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Research Article

Convergence Theorem for a Family of Generalized Asymptotically Nonexpansive Semigroup in Banach Spaces

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Let E be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $\mathfrak{J} = \{T(t) : t \geq 0\}$ be a family of uniformly asymptotically regular generalized asymptotically nonexpansive semigroup of E, with functions $u, v : [0,\infty) \to [0,\infty)$. Let $F := F(\mathfrak{J}) = \cap_{t\geq 0} F(T(t)) \neq \emptyset$ and $f : K \to K$ be a weakly contractive map. For some positive real numbers λ and δ satisfying $\delta + \lambda > 1$, let $G : E \to E$ be a δ-strongly accretive and λ-strictly pseudocontractive map. Let $\{t_n\}$ be an increasing sequence in $[0,\infty)$ with $\lim_{n\to\infty} t_n = \infty$, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1]$ satisfying some conditions. Strong convergence of a viscosity iterative sequence to common fixed points of the family $\mathfrak{J}$ of uniformly asymptotically regular asymptotically nonexpansive semigroup, which also solves the variational inequality $\langle (G-\gamma f)p, j(p-x)\rangle \leq 0$, for all $x \in F$, is proved in a framework of a real Banach space.

1. Introduction

Let E be a real Banach space. We denote by J the normalized duality map from E to 2^{E^*} (E^* is the dual space of E), and it is defined by

$$J(x) = \left\{ f \in E^* : \langle x, f\rangle = \|x\|^2 = \left\|f\right\|^2 \right\}. \tag{1.1}$$

A mapping $T : E \to E$ is said to be contractive if $\|Tx - Ty\| \leq \alpha\|x-y\|$, for $x, y \in E$, and some constant $\alpha \in [0,1)$. It is said to be weakly contractive if there exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ satisfying $\psi(t) = 0$ if and only if $t = 0$ and $\|Tx - Ty\| \leq \|x-y\| - \psi(\|x-y\|)$, for all $x, y \in E$. It is known that the class of weakly contractive maps contain properly the class of contractive ones, see [1, 2]. A mapping $T : E \to E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in E$ and asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and $\|T^n x - T^n y\| \leq k_n \|x - y\|$, for all $x, y \in E$. We denote by $F(T) = \{x \in K : Tx = x\}$ the set of fixed points of a map T.

A mapping $T : E \to E$ is said to be total asymptotically nonexpansive (see [3]) if there exist nonnegative real sequences $\{u_n\}$ and $\{v_n\}$, with $u_n \to 0$ and $v_n \to 0$ as $n \to \infty$ and strictly increasing and continuous functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that

$$\left\|T^n x - T^n y\right\| \leq \left\|x - y\right\| + u_n \psi\left(\left\|x - y\right\|\right) + v_n, \quad \forall x, y \in K. \tag{1.2}$$

Remark 1.1. If $\psi(\lambda) = \lambda$, the total asymptotically nonexpansive mapping coincides with generalized asymptotically nonexpansive mapping. In addition, for all $n \in \mathbb{N}$, if $v_n = 0$, then generalized asymptotically nonexpansive mapping coincides with asymptotically nonexpansive mapping; if $u_n = 0,\ v_n = \max\{0, p_n\}$ where $p_n := \sup_{x,y \in K}(\|T_n x - T_n y\| - \|x - y\|)$, then generalized asymptotically nonexpansive mapping coincide with asymptotically nonexpansive mapping in the intermediate sense; if $u_n = 0$, and $v_n = 0$ then we obtain from (1.2) the class of nonexpansive mapping.

A one-parameter family of generalized asymptotically nonexpansive semigroup is a family $\mathfrak{J} = \{T(t) : t \geq 0\}$ of self-mapping of E such that

(i) $T(0)x = x$ for $x \in E$,

(ii) $T(s + t)x = T(s)T(t)x$ for all $t, s \geq 0$ and $x \in E$,

(iii) $\lim_{t \to 0} T(t)x = x$ for $x \in E$,

(iv) there exist functions $u, v : [0, \infty) \to [0, \infty)$ such that $u(t) \to 0, v(t) \to 0$ as $t \to \infty$, and

$$\left\|T(t)x - T(t)y\right\| \leq (1 + u(t))\left\|x - y\right\| + v(t) \quad \forall x, y \in E. \tag{1.3}$$

We will denote by F the common fixed-point set of $\mathfrak{J}$, that is,

$$F := \mathrm{Fix}(\mathfrak{J}) = \{x \in E : T(t)x = x, t \geq 0\} = \bigcap_{t \geq 0} \mathrm{Fix}(T(t)). \tag{1.4}$$

