the space of all $C^\infty(\mathbb{R})$ functions. By $\mathscr{K}'_M(\mathbb{R})$, we mean the space of continuous linear functionals on $\mathscr{K}_M(\mathbb{R})$.

Definition 1. We say that the elements of $\mathscr{K}'_M(\mathbb{R})$ are generalized tempered distributions.

Clearly, when $M(x) = \log(1 + |x|)$, $\mathscr{K}'_M(\mathbb{R})$ are tempered distributions (Schwarz distributions), $\mathscr{S}(\mathbb{R})$. When $M(x) = |x|$, $\mathscr{K}'_M(\mathbb{R})$ are tempered distributions, $\mathscr{K}'_1(\mathbb{R})$, which are introduced and characterized by Yoshinaga [6] and Hasumi [5], independently. When $M(x) = |x|^p$, $p > 1$, $\mathscr{K}'_M(\mathbb{R})$ are tempered distributions, $\mathscr{K}'_p(\mathbb{R})$, which are introduced and characterized by Sznajder and Zieleżny [7, 8]. For details about $\mathscr{K}'_M(\mathbb{R})$, we refer to [9, 10].

For a natural number r, we define by $\mathscr{K}^r_M(\mathbb{R})$ the space of all $\phi \in C^r(\mathbb{R})$ such that

$$\begin{aligned} &\|\phi\|_{\mathscr{K}^r_M} = \sup_{x\in\mathbb{R},\alpha\leq r} e^{M(rx)} \left| \frac{d^\alpha}{dx^\alpha}\varphi(x)\right| < \infty, \\ &\lim_{|x|\to\infty} \sup_{0\leq\alpha\leq r} e^{M(rx)} \left| \frac{d^\alpha}{dx^\alpha}\varphi(x)\right| = 0. \end{aligned} \tag{4}$$

The topology of $\mathscr{K}^r_M(\mathbb{R})$ is defined by the family of $\|\cdot\|_{\mathscr{K}^r_M}$ and the dual of $\mathscr{K}^r_M(\mathbb{R})$ is denoted by $\mathscr{K}^r_M{}'(\mathbb{R})$. Clearly, $\mathscr{K}_M(\mathbb{R})$ is the projective limit of $\mathscr{K}^r_M(\mathbb{R})$ when $r \to \infty$ and $\mathscr{K}'_M(\mathbb{R}) = \bigcup_{r\in\mathbb{N}} \mathscr{K}^r_M{}'(\mathbb{R})$. Also, we have continuous and dense inclusion mapping as following:

$$\begin{aligned} &\mathscr{K}_M(\mathbb{R}) \hookrightarrow \cdots \hookrightarrow \mathscr{K}^{r+1}_M(\mathbb{R}) \hookrightarrow \mathscr{K}^r_M(\mathbb{R}) \hookrightarrow \cdots \\ &\hookrightarrow \mathscr{K}^r_M{}'(\mathbb{R}) \hookrightarrow \mathscr{K}^{r+1}_M{}'(\mathbb{R}) \hookrightarrow \cdots \hookrightarrow \mathscr{K}'_M(\mathbb{R}). \end{aligned} \tag{5}$$

Definition 2. We say that the elements of $\mathscr{K}^r_M{}'(\mathbb{R})$ are generalized tempered distributions of order r.

We define by $\widetilde{\mathscr{K}}^r_M(\mathbb{R})$ the space of all $\psi \in C^r(\mathbb{R})$ such that

$$\|\psi\|_{\widetilde{\mathscr{K}}^r_M} = \sup_{x\in\mathbb{R},0\leq\alpha\leq r} e^{M(lx)} \left| \frac{d^\alpha}{dx^\alpha}\psi(x)\right| < \infty, \quad l = 1, 2, \ldots. \tag{6}$$

The topology of $\widetilde{\mathscr{K}}^r_M(\mathbb{R})$ is defined by the family of $\|\cdot\|_{\widetilde{\mathscr{K}}^r_M}$ and the dual of $\widetilde{\mathscr{K}}^r_M(\mathbb{R})$ is denoted by $\widetilde{\mathscr{K}}^r_M{}'(\mathbb{R})$. Obviously, $\widetilde{\mathscr{K}}^r_M(\mathbb{R}) \subset \mathscr{K}^r_M(\mathbb{R})$.

Now, we give a theorem that will be used later.

Theorem 3. *Let ϕ and sequence $\{\phi_n\}_{n\in\mathbb{N}}$ be given in $\mathscr{K}^{r+1}_M(\mathbb{R})$ such that $\{(d^\alpha/dx^\alpha)\phi_n\}_{n\in\mathbb{N}}$ converges uniformly to $(d^\alpha/dx^\alpha)\phi$ on every compact set $K \subset \mathbb{R}$ and for $\alpha = 0, 1, \ldots, r$. If $\{\phi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathscr{K}^{r+1}_M(\mathbb{R})$, then the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ converges to $\phi \in \mathscr{K}^{r+1}_M(\mathbb{R})$ in $\mathscr{K}^r_M(\mathbb{R})$.*