The family $\mathfrak{J} = \{T(t) : t \geq 0\}$ is said to be asymptotically regular if

$$\lim_{s \to \infty} \|T(s + t)x - T(s)x\| = 0, \tag{1.5}$$

for all $t \in [0, \infty)$ and $x \in E$. It is said to be uniformly asymptotically regular if, for any $t \geq 0$ and for any bounded subset C of E,

$$\lim_{s \to \infty} \sup_{x \in C} \|T(s + t)x - T(s)x\| = 0. \tag{1.6}$$

For some positive real numbers δ and λ, a mapping $G : E \to E$ is said to be δ-strongly accretive if for any $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Gx - Gy, j(x-y)\rangle \geq \delta \|x-y\|^2, \tag{1.7}$$

and it is called λ-strictly pseudocontractive if

$$\langle Gx - Gy, j(x-y)\rangle \leq \|x-y\|^2 - \lambda \|(I-G)x - (I-G)y\|^2. \tag{1.8}$$

Let E be a real Banach space, and let δ, λ, and τ be positive real numbers satisfying $\delta + \lambda > 1$ and $\tau \in (0,1)$. Let $G : E \to E$ be a δ-strongly accretive and λ-strictly pseudocontractive, then the following holds, see [4], for $x, y \in E$:

$$\begin{gathered} \|(I-G)x - (I-G)y\| \leq \left(\sqrt{\frac{1-\delta}{\lambda}}\right)\|x-y\|, \\ \|(I-\tau G)x - (I-\tau G)y\| \leq 1 - \tau\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\|x-y\|, \end{gathered} \tag{1.9}$$

that is, $(I-G)$ and $(I-\tau G)$ are contractive mappings.

Let C be a nonempty closed-convex subset of E and $T : E \to E$ a map. Then, a variational inequality problem with respect to C and T is found to be $x^* \in C$ such that

$$\langle Tx^*, j(y-x^*)\rangle \geq 0, \quad \forall y \in C,\ j(y-x^*) \in J(y-x^*). \tag{1.10}$$

Recently, convergence theorems for fixed points of nonexpansive mappings, common fixed points of family of nonexpansive mappings, nonexpansive semigroup, and their generalisation have been studied by numerous authors (see, e.g., [5–21]).

Acedo and Suzuki [22], recently, proved the strong convergence of the Browder's implicit scheme, $x_0, u \in C$,

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad n \geq 0, \tag{1.11}$$

to a common fixed point of a uniformly asymptotically regular family $\{T(t) : t \geq 0\}$ of nonexpansive semigroup in the framework of a real Hilbert space.

Li et al. [23] proved strong convergence theorems for implicit viscosity schemes for common fixed points of family of generalized asymptotically nonexpansive semigroups in Banach spaces.

Let S be a semigroup and $B(S)$ the subspace of all bounded real-valued functions defined on S with supremum norm. For each $s \in S$, the left translator operator $l(s)$ on $B(S)$ is defined by $(l(s)f)(t) = f(st)$ for each $t \in S$ and $f \in B(S)$. Let X be a subspace of $B(S)$ containing 1, and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. Let X be l_s invariant, that is, $l_s(X) \subset X$ for each $s \in S$. A mean μ on X is said to be left invariant if $\mu(l_s f) = \mu(f)$ for each $s \in S$ and $f \in X$.

Recently, Saeidi and Naseri [24] studied the problem of approximating common fixed point of a family of nonexpansive semigroup and solution of some variational inequality problem in a real Hilbert space. They proved the following theorem.

Theorem 1.2 (Saeidi and Naseri [24]). *Let $\mathfrak{J} = \{T(t) : t \in S\}$ be a nonexpansive semigroup in a real Hilbert space H such that $F(\mathfrak{J}) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T(t)x, y\rangle$ is an element of X for each $x, y \in H$. Let $f : E \to E$ be a contraction with constant α, and let $G : H \to H$ be strongly positive map with constant $\overline{\gamma} > 0$. Let $\{\mu_n\}$ be a left regular sequence of means on X, and let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Let $\gamma \in (0, \overline{\gamma}/\alpha)$, and let $\{x_n\}$ be a sequence generated by $x_0 \in H$,*

$$x_{n+1} = (I - \alpha_n G)T(\mu_n)x_n + \alpha_n\gamma f(x_n), \quad n \geq 0. \tag{1.12}$$

Then, $\{x_n\}$ converges strongly to a common fixed point of the family $\mathfrak{J}$ which is the unique solution of the variational inequality $\langle (G - \gamma f)x^*, j(x - x^*)\rangle \geq 0$ for all $x \in F(\mathfrak{J})$. Equivalently one has $P_{F(\mathfrak{J})}(I - G + \gamma f)x^* = x^*$.

More recently, as commented by Golkarmanesh and Naseri [25], Piri and Vaezi [4] gave a minor variation of Theorem 1.2 as follows.

Theorem 1.3 (Piri and Vaezi [4]). *Let $\mathfrak{J} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on a real Hilbert space H such that $F(\mathfrak{J}) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T(t)x, y\rangle$ is an element of X for each $x, y \in H$. Let $f : E \to E$ be a contraction with constant α, and let $G : H \to H$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Let $\{\mu_n\}$ be a left regular sequence of means on X, and let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Let $\{x_n\}$ be a sequence generated by $x_0 \in H$,*

$$x_{n+1} = (I - \alpha_n G)T(\mu_n)x_n + \alpha_n\gamma f(x_n), \quad n \geq 0, \tag{1.13}$$

where $0 < \gamma < (1 - \sqrt{(1 - \delta/\lambda)})/\alpha$, then, $\{x_n\}$ converges strongly to a common fixed point of the family $F(\mathfrak{J})$ which is the unique solution of the variational inequality $\langle (G - \gamma f)x^*, j(x - x^*)\rangle \geq 0$ for all $x \in F(\mathfrak{J})$. Equivalently one has $P_{F(\mathfrak{J})}(I - G + \gamma f)x^* = x^*$.