Proof. Let $\epsilon > 0$ be given and let $\alpha \in \{0, 1, \ldots, r\}$. Then there exist N such that

$$\sup_{x\in K} e^{M(rx)} \left| \frac{d^\alpha}{dx^\alpha}(\phi_n - \phi)(x)\right| < \epsilon, \quad n \geq N, \tag{7}$$

for arbitrary $K \subset \mathbb{R}$. Also, since the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathscr{K}^{r+1}_M(\mathbb{R})$, we can take a positive number $A > 0$ and a compact set K such that $|x| > A$ when $x \notin K$ and

$$\begin{aligned} &\sup_{x\notin K} e^{M(rx)} \left| \frac{d^\alpha}{dx^\alpha}(\phi_n - \phi)(x)\right| \\ &= \sup_{x\notin K} e^{M(rx)} e^{-M((r+1)x)} e^{M((r+1)x)} \left| \frac{d^\alpha}{dx^\alpha}(\phi_n - \phi)(x)\right| \\ &\leq \sup_{x\notin K} e^{-M(x)} e^{M((r+1)x)} \left| \frac{d^\alpha}{dx^\alpha}(\phi_n - \phi)(x)\right| \\ &\leq e^{-M(A)} \sup_{x\in\mathbb{R}} e^{M((r+1)x)} \left(\left| \frac{d^\alpha}{dx^\alpha}\phi_n(x)\right| + \left| \frac{d^\alpha}{dx^\alpha}\phi(x)\right|\right) \\ &< Ce^{-M(A)} < \epsilon. \end{aligned} \tag{8}$$

From (7) and (8), we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} e^{M(rx)} \left| \frac{d^\alpha}{dx^\alpha}(\phi_n - \phi)(x)\right| = 0, \quad 0 \leq \alpha \leq r. \tag{9}$$

□

3. Multiresolution Expansion of $\mathscr{K}'_M(\mathbb{R})$

Definition 4. A multiresolution analysis (shortly MRA) consists of a sequence of closed subspaces $V_n, n \in \mathbb{Z}$, of $L^2(\mathbb{R})$ satisfying the following:

(i) $\{\psi(t - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of V_0,

(ii) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$,

(iii) $f \in V_n \Leftrightarrow f(2\cdot) \in V_{n+1}$,

(iv) $\bigcap_n V_n = \{0\}$, $\overline{\bigcup_n V_n} = L^2(\mathbb{R})$.

The function ψ whose existence is asserted in (i) is called a scaling function of the given MRA.

Definition 5. We say that a multiresolution analysis V_n, $n \in \mathbb{Z}$, is $(M, r\text{-})$ regular MRA of $L^2(\mathbb{R})$ if the scaling function ψ is in $\widetilde{\mathcal{K}}^r_M(\mathbb{R})$.

Example 6. It is impossible that the scaling function ψ has exponential decay and $\psi \in C^\infty(\mathbb{R})$, with all derivatives bounded, unless $\psi = 0$. Refer to [13, Corollary 5.5.3]. So we will restrict our attention to $\mathcal{K}^r_M(\mathbb{R})$ or $\widetilde{\mathcal{K}}^r_M(\mathbb{R})$. From the remark in [13] or, page 152 [2, Example 4, page 48], Battle-Lemarié's wavelets are in $\mathcal{K}^r_M(\mathbb{R})$ for some $r \in \mathbb{N}$ when $M(x) = |x|$, but not in $\widetilde{\mathcal{K}}^r_M(\mathbb{R})$ even if they have exponential decay and smoothness. In [13], Daubechies shown that for an arbitrary nonnegative integer r, there exists an $(M, r\text{-})$ regular MRA of $L^2(\mathbb{R})$ such that the scaling function ψ has compact supports.

Let V_j be an $(M, r\text{-})$ regular MRA of $L^2(\mathbb{R})$ and let ψ be a scaling function. The reproducing kernel of V_0 is given by

$$q_0(x, y) = \sum_{n\in\mathbb{Z}} \psi(x-n)\overline{\psi(y-n)}. \tag{10}$$

The series and its derivatives with respect to x or y of order $\leq r$ converge uniformly on $\mathbb{R}$ because of the regularity of $\psi \in \widetilde{\mathcal{K}}^r_M(\mathbb{R})$. The reproducing kernel of the projection operator onto V_j is

$$q_j(x, y) = 2^j q_0\left(2^j x, 2^j y\right), \quad x, y \in \mathbb{R}, \tag{11}$$

and the projection of $f \in L^2(\mathbb{R})$ onto V_j is given by

$$\begin{aligned} q_j f(x) &= \left\langle f(y), q_j(x, y)\right\rangle \\ &= \int f(y)\, q_j(x, y)\, dy, \quad x \in \mathbb{R}. \end{aligned} \tag{12}$$

The sequence $\{q_j\}_{j\in\mathbb{Z}}$, given in (12), is called the multiresolution expansion of $f \in L^2(\mathbb{R})$.

Definition 7. For a given $f \in \mathcal{K}^{r\,\prime}_M(\mathbb{R})$, the sequence $\{q_j\}_{j\in\mathbb{Z}}$ defined by

$$\left\langle q_j f, \phi\right\rangle = \left\langle f, q_j\phi\right\rangle, \quad \phi \in \mathcal{K}^r_M(\mathbb{R}) \tag{13}$$

is called the multiresolution expansion of $f \in \mathcal{K}^{r\,\prime}_M(\mathbb{R})$.