Very recently, Ali [26] continued the study of the problem in [4, 24] and proved a strong convergence theorem in a Banach space setting much more general than Hilbert space. He actually proved the following theorem.

Theorem 1.4 (Ali [26]). *Let E be a real Banach space with local uniform Opial's property whose duality mapping is sequentially continuous. Let $\mathfrak{J} = \{T(t) : t \geq 0\}$ be a uniformly asymptotically regular family of asymptotically nonexpansive semigroup of E with function $k : [0,\infty) \to [0,\infty)$ and $F := F(\mathfrak{J}) = \cap_{t\geq 0}F(T(t)) \neq \emptyset$. Let $f : E \to E$ be weakly contractive, and let $G : E \to E$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Let $\eta := (1 - \sqrt{(1 - \delta)/\lambda})$ and $\gamma \in (0, \min\{\eta, \delta/2\})$. Let $\{\beta_n\}$ and $\{\alpha_n\}$ be sequences in $(0,1]$, and let $\{t_n\}$ be an increasing sequence in $[0,\infty)$ satisfying the following conditions:*

$$\lim_{n\to\infty}\alpha_n = 0, \qquad \lim_{n\to\infty}\frac{k_n}{\alpha_n} = 0, \qquad \sum_{n=1}^{\infty}\alpha_n = \infty, \qquad 0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1. \tag{1.14}$$

Define a sequence $\{x_n\}$ *by* $x_0 \in E$,

$$x_{n+1} = \beta_n x_n + (1-\beta_n) y_n,$$
$$y_n = (I - \alpha_n G) T(t_n) x_n + \alpha_n \gamma_n f(x_n), \quad n \geq 0. \tag{1.15}$$

Then, the sequence $\{x_n\}$ *converges strongly to a common fixed point of the family* $\mathfrak{J}$ *which solves the variational inequality*

$$\langle (G - \gamma f) q, j(x-q) \rangle \geq 0, \quad \forall x \in F. \tag{1.16}$$

Remark 1.5. It is well known that all l^p $(1 < p < \infty)$ spaces satisfy Opial's condition and possess a weakly sequentially continuous duality mapping. However, L^p $(1 < p < \infty)$ spaces and consequently all Sobolev spaces do not satisfy either of the properties.

It is our purpose in this paper to prove a strong convergence theorem for approximating common fixed points of family of uniformly asymptotically regular generalized asymptotically nonexpansive semigroup in a real reflexive and strictly convex Banach space E with a uniformly Gâteaux differentiable norm. Our theorem is applicable in $L_p(\ell_p)$ spaces, $1 < p < \infty$ (and consequently in sobolev spaces). Our theorem extends and improves some recent important results. For instance, our theorem presents a convergence of an explicit scheme that extends Theorem 1.4 to a more general setting of Banach spaces that includes L^p $(1 < p < \infty)$ spaces on one hand and for more general class of maps on the other hand.

2. Preliminaries

Let $S := \{x \in E : \|x\| = 1\}$ denote the unit sphere of a real Banach space E. E is said to have a *Gâteaux differentiable* norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S$; E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$, the limit is attained uniformly for $x \in S$. A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ for $x \neq y$ and $\|x\| = \|y\| = 1$.

Let K be a nonempty, closed, convex, and bounded subset of a real Banach space E, and let the diameter of K be defined by $d(K) := \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := \sup\{\|x - y\| : y \in K\}$ and $r(K) := \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (introduced in 1980 by Bynum [27], see also Lim [28] and the references contained therein) is defined by $N(E) := \inf\{(d(K)/r(K)) : K$ is a closed convex and bounded subset of E with $d(K) > 0\}$. A space E such that $N(E) > 1$ is said to have *uniform normal structures*. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g., [29]).

Let E be a real Banach space with uniformly Gâteaux differentiable norm, then the normalized duality mapping $J : E \to 2^{E^*}$, defined by (1.1), is singled valued and uniformly continuous from the norm topology of E to the weak* topology of E^* on each bounded subset of E, see, for example [30].

Definition 2.1. Let μ be a continuous linear functional on l^∞, and let $(a_0, a_1, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(a_0, a_1, \ldots)$. The function μ is called a Banach limit when μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for each $(a_0, a_1, \ldots) \in l^\infty$.

For a Banach limit μ, it is known that $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for every $a = (a_0, a_1, \ldots) \in l^\infty$. So if $a = (a_0, a_1, \ldots) \in l^\infty$ and $a_n - b_n \to 0$ as $n \to \infty$, we have $\mu_n(a_n) = \mu_n(b_n)$.

We will make use of the following well-known result.

Lemma 2.2. *Let E be a real-normed linear space. Then, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle \quad \forall x, y \in E, j(x + y) \in J(x + y). \tag{2.2}$$

In the sequel, we shall also make use of the following lemmas.

Lemma 2.3 (Suzuki [31]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a real Banach space E, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integer $n \geq 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.4 (Shioji and Takahashi [32]). *Let $(a_0, a_1, a_2, \ldots) \in l^\infty$ be such that $\mu_n a_n \leq 0$ for all Banach limits μ. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq 0$.*

Lemma 2.5 (Xu [33]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0, \tag{2.3}$$

where (i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ (ii) $\limsup_{n\to\infty}\sigma_n \leq 0$ (iii) $\gamma_n \geq 0$ *and* $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$*. Then, $a_n \to 0$ as $n \to \infty$.*

3. Main Results

Theorem 3.1. *Let E be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and let $\mathfrak{J} = \{T(t) : t \geq 0\}$ be uniformly asymptotically regular family of generalized asymptotically nonexpansive semigroup of E, with functions $u, v : [0,\infty) \to [0,\infty)$ and $F := F(\mathfrak{J}) = \cap_{t\geq 0} F(T(t)) \neq \emptyset$. Let $f : E \to E$ be weakly contractive, and let $G : E \to E$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Let $\eta := (1 - \sqrt{(1-\delta)/\lambda})$ and $\gamma \in (0, \min\{\delta, \eta/2\})$. Let $\{\beta_n\}$ and $\{\alpha_n\}$ be sequences in $(0,1]$ and $\{t_n\}$ an increasing sequence in $[0,\infty)$ satisfying the following conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \lim_{n\to\infty} \frac{u(t_n)}{\alpha_n} = 0, \qquad \lim_{n\to\infty} \frac{v(t_n)}{\alpha_n} = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty,$$
$$0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1, \qquad \lim_{n\to\infty} t_n = \infty. \tag{3.1}$$

Define a sequence $\{x_n\}$ *by* $x_0 \in E$,

$$x_{n+1} = \beta_n x_n + (1-\beta_n) y_n,$$
$$y_n = (I - \alpha_n G) T(t_n) x_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{3.2}$$

Then, the sequence $\{x_n\}$ *converges strongly to a common fixed point of the family* $\mathfrak{J}$ *which solves the variational inequality*

$$\langle (G - \gamma f) q, j(x - q) \rangle \geq 0, \quad \forall x \in F. \tag{3.3}$$

Proof. We start by showing that solution of the variational inequality (3.3) in F is at most one. Assume that $q, p \in F$ are solutions of the variational inequality (3.3), then

$$\langle (G - \gamma f) p, j(q - p) \rangle \geq 0, \qquad \langle (G - \gamma f) q, j(p - q) \rangle \geq 0. \tag{3.4}$$

Adding these two inequalities, we get

$$\langle (G - \gamma f) p - (G - \gamma f) q, j(p - q) \rangle \leq 0. \tag{3.5}$$

Therefore,

$$\begin{aligned} 0 &\geq \langle (G - \gamma f) p - (G - \gamma f) q, j(p - q) \rangle \\ &= \langle G(p) - G(q), j(p - q) \rangle - \gamma \langle f(p) - f(q), j(p - q) \rangle \\ &\geq \delta \|p - q\|^2 - \gamma \|f(p) - f(q)\| \|p - q\| \\ &\geq \delta \|p - q\|^2 + \gamma \psi(\|p - q\|) \|p - q\| - \gamma \|p - q\|^2 \\ &= (\delta - \gamma) \|p - q\|^2 + \gamma \psi(\|p - q\|) \|p - q\|. \end{aligned} \tag{3.6}$$

Since $\delta > \gamma$, we obtain that $p = q$, and so the solution is unique in F.

Now, let $p \in F$, since $(1 - \alpha_n \eta)(u(t_n)/\alpha_n) \to 0$ and $(1 - \alpha_n \eta)(v(t_n)/\alpha_n) \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $(1 - \alpha_n \eta)(u(t_n)/\alpha_n) < (\eta - \gamma)/2$ and $(1 - \alpha_n \eta)(v(t_n)/\alpha_n) < (\eta - \gamma)/2$ for all $n \geq n_0$. Hence, for $n \geq n_0$, we have the following:

$$\begin{aligned} \|y_n - p\| &\leq \|(I - \alpha_n G)(T(t_n) x_n - p)\| + \alpha_n \|\gamma f(x_n) - G(p)\| \\ &\leq (1 - \alpha_n \eta) [(1 + u(t_n)) \|x_n - p\| + v(t_n)] + \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - G(p)\| \\ &\leq [1 - \alpha_n (\eta - \gamma) + (1 - \alpha_n \eta) u(t_n)] \|x_n - p\| + (1 - \alpha_n \eta) v(t_n) + \alpha_n \|\gamma f(p) - G(p)\|, \end{aligned} \tag{3.7}$$