We deduce the following properties of the reproducing kernel q_0 with scaling function $\psi \in \widetilde{\mathcal{K}}^r_M(\mathbb{R})$:

(a) $q_0(x, y) = q_0(y, x)$ and $q_0(x+k, y+k) = q_0(x, y)$ for all $k \in \mathbb{Z}$.

(b) For every $l \in \mathbb{N}$ and $0 \leq \alpha,\ \beta \leq r$, there exist $C'_l > 0$ such that

$$\begin{aligned} \left|\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial y^\beta} q_0(x, y)\right| &\leq \sum_j \left|\frac{d^\alpha}{dx^\alpha}\psi(x-j)\right|\left|\frac{d^\beta}{dx^\beta}\psi(y-j)\right| \\ &\leq \sum_j C_l e^{-M((2l+1)(x-j))} e^{-M((2l+1)(y-j))} \\ &\leq \sum_j C_l e^{-M(2l(x-j))} e^{-M(x-j)} \\ &\quad \times e^{-M(2l(j-y))} e^{-M(j-y)} \\ &\leq C_l e^{-M(l(x-y))} \sum_j C_l e^{-M(x-j)} e^{-M(j-y)} \\ &\leq C'_l e^{-M(l(x-y))}, \end{aligned} \tag{14}$$

where we used the properties (2).

(c) $\int_{-\infty}^{\infty} q_0(x, y) y^\alpha dy = x^\alpha,\ y \in \mathbb{R},\ 0 \leq \alpha \leq r$.

Let V_j be an $(M, r\text{-})$ regular MRA of $L^2(\mathbb{R})$. We fix a function $g \in \mathcal{D}(\mathbb{R})$ with $\int g(x)dx = 1$. We let g_j denote the function $2^j g(2^j x)$ and let G_j denote the operation of convolution by g_j. For each fixed x, we consider the function $\partial^\alpha_x q_0(x, y)$ of the variable y. From (c), we have

$$\int \partial^\alpha_x q_0(x, y)\, y^\beta dy = 0, \tag{15}$$

for $0 \leq \beta < \alpha$, whereas

$$\int \partial^\alpha_x q_0(x, y)\, y^\alpha dy = \alpha!. \tag{16}$$

Now, it follows from integration by parts that the kernal $g(x-y)$ of the operator G shares these properties (15) and (16) with $q_0(x, y)$.

Let

$$R^\alpha(x, y) = \partial^\alpha_x q_0(x, y) - \partial^\alpha_x g(x-y). \tag{17}$$

From (b) and the fact that $g \in \mathcal{D}(\mathbb{R}) \subset \mathcal{K}^r_M(\mathbb{R})$, we have

$$\left|R^\alpha(x, y)\right| \leq c_k e^{-M(k(x-y))}, \quad x, y \in \mathbb{R},\ k \in \mathbb{N}, \tag{18}$$

and these functions also satisfy

$$\int R^\alpha(x, y)\, dy = 0 \tag{19}$$

identically in x for every $\alpha = 1, 2, \ldots, r$. They, for every $j \in \mathbb{Z}$ and $f \in C^r(\mathbb{R})$ with at most $e^{M(x)}$-growth, define operator R^α_j by

$$R^\alpha_j f(x) = 2^j \int R^\alpha\left(2^j x, 2^j y\right) f(y)\, dy \tag{20}$$

which are such that

$$q_j \frac{d^\alpha}{dx^\alpha} f(x) = G_j \frac{d^\alpha}{dy^\alpha} f(y) + R_j^\alpha \frac{d^\alpha}{dy^\alpha} f(y), \quad (21)$$

that is,

$$\begin{aligned} &\int q_j(x,y) \frac{d^\alpha}{dy^\alpha} f(y)\, dy \\ &\quad = 2^j \int g\left(2^j(x-y)\right) \frac{d^\alpha}{dy^\alpha} f(y)\, dy \\ &\quad + 2^j \int R^\alpha\left(2^j x, 2^j y\right) \frac{d^\alpha}{dy^\alpha} f(y)\, dy. \end{aligned} \quad (22)$$

From Theorem 1.1 in [14], we have

$$\lim_{j\to\infty} G_j \frac{d^\alpha}{dy^\alpha} f(y)\, dy = \frac{d^\alpha}{dx^\alpha} f(x), \quad x \in \mathbb{R},\ \alpha \geq 0, \quad (23)$$

uniformly on compact sets. Now we will show the uniform convergence on compact sets of the derivatives of multiresolution expansions of $\mathscr{K}_M^{r\,\prime}(\mathbb{R})$.