so that

$$\begin{aligned}
\|x_{n+1}-p\| &\le \beta_n\|x_n-p\| + (1-\beta_n)\|y_n-p\| \\
&\le \left[\beta_n + (1-\beta_n)\left[1-\alpha_n(\eta-\gamma) + (1-\alpha_n\eta)u(t_n)\right]\right]\|x_n-p\| \\
&\quad + (1-\alpha_n\eta)(1-\beta_n)v(t_n) + \alpha_n(1-\beta_n)\|\gamma f(p)-G(p)\| \\
&\le \left[1-\alpha_n(1-\beta_n)\left((\eta-\gamma)-(1-\alpha_n\eta)\frac{u(t_n)}{\alpha_n}\right)\right]\|x_n-p\| \\
&\quad + \alpha_n(1-\beta_n)\left[\|\gamma f(p)-G(p)\| + (1-\alpha_n\eta)\frac{v(t_n)}{\alpha_n}\right] \\
&\le \left[1-\alpha_n(1-\beta_n)\left((\eta-\gamma)-(1-\alpha_n\eta)\frac{u(t_n)}{\alpha_n}\right)\right]\|x_n-p\| \\
&\quad + \alpha_n(1-\beta_n)\left((\eta-\gamma)-(1-\alpha_n\eta)\frac{u(t_n)}{\alpha_n}\right) \\
&\quad \times \frac{2\left[\|\gamma f(p)-G(p)\| + (1-\alpha_n\eta)(v(t_n)/\alpha_n)\ \right]}{\eta-\gamma} \\
&\le \max\left\{\|x_n-p\|, \frac{2\|\gamma f(p)-G(p)\|}{\eta-\gamma}+1\right\}.
\end{aligned} \tag{3.8}$$

By induction, we have

$$\|x_n-p\| \le \max\left\{\|x_{n_0}-p\|, \frac{2\|\gamma f(p)-G(p)\|}{\eta-\gamma}+1\right\}, \quad \forall n \ge 0. \tag{3.9}$$

Thus, $\{x_n\}$ is bounded and so are $\{T(t_n)x_n\}, \{GT(t_n)x_n\}, \{y_n\}$, and $\{f(x_n)\}$. Observe that

$$\begin{aligned}
y_{n+1}-y_n &= ((I-\alpha_{n+1}G)T(t_{n+1})x_{n+1} - (I-\alpha_{n+1}G)T(t_{n+1})x_n) \\
&\quad + ((I-\alpha_{n+1}G)T(t_{n+1})x_n - (I-\alpha_n G)T(t_{n+1})x_n) \\
&\quad + ((I-\alpha_n G)T(t_{n+1})x_n - (I-\alpha_n G)T(t_n)x_n) \\
&\quad + \left(\alpha_{n+1}\gamma f(x_{n+1}) - \alpha_{n+1}\gamma f(x_n)\right) + \left(\alpha_{n+1}\gamma f(x_n) - \alpha_n\gamma f(x_n)\right),
\end{aligned} \tag{3.10}$$

so that

$$\begin{aligned}\|y_{n+1}-y_n\| &\leq (1-\alpha_{n+1}\eta)(1+u(t_{n+1}))\|x_{n+1}-x_n\|+(1-\alpha_{n+1}\eta)v(t_{n+1})\\ &\quad+|\alpha_n-\alpha_{n+1}|\|GT(t_{n+1})x_n\|+(1-\alpha_n\eta)\|T((t_{n+1}-t_n)+t_n)x_n-T(t_n)x_n\|\\ &\quad+\alpha_{n+1}\gamma\|f(x_{n+1})-f(x_n)\|+|\alpha_{n+1}-\alpha_n|\gamma\|f(x_n)\|\\ &\leq (1-\alpha_{n+1}\eta)(1+u(t_{n+1}))\|x_{n+1}-x_n\|+(1-\alpha_{n+1}\eta)v(t_{n+1})\\ &\quad+|\alpha_n-\alpha_{n+1}|\|GT(t_{n+1})x_n\|\\ &\quad+(1-\alpha_n\eta)\sup_{z\in\{x_n\},s\in\mathbb{R}^+}\|T(s+t_n)z-T(t_n)z\|\\ &\quad+\alpha_{n+1}\gamma\|f(x_{n+1})-f(x_n)\|+|\alpha_{n+1}-\alpha_n|\gamma\|f(x_n)\|.\end{aligned} \tag{3.11}$$

From this, we obtain that

$$\begin{aligned}\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| &\leq [(1-\alpha_{n+1}\eta)(1+u(t_{n+1}))-1]\|x_{n+1}-x_n\|\\ &\quad+(1-\alpha_{n+1}\eta)v(t_{n+1})+|\alpha_n-\alpha_{n+1}|\|GT(t_{n+1})x_n\|\\ &\quad+(1-\alpha_n\eta)\sup_{z\in\{x_n\},s\in\mathbb{R}^+}\|T(s+t_n)z-T(t_n)z\|\\ &\quad+\alpha_{n+1}\gamma\|f(x_{n+1})-f(x_n)\|+|\alpha_{n+1}-\alpha_n|\gamma\|f(x_n)\|,\end{aligned} \tag{3.12}$$

which implies that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0, \tag{3.13}$$

and by Lemma 2.3,

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.14}$$

Thus,

$$\|x_{n+1}-x_n\|=(1-\beta_n)\|y_n-x_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.15}$$