Theorem 8. *Let $f \in C^r(\mathbb{R})$ such that the corresponding derivatives $(d^\alpha/dx^\alpha)f$ are bounded by a $e^{M(k_0 x)}$ when $|x| \to \infty$, for every $\alpha = 0, 1, \ldots, r$ and some $k_0 \in \mathbb{N}$. If $q_j f$, given by (12), be the projection of f onto an $(M, r$-$)$ regular MRA of $L^2(\mathbb{R})$, then the sequence $\{(d^\alpha/dx^\alpha) q_j f\}_{j\in\mathbb{Z}}$ converges uniformly on compact sets to $(d^\alpha/dx^\alpha)f$ as $j \to \infty$, for every $\alpha = 0, 1, \ldots, r$.*

Proof. If $|y - x| \leq c$, we have

$$\left| \frac{d^\alpha}{dy^\alpha} f(y) - \frac{d^\alpha}{dy^\alpha} f(y) \Big|_{y=x} \right| \leq e_\alpha^M(y - x), \quad (24)$$

where $e_\alpha^M(x)$ is a continuous function with $e^{M(k_0 x)}$ growth and $e_\alpha^M(0) = 0$. From (18), given a compact set K, we have

$$\begin{aligned} &2^j \left| \int R^\alpha\left(2^j x, 2^j y\right) \frac{d^\alpha}{dy^\alpha} f(y)\, dy \right| \\ &\quad \leq 2^j \int \left| R^\alpha\left(2^j x, 2^j y\right) \left(\frac{d^\alpha}{dy^\alpha} f(y) - \frac{d^\alpha}{dy^\alpha} f(y)\Big|_{y=x} \right) \right| dy \\ &\quad \leq 2^j \int c_k e^{-M(2^j k(x-y))} \left| e_\alpha^M(y-x) \right| dy, \end{aligned} \quad (25)$$

for large enough j and $x \in K$. Since k can be chosen arbitrary, we obtain by dominated convergence theorem,

$$\begin{aligned} &\lim_{j\to\infty} 2^j \left| \int R^\alpha\left(2^j x, 2^j y\right) \frac{d^\alpha}{dy^\alpha} f(y)\, dy \right| \\ &\quad \leq \int \lim_{j\to\infty} 2^j c_k e^{-M(2^j k(x-y))} \left| e_\alpha^M(y-x) \right| dy = 0 \end{aligned} \quad (26)$$

uniformly for $x \in K$. From (21) and (23), we have the conclusion. □

We now ready to show the main theorem.

Theorem 9. *Let $\phi \in \widetilde{\mathscr{K}}_M^r(\mathbb{R})$ and let $q_j\phi(x)$, given by (7), be a projection of ϕ onto an $(M, r$-$)$ regular MRA of $L^2(\mathbb{R})$. If $\phi \in \mathscr{K}_M^{2(r+1)}(\mathbb{R})$, then the sequence $\{q_j\phi(x)\}$ converges to $\phi(x)$ in $\mathscr{K}_M^r(\mathbb{R})$ as $j \to \infty$.*

Proof. Let g and R^α be given in (21) such that $g \in \mathscr{D}(\mathbb{R})$ and $\int g(x)dx = 1$. From Theorems 3 and 8 and (21), it suffices to show that

$$\begin{aligned} &\sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h} \left| \int q_0\left(\frac{x}{h}, \frac{y}{h}\right) \phi(y)\, dy \right| \\ &\quad = \sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h} \left| \int g\left(\frac{x-y}{h}\right) \frac{d^\alpha}{dy^\alpha} \phi(y)\, dy \right. \\ &\qquad \left. + \int R^\alpha\left(\frac{x}{h}, \frac{y}{h}\right) \frac{d^\alpha}{dy^\alpha} \phi(y)\, dy \right| \end{aligned} \quad (27)$$

is bounded for every $\alpha \in \{0, 1, \ldots, r\}$ and $h > 0$. Since g has a compact support, then

$$\begin{aligned} &\sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h} \left| \int g\left(\frac{x-y}{h}\right) \frac{d^\alpha}{dy^\alpha} \phi(y)\, dy \right| \\ &\quad \leq \sup_{x\in\mathbb{R}} \frac{1}{h} \left| \int g\left(\frac{x-y}{h}\right) e^{M((r+1)x)} e^{-M(2(r+1)y)} dy \right| \\ &\quad \leq \sup_{x\in\mathbb{R}} \frac{1}{h} \left| \int g\left(\frac{x-y}{h}\right) e^{M(2(r+1)(x-y))} dy \right| \leq C. \end{aligned} \quad (28)$$

Hence we have only to show that

$$K = \sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h} \left| \int R^\alpha\left(\frac{x}{h}, \frac{y}{h}\right) \frac{d^\alpha}{dy^\alpha} \phi(y)\, dy \right| \leq C, \quad x \in \mathbb{R}, \quad (29)$$

for every $\alpha \in \{0, 1, \ldots, r\}$ and $h > 0$. Let $S_1 = \{y : |x - y| \leq 1\}$, $S_2 = \{y : |x - y| > 1 \text{ and } (1/2)|x| \leq |y|\}$ and $S_3 = \{y : |x - y| > 1 \text{ and } (1/2)|x| > |y|\}$. Then, by (18), we have

$$\begin{aligned} I &= \sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h} \left| \int R^\alpha\left(\frac{x}{h}, \frac{y}{h}\right) \frac{d^\alpha}{dy^\alpha} \phi(y)\, dy \right| \\ &\leq c_l \sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h} \int e^{-M(l((x-y)/h))} e^{-M(2(r+1)y)} dy \\ &= c_l \sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h} \left(\int_{S_1} + \int_{S_2} + \int_{S_3} \right) \\ &\quad \times e^{-M(l((x-y)/h))} e^{-M(2(r+1)y)} dy \\ &= c_l \left(I_1 + I_2 + I_3 \right). \end{aligned} \quad (30)$$

By a simple change of variable, we have

$$\begin{aligned} I_1 &= \sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h}\int_{S_1} e^{-M(l((x-y)/h))} e^{-M(2(r+1)y)} dy \\ &\le \sup_{x\in\mathbb{R}} \frac{1}{h}\int_{S_1} e^{-M(l((x-y)/h))} e^{M(2(r+1)(x-y))} dy \\ &\le 2e^{M(2(r+1))} \int_0^1 \frac{1}{h} e^{-M(l(t/h))} dt = 2e^{M(2(r+1))} \\ &\quad \times \int_0^{1/h} e^{-M(lu)} du \le C_1. \end{aligned} \tag{31}$$

Since $(1/2)|x| < |y|$ and $(1/2)|y| \le |x-y|$ on S_2, then

$$\begin{aligned} I_2 &= \sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h}\int_{S_2} e^{-M(l((x-y)/h))} e^{-M(2(r+1)y)} dy \\ &\le \sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h}\int_{S_2} e^{-M(l((x-y)/h))} e^{-M((r+1)x)} dy \\ &\le \frac{1}{h}\int_{S_2} e^{-M((l/2h)y)} dy \le C_2, \end{aligned} \tag{32}$$

for sufficiently large l. Since $(1/2)|x| > |x-y|$ on S_3, then

$$\begin{aligned} I_3 &= \sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h}\int_{S_3} e^{-M(l((x-y)/h))} e^{-M(2(r+1)y)} dy \\ &\le \sup_{x\in\mathbb{R}} e^{M((r+1)x)} \frac{1}{h} e^{-M((l/2)(x/h))} \int_{S_3} e^{-M((r+1)y)} dy \\ &\le C_3' \frac{1}{h}\int_{S_3} e^{-M(2(r+1)y)} dy \le C_3, \end{aligned} \tag{33}$$

for sufficiently large l. □

4. Approximation Order of $\mathscr{K}'_M(\mathbb{R})$

A space of functions S is called shift invariant if it is invariant under all integer translate, that is,

$$f \in S \Longleftrightarrow f(\cdot + k) \in S \quad \forall k \in \mathbb{Z}. \tag{34}$$

The principal shift-invariant subspaces $S = S(\phi)$ are generated by the closure of the linear span of the shifts of ϕ. The stationary ladder of spaces $\{S^h(\phi) : h > 0\}$ is given by

$$S^h(\phi) = \left\{ f\left(\frac{\cdot}{h}\right) : f \in S \right\}. \tag{35}$$

To rate the efficiency for approximation of such spaces, the concept of approximation order is widely used. We say that the scale of the space $S^h(\phi)$ provides approximation order k in F if for every sufficiently smooth f,

$$\inf_{g\in S^h(\phi)} \|f-g\|_F \le Ch^k, \quad h > 0, \tag{36}$$

where $C = C(f) > 0$. For further details about the theory on the approximation order provided by shift-invariant spaces, we refer to [15, 16]. We will focus our attention to the so-called approximation order of an integral operator.

Let K be an integral operator of the following form

$$(Kf)(x) = \int K(x,y) f(y)\, dy, \quad x \in \mathbb{R}. \tag{37}$$

We assume that $K(x-k, y) = K(x, y+k)$, $h \in \mathbb{Z}$, $x, y \in \mathbb{R}$. For $h > 0$, we define

$$K_h = \varrho_h K \varrho_{1/h}, \tag{38}$$

where ϱ is the scaling operator $\varrho_h f = f(\cdot/h)$. We say that the integral operator K defined by (37) provides approximation order k in F if for every sufficiently smooth f,

$$\|K_h f - f\|_F \le Ch^k, \quad h > 0, \tag{39}$$

where $C = C(f) > 0$. For further details about the theory on the approximation order provided by integral or kernel operator, we refer to [17, 18].

Definition 10 (see [4]). Let $f \in \mathscr{K}^{r}_M{}'(\mathbb{R})$. Let $K(x,y)$, $x, y, \in \mathbb{R}$, be the kernel of an integral operator $K : \mathscr{K}^{r}_M{}'(\mathbb{R}) \to \mathscr{K}^{r}_M{}'(\mathbb{R})$. Kf is given by $\langle Kf, \phi\rangle = \langle f, K\phi\rangle$. We say that the operator K provides approximation order k in $\mathscr{K}^{r}_M{}'(\mathbb{R})$ if

$$\begin{aligned} \|K_h f - f\|_{\mathscr{K}^{r+k}_M{}'} = \sup_{\|\phi\|_{\mathscr{K}^{r+k}_M = 1}} \left|\langle K_h f, \phi\rangle - \langle f, \phi\rangle\right| \le Ch^k, \\ h > 0, \end{aligned} \tag{40}$$

where the constant $C = C(f) > 0$.

We will now show that the kernel of an integral operator $K : \mathscr{K}^{r}_M{}'(\mathbb{R}) \to \mathscr{K}^{r}_M{}'(\mathbb{R})$ provides approximation order in $\mathscr{K}^{r}_M{}'(\mathbb{R})$.