Next, we show that $\lim_{n\to\infty}\|y_n-T(t)y_n\|=0$, for all $t\geq 0$.
Since

$$\begin{aligned}\|x_n-T(t_n)x_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T(t_n)x_n\|\\ &\leq \|x_n-x_{n+1}\|+\beta_n\|x_n-T(t_n)x_n\|+(1-\beta_n)\|y_n-T(t_n)x_n\|,\end{aligned} \tag{3.16}$$

we have

$$\begin{aligned}(1-\beta_n)\|x_n - T(t_n)x_n\| &\le \|x_n - x_{n+1}\| + (1-\beta_n)\|y_n - T(t_n)x_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n(1-\beta_n)\|\gamma f(x_n) - GT(t_n)x_n\|.\end{aligned} \tag{3.17}$$

From $\alpha_n \to 0$ as $n \to \infty$ and (3.15), we obtain

$$\lim_{n\to\infty}\|x_n - T(t_n)x_n\| = 0. \tag{3.18}$$

Also,

$$\begin{aligned}\|y_n - T(t_n)y_n\| &\le \|y_n - x_n\| + \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - T(t_n)y_n\| \\ &\le (2+u(t_n))\|y_n - x_n\| + v(t_n) + \|x_n - T(t_n)x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.19}$$

Since $\lim_{n\to\infty} t_n = \infty$ and $\{T(t) : t \ge 0\}$ is uniformly asymptotically regular,

$$\begin{aligned}\lim_{n\to\infty}\|T(t)T(t_n)x_n - T(t_n)x_n\| &\le \lim_{n\to\infty}\sup_{x\in C}\|T(t)T(t_n)x - T(t_n)x\| = 0, \\ \lim_{n\to\infty}\|T(t)T(t_n)y_n - T(t_n)y_n\| &\le \lim_{n\to\infty}\sup_{y\in C}\|T(t)T(t_n)y - T(t_n)y\| = 0,\end{aligned} \tag{3.20}$$

where C is any bounded subset of E containing $\{x_n\}$. Since $\{T(t)\}$ is continuous, we get that

$$\begin{aligned}\|y_n - T(t)y_n\| \le{}& \|y_n - T(t_n)y_n\| + \|T(t_n)y_n - T(t)(T(t_n)y_n)\| \\ &+ \|T(t)(T(t_n)y_n) - T(t)y_n\|.\end{aligned} \tag{3.21}$$

This implies that

$$\lim_{n\to\infty}\|y_n - T(t)y_n\| = 0, \quad \forall t \ge 0. \tag{3.22}$$

Next, we show that

$$\limsup_{n\to\infty}\langle(\gamma f - G)p, j(y_n - p)\rangle \le 0. \tag{3.23}$$

Define a map $\phi : E \to \mathbb{R}$ by

$$\phi(y) := \mu_n\|y_n - y\|^2, \quad \forall y \in E. \tag{3.24}$$

Then, $\phi(y) \to \infty$ as $\|y\| \to \infty$, ϕ is continuous and convex, so as E is reflexive, there exists $q \in E$ such that $\phi(q) = \min_{u\in E}\phi(u)$. Hence, the set

$$K^* := \left\{y \in E : \phi(y) = \min_{u\in E}\phi(u)\right\} \ne \emptyset. \tag{3.25}$$

Since $\lim_{n\to\infty}\|y_n - T(t)y_n\| = 0$, $\lim_{t\to\infty}u(t) = 0$, $\lim_{t\to\infty}v(t) = 0$, and ϕ is continuous for all $z \in K^*$, we have

$$\begin{aligned}\phi\left(\lim_{t\to\infty}T(t)z\right) &= \lim_{t\to\infty}\phi(T(t)z) = \lim_{t\to\infty}\mu_n\|y_n - T(t)z\|^2 \\ &\le \lim_{t\to\infty}\mu_n\big((1+u(t))\|y_n - z\| + (v(t))\big)^2 = \mu_n\|y_n - z\|^2 = \phi(z).\end{aligned} \tag{3.26}$$

Hence, $\lim_{t\to\infty}T(t)z \in K^*$.

Let $p \in F$. Since K^* is a closed-convex set, there exists a unique $q \in K^*$ such that

$$\|p - q\| = \min_{x\in K^*}\|p - x\|. \tag{3.27}$$

Since $p = \lim_{t\to\infty}T(t)p$ and $\lim_{t\to\infty}T(t)q \in K^*$,

$$\begin{aligned}\left\|p - \lim_{t\to\infty}T(t)q\right\| &= \left\|\lim_{t\to\infty}T(t)p - \lim_{t\to\infty}T(t)q\right\| \\ &= \lim_{t\to\infty}\|T(t)p - T(t)q\| \\ &\le \lim_{t\to\infty}\big((1+u(t))\|p - q\| + v(t)\big) \\ &\le \|p - q\|.\end{aligned} \tag{3.28}$$

Therefore, $\lim_{t\to\infty}T(t)q = q$. Since $T(s+h)x = T(s)T(h)x$ for all $x \in E$ and $s \ge 0$, we have

$$\begin{aligned}q &= \lim_{t\to\infty}T(t)q = \lim_{t\to\infty}T(s+t)q = \lim_{t\to\infty}T(s)T(t)q \\ &= T(s)\lim_{t\to\infty}T(t)q = T(s)q.\end{aligned} \tag{3.29}$$

Therefore, $q \in F$ and so $K^* \cap F \ne \emptyset$.