Theorem 11. *Let $\phi \in \widetilde{\mathscr{K}}^{r+k}_M(\mathbb{R})$ with compact support such that the integer shifts of ϕ form an orthogonal basis of $S(\phi)$ with respect to the inner product in $L^2(\mathbb{R})$. Assume that $\phi(x) = \sum_{n\in\mathbb{N}} c_n \phi(2x-k)$ for some sequence $\{c_n\}_{n\in\mathbb{N}}$. Let*

$$K(x,y) = \sum_{l\in\mathbb{Z}} \phi(x-l)\overline{\phi}(y-l), \quad x, y \in \mathbb{R} \tag{41}$$

be the kernel of the integral operator given by (37). *Then K provides approximation order k in $\mathscr{K}^{r}_M{}'(\mathbb{R})$.*

Proof. Firstly, we will show that

$$\begin{aligned} J &= \|K_h\phi - \phi\|_{\mathscr{K}^r_M} \\ &= \sup_{x\in\mathbb{R}} e^{M(rx)} \left| \frac{d^\alpha}{dx^\alpha} \left(\langle K_h(x,y), \phi(y)\rangle - \phi(x) \right) \right| \\ &\le C\|\phi\|_{\mathscr{K}^{r+k}_M} h^k, \end{aligned} \tag{42}$$

where $\phi \in \mathscr{K}_M^{r+k}$, $k \in \mathbb{N}$. If we accept the result (42) for a moment, it follows that for $f \in \mathscr{K}_M^{r}{}'(\mathbb{R}) \subset \mathscr{K}_M^{r+k}{}'(\mathbb{R})$, we have

$$\begin{aligned} \left|\langle K_h f - f, \phi\rangle\right| &= \left|\langle f, K_h\phi - \phi\rangle\right| \\ &\leq \|f\|_{\mathscr{K}_M^{r}{}'} \|K_h\phi - \phi\|_{\mathscr{K}_M^{r}} \leq Ch^k \|\phi\|_{\mathscr{K}_M^{r+k}{}'} \end{aligned} \tag{43}$$

hence

$$\|K_h f - f\|_{\mathscr{K}_M^{r+k}{}'} = \sup_{\|\phi\|_{\mathscr{K}_M^{r+k}=1}} \left|\langle K_h f, \phi\rangle - \langle f, \phi\rangle\right| \leq Ch^k, \quad h > 0, \tag{44}$$

which implies the conclusion.

Since $\{S^{2^{-j}}(\phi) : j \in \mathbb{Z}\}$ satisfy the conditions of $(M, r\text{-})$ regular MRA of $L^2(\mathbb{R})$ with $S(\phi) = V_0$, we can apply (21) to the operator K, that is,

$$\begin{aligned} \frac{d^\alpha}{dx^\alpha} K_h f(x) &= \frac{1}{h}\int g\left(\frac{x-y}{h}\right)\frac{d^\alpha}{dy^\alpha} f(y)\,dy \\ &+ \frac{1}{h}\int R^\alpha\left(\frac{x}{h}, \frac{y}{h}\right)\frac{d^\alpha}{dy^\alpha} f(y)\,dy, \end{aligned} \tag{45}$$

where g and R^α are given in (21). For $0 \leq \alpha \leq r$,

$$\begin{aligned} J &= \sup_{x\in\mathbb{R}} e^{M(rx)} \left|\frac{d^\alpha}{dx^\alpha}\left(\langle K_h(x,y), \phi(y)\rangle - \phi(x)\right)\right| \\ &\leq \sup_{x\in\mathbb{R}} e^{M(rx)} \left|\frac{1}{h}\int g\left(\frac{x-y}{h}\right) \times \left(\frac{d^\alpha}{dy^\alpha}\phi(y) - \frac{d^\alpha}{dy^\alpha}\phi(y)\Big|_{y=x}\right)dy\right| \\ &+ \sup_{x\in\mathbb{R}} e^{M(rx)} \left|\frac{1}{h}\int R^\alpha\left(\frac{x-y}{h}\right) \times \left(\frac{d^\alpha}{dy^\alpha}\phi(y) - \frac{d^\alpha}{dy^\alpha}\phi(y)\Big|_{y=x}\right)dy\right| \\ &= J_1 + J_2. \end{aligned} \tag{46}$$

In order to estimate J_1, we consider $g \in \mathscr{D}(\mathbb{R})$ with $\int g(x)dx = 1$ and $\int g(x)x^\alpha dx = 0, 0 < |\alpha| < \max\{r, k-1\}$. Let c be a constant such that $\sup g \subset [-c, c]$. If we assume $h \in (0,1)$, the smoothness of $\phi \in \mathscr{K}_M^{r+k}(\mathbb{R}) \subset C^{r+k}(\mathbb{R})$ implies