Let $p \in K^* \cap F(T)$ and $\tau \in (0,1)$. Then, it follows that $\phi(p) \le \phi(p - \tau(G - \gamma f)p)$, and using Lemma 2.2, we obtain that

$$\|y_n - p + \tau(G - \gamma f)p\|^2 \le \|y_n - p\|^2 + 2\tau\big\langle (G - \gamma f)p, j\big(y_n - p + \tau(G - \gamma f)p\big)\big\rangle, \tag{3.30}$$

which implies that

$$\mu_n\big\langle (\gamma f - G)p, j\big(y_n - p + \tau(G - \gamma f)p\big)\big\rangle \le 0. \tag{3.31}$$

Moreover,

$$\begin{aligned}\mu_n\langle(\gamma f-G)p,j(y_n-p)\rangle &= \mu_n\langle(\gamma f-G)p,j(y_n-p)-j(y_n-p+\tau(G-\gamma f)p)\rangle\\&\quad+\mu_n\langle(\gamma f-G)p,j(y_n-p+\tau(G-\gamma f)p)\rangle\\&\le\mu_n\langle(\gamma f-G)p,j(y_n-p)-j(y_n-p+\tau(G-\gamma f)p)\rangle.\end{aligned}\tag{3.32}$$

Since j is norm-to-weak* uniformly continuous on bounded subsets of E, we have that

$$\mu_n\langle(\gamma f-G)p,j(y_n-p)\rangle\le 0.\tag{3.33}$$

Observe that from (3.14) and (3.15), we have

$$\lim_{n\to\infty}\|y_{n+1}-y_n\|=0.\tag{3.34}$$

This implies that

$$\limsup_{n\to\infty}\left[\langle(\gamma f-G)p,j(y_n-p)\rangle-\langle(\gamma f-G)p,j(y_{n+1}-p)\rangle\right]\le 0,\tag{3.35}$$

and so we obtain by Lemma 2.4 that

$$\limsup_{n\to\infty}\langle(\gamma f-G)p,j(y_n-p)\rangle\le 0.\tag{3.36}$$

Finally, we show that $x_n\to p$ as $n\to\infty$. Since $\lim_{n\to\infty}(u(t_n)/\alpha_n)=0$, if we denote by $\sigma(t_n)$ the value $2u(t_n)+u(t_n)^2$, then we clearly have $\lim_{n\to\infty}(\sigma(t_n)/\alpha_n)=0$. Let $N_0\in\mathbb{N}$ be large enough such that $(1-\alpha_n\eta)(\sigma(t_n)/\alpha_n)<(\eta-2\gamma)/2$, for all $n\ge N_0$, and let M be a positive real number such that $||x_n - p|| \leq M$ for all $n \geq 0$. Then, using the recursion formula (3.2) and for $n \geq N_0$, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(\gamma f(x_n) - G(p)) + (I - \alpha_n G)(T(t_n)x_n - p)\|^2 \\
&\leq (1-\alpha_n\eta)\|T(t_n)x_n - p\|^2 + 2\alpha_n\langle \gamma f(x_n) - G(p), j(y_n - p)\rangle \\
&\leq (1-\alpha_n\eta)\left[(1+u(t_n))\|x_n - p\| + v(t_n)\right]^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - \gamma f(p) + \gamma f(p) - G(p), j(y_n - p)\rangle \\
&\leq (1-\alpha_n\eta)\left[(1+u(t_n))^2\|x_n - p\|^2 + 2(1+u(t_n))v(t_n)\|x_n - p\|^2 + v(t_n)^2\right] \\
&\quad + 2\alpha_n\langle \gamma f(p) - G(p), j(y_n - p)\rangle - 2\alpha_n\gamma\|y_n - p\|\psi(\|x_n - p\|) \\
&\quad + 2\alpha_n\gamma\|(y_n - x_n) + (x_n - p)\|\|x_n - p\| \\
&\leq \left[(1-\alpha_n\eta)(1+\sigma(t_n)) + 2\alpha_n\gamma\right]\|x_n - p\|^2 \\
&\quad + \alpha_n\left[2\langle \gamma f(p) - G(p), j(y_n - p)\rangle + 2(1-\alpha_n\eta)(1+u(t_n))\frac{v(t_n)}{\alpha_n}\|x_n - p\|^2\right. \\
&\qquad \left. + (1-\alpha_n\eta)\frac{v(t_n)^2}{\alpha_n} + 2\gamma\|y_n - x_n\|\|x_n - p\|\right] \\
&= \left[1 - \alpha_n\left((\eta - 2\gamma) - (1-\alpha_n\eta)\frac{\sigma_n}{\alpha_n}\right)\right]\|x_n - p\|^2 \\
&\quad + \alpha_n\left[2\langle \gamma f(p) - G(p), j(y_n - p)\rangle + 2(1-\alpha_n\eta)(1+u(t_n))\frac{v(t_n)}{\alpha_n}\|x_n - p\|^2\right. \\
&\qquad \left. + (1-\alpha_n\eta)\frac{v(t_n)^2}{\alpha_n} + 2\gamma\|y_n - x_n\|\|x_n - p\|\right],
\end{aligned} \tag{3.37}$$