$$\begin{aligned} J_1 &= \sup_{hx\in\mathbb{R}} e^{M(rhx)} \left|\int_{|x-y|\leq c} g(x-y) \times \left(\frac{d^\alpha}{dy^\alpha}\phi(y)\Big|_{y=hy} - \frac{d^\alpha}{dy^\alpha}\phi(y)\Big|_{y=hx}\right)dy\right| \\ &= \sup_{hx\in\mathbb{R}} e^{M(rhx)} \left|\int_{|x-y|\leq c} g(x-y) \times \left((y-x) \times h\frac{d^{\alpha+1}}{dy^{\alpha+1}}\phi(y)\Big|_{y=hx} + \cdots + \frac{(y-x)^{k-1}}{(k-1)!} \times h^{k-1}\frac{d^{\alpha+k-1}}{dy^{\alpha+k-1}}\phi(y)\Big|_{y=hx} + \frac{(y-x)^k}{k!} \times h^k\frac{d^{\alpha+k}}{dy^{\alpha+k}}\phi(y)\Big|_{y=\xi(y)}\right)dy\right| \\ &\leq \sup_{hx\in\mathbb{R}} e^{M(rhx)} \left|\int_{|x-y|\leq c} g(x-y)\frac{(y-x)^k}{k!} \times h^k\frac{d^{\alpha+k}}{dy^{\alpha+k}}\phi(y)\Big|_{y=\xi(y)}dy\right| \\ &\leq \sup_{hx\in\mathbb{R}} e^{M(rhx)} C\frac{c^k}{k!}h^k \sup_{\xi(y)\in[hx-hc, hx+hc]}\left|\frac{d^{\alpha+k}}{dy^{\alpha+k}}\phi(y)\right| \\ &= C\frac{c^k}{k!}h^k \sup_{s\in\mathbb{R}} e^{M(rs)} \sup_{t\in[s-c,s+c]}\left|\frac{d^{\alpha+k}}{dt^{\alpha+k}}\phi(t)\right| \\ &= C\frac{c^k}{k!}h^k \sup_{s\in\mathbb{R}} e^{M(rs)} \sup_{t\in[s-c,s+c]}\left(e^{-M(rt)}e^{M(rt)}\left|\frac{d^{\alpha+k}}{dt^{\alpha+k}}\phi(t)\right|\right) \\ &\leq C\frac{c^k}{k!}h^k \sup_{s\in\mathbb{R}} e^{M(rs)}e^{-M(r(|s|-c))} \sup_{t\in\mathbb{R}}\left(e^{M(rt)}\left|\frac{d^{\alpha+k}}{dt^{\alpha+k}}\phi(t)\right|\right) \\ &= C_1\|\phi\|_{\mathscr{K}_M^{r+k}}, \end{aligned} \tag{47}$$

where $\xi(y) = hx + \theta h(y-x) \in [hx - hc, hx + hc]$ for some $\theta \in (0,1)$ and $C_1 = C(c^k/k!)h^k \sup_{s\in\mathbb{R}} e^{M(rs)}e^{-M(r(|s|-c))} < \infty$. To show the finiteness of C_1 in the last statement, we use

$$\sup_{s\in\mathbb{R}} e^{M(rs)}e^{-M(r(|s|-c))} \leq \sup_{|s|\leq c} e^{M(rs)}e^{-M(r(|s|-c))}$$

$$+\sup_{|s|>c} e^{M(rs)} e^{-M(r(|s|-c))}$$
$$\leq \sup_{|s|\leq c} e^{M(rs)} e^{-M(r(|s|-c))} + \sup_{|s|>c} e^{M(rs)} e^{-M(r(|s|-c))}$$
$$\leq e^{M(rc)} + e^{-M(rc)}. \tag{48}$$

We will estimate J_2 by using the following facts. Since ϕ has a compact support, there exists $M > 0$ such that $K(x, y) = 0$ for $|x-y| > M$. Also, by the choice of g and property (c) of the reproducing kernel q_0, we have

$$\begin{aligned}\int R^{\alpha}(x,y)\, y^s dy &= \frac{d^{\alpha}}{dx^{\alpha}} \int K(x,y)\, y^s dy - \frac{d^{\alpha}}{dx^{\alpha}} \int g(x-y)\, y^s dy \\ &= \frac{d^{\alpha}}{dx^{\alpha}} x^s - \frac{d^{\alpha}}{dx^{\alpha}} \int g(t)\, x^s dt \\ &= \frac{d^{\alpha}}{dx^{\alpha}} x^s - \frac{d^{\alpha}}{dx^{\alpha}} x^s = 0, \quad 0 \leq s \leq r+k-1.\end{aligned} \tag{49}$$

Hence

$$\begin{aligned}J_2 &= \sup_{hx\in\mathbb{R}} e^{M(rhx)} \left| \int R^{\alpha}(x,y) \times \left(\frac{d^{\alpha}}{dy^{\alpha}} \phi(y)\Big|_{y=hy} - \frac{d^{\alpha}}{dy^{\alpha}} \phi(y)\Big|_{y=hx} \right) dy \right| \\ &= \sup_{hx\in\mathbb{R}} e^{M(rhx)} \left| \int R^{\alpha}(x,y) \times \left((y-x) \times h \frac{d^{\alpha+1}}{dy^{\alpha+1}} \phi(y)\Big|_{y=hx} + \cdots + \frac{(y-x)^{k-1}}{(k-1)!} \times h^{k-1} \frac{d^{\alpha+k-1}}{dy^{\alpha+k-1}} \phi(y)\Big|_{y=hx} + \frac{(y-x)^k}{k!} \times h^k \frac{d^{\alpha+k}}{dy^{\alpha+k}} \phi(y)\Big|_{y=\xi(y)} \right) dy \right| \\ &= \frac{h^k}{k!} \sup_{hx\in\mathbb{R}} e^{M(rhx)} \left| \int_{|x-y|\leq M} R^{\alpha}(x,y)(y-x)^k \times \frac{d^{\alpha+k}}{dy^{\alpha+k}} \phi(y)\Big|_{y=\xi(y)} dy \right| \\ &\leq \frac{M^k h^k}{k!} \sup_{hx\in\mathbb{R}} \sup_{hx<\xi(y)<hy} e^{M(rhx)} \left| \frac{d^{\alpha+k}}{dy^{\alpha+k}} \phi(y)\Big|_{y=\xi(y)} \right| \times \left| \int_{|x-y|\leq M} R^{\alpha}(x,y)(y-x)^k dy \right| \\ &\leq C' \frac{c^k}{k!} h^k \sup_{s\in\mathbb{R}} e^{M(rs)} e^{-M(r(|s|-c))} \sup_{t\in\mathbb{R}} \left(e^{M(rt)} \left| \frac{d^{\alpha+k}}{dt^{\alpha+k}} \phi(t) \right| \right) \\ &= C_2 \|\phi\|_{\mathcal{K}_M^{r+k}}.\end{aligned} \tag{50}$$

□

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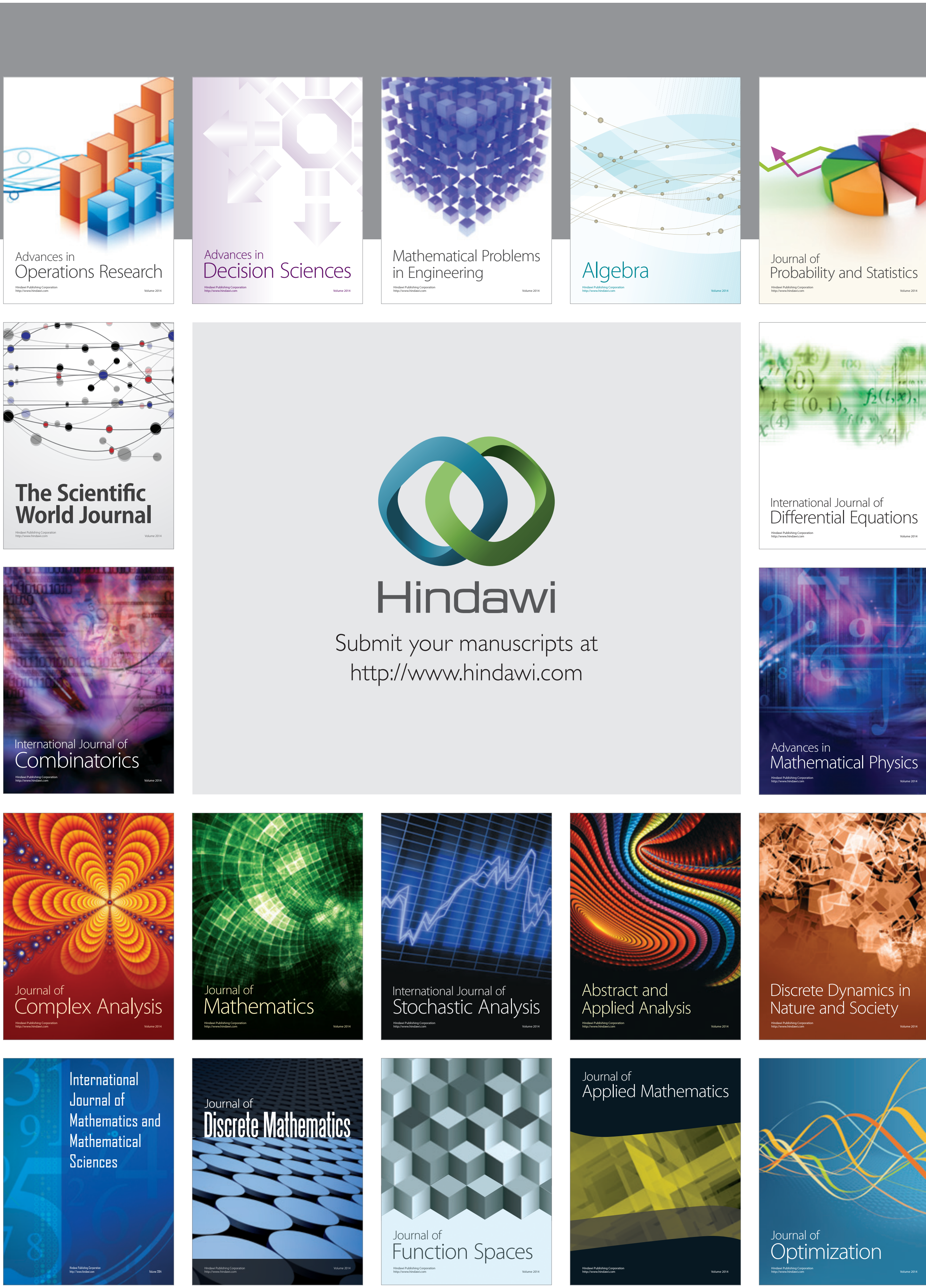
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