so that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\
&\leq \left(\beta_n + (1-\beta_n)\left[1 - \alpha_n\left((\eta - 2\gamma) - (1-\alpha_n\eta)\frac{\sigma_n}{\alpha_n}\right)\right]\right)\|x_n - p\|^2 \\
&\quad + \alpha_n(1-\beta_n)\left[2\langle \gamma f(p) - G(p), j(y_n - p)\rangle + 2(1-\alpha_n\eta)(1+u(t_n))\right. \\
&\qquad \left. \times\frac{v(t_n)}{\alpha_n}\|x_n - p\|^2 + (1-\alpha_n\eta)\frac{v(t_n)^2}{\alpha_n} + 2\gamma\|y_n - x_n\|\|x_n - p\|\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \left[1-\alpha_n(1-\beta_n)\left((\eta-2\gamma)-(1-\alpha_n\eta)\frac{\sigma_n}{\alpha_n}\right)\right]\|x_n-p\|^2 \\
&\quad+\alpha_n(1-\beta_n)\left[2\langle\gamma f(p)-G(p),j(y_n-p)\rangle+2(1-\alpha_n\eta)(1+u(t_n))\frac{v(t_n)}{\alpha_n}M^2\right. \\
&\qquad\left.+(1-\alpha_n\eta)\frac{v(t_n)^2}{\alpha_n}+2\gamma\|y_n-x_n\|M\right] \\
&= \left[1-\alpha_n(1-\beta_n)\left((\eta-2\gamma)-(1-\alpha_n\eta)\frac{\sigma_n}{\alpha_n}\right)\right]\|x_n-p\|^2 \\
&\quad+\alpha_n(1-\beta_n)\left((\eta-2\gamma)-(1-\alpha_n\eta)\frac{\sigma_n}{\alpha_n}\right) \\
&\quad\times\frac{\left[2\langle\gamma f(p)-G(p),j(y_n-p)\rangle+2(1-\alpha_n\eta)(1+u(t_n))\left(\dfrac{v(t_n)}{\alpha_n}\right)M^2+\mathcal{A}_n\right]}{\left((\eta-2\gamma)-(1-\alpha_n\eta)\left(\dfrac{\sigma_n}{\alpha_n}\right)\right)},
\end{aligned} \tag{3.38}$$

where $\mathcal{A}_n$ denotes $(1-\alpha_n\eta)(v(t_n)^2/\alpha_n)+2\gamma\|y_n-x_n\|M$.

Observe that $\sum_{n=1}^{\infty}\alpha_n(1-\beta_n)((\eta-2\gamma)-(1-\alpha_n\eta)(\sigma_n/\alpha_n))=\infty$ and

$$\limsup_{n\to\infty}\left(\frac{2\langle\gamma f(p)-G(p),j(y_n-p)\rangle+2(1-\alpha_n\eta)(1+u(t_n))(v(t_n)/\alpha_n)\ M^2+\mathcal{A}_n}{((\eta-2\gamma)-(1-\alpha_n\eta)(\sigma_n/\alpha_n))}\right)\leq 0. \tag{3.39}$$

Applying Lemma 2.5, we obtain $\|x_n-p\|\to 0$ as $n\to\infty$. This completes the proof. □

The following corollaries follow from Theorem 3.1.

Corollary 3.2. *Let E be a real uniformly convex and uniformly smooth Banach space, $\mathfrak{J}=\{T(t):t\geq 0\}$, and let $F,f,G,\delta,\lambda,\eta,\gamma,\{\beta_n\},\{\alpha_n\},\{t_n\}$ and $\{x_n\}$ be as in Theorem 3.1. Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathfrak{J}$ which solves the variational inequality* (3.3).

Corollary 3.3. *Let $E=H$ be a real Hilbert space, and let $\mathfrak{J}=\{T(t):t\geq 0\}$, $F,f,G,\delta,\lambda,\eta,\gamma,\{\beta_n\},\{\alpha_n\},\{t_n\}$ and $\{x_n\}$ be as in Theorem 3.1. Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathfrak{J}$ which solves the variational inequality*

$$\langle(G-\gamma f)q,x-q\rangle\geq 0,\quad\forall x\in F. \tag{3.40}$$

Corollary 3.4. *Let $\mathfrak{J}=\{T(t):t\geq 0\}$ be a family of nonexpansive semigroup of a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm E, and let $F,f,G,\delta,\lambda,\eta,\gamma,\{\beta_n\},\{\alpha_n\},\{t_n\}$, and $\{x_n\}$ be as in Theorem 3.1. Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathfrak{J}$ which solves the variational inequality* (3.3).

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