Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 856927, 14 pages
doi:10.1155/2012/856927

Research Article

On Heat Conduction in Domains Containing Noncoaxial Cylinders

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Received 28 November 2011; Accepted 16 January 2012

Academic Editor: Adolfo Ballester-Bolinches

We consider heat conduction in domains containing noncoaxial cylinders. In particular, we present some regularity results for the solution and consider criteria which ensure the single valueness of the corresponding complex potential. Examples are discussed. In addition, we present some classes of cases where the parameters describing the solution are rational. Alternative ways of calculating the heat flux are also discussed.

1. Introduction

Let D be the unit-disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$ and let $B \subset D$ be some disk of the form $B = \{(x, y) : (x - x_0)^2 + y^2 \leq R^2\}$. Moreover, let the conductivity $\lambda = \lambda(x, y)$ be defined on the unit-disk as follows:

$$\lambda(x, y) = \begin{cases} k & \text{if } (x, y) \in B, \\ 1 & \text{if } (x, y) \in D \setminus B, \end{cases} \tag{1.1}$$

where $k > 0$ is some constant. We focus on the stationary heat conduction problems of the form:

$$\begin{gathered} \operatorname{div}(\lambda \operatorname{grad} u) = 0 \quad \text{on } \Omega \setminus \partial B, \\ u = g \quad \text{on } \partial\Omega, \\ u \text{ and } \frac{\lambda \partial u}{\partial n} \text{ are continuous through } \partial B. \end{gathered} \tag{1.2}$$

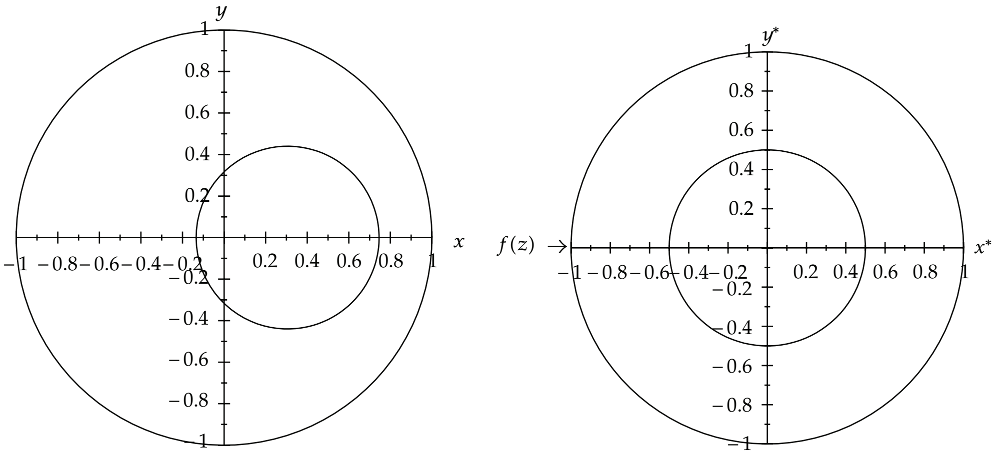


Figure 1: The Möbius transformation.

Here, $\Omega \subset D$ is a region with smooth boundary $\partial\Omega$, g is a fixed smooth function defined on $\partial\Omega$, u is the temperature, and n is the outward unit normal of the surface ∂B of B. Since $u(x, y)$ is piecewise harmonic (see Proposition 2.1), it is the real part (or imaginary part) of some piecewise analytic function $F(z)$, $z = x + iy$, the associated *complex potential* (Figure 1). Using a particular Möbius transformation $z = x + iy \to z^* = x^* + iy^*$ of the type

$$z^* = f(z) = \frac{z - b}{bz - 1}, \tag{1.3}$$

which maps D onto D and B onto a disk $B^* \subset D$ with center at 0 and some radius R^* (for a particular choice of the constant b,$-1 < b < 1$), we obtain that $F(z) = F^*(z^*)$, where $F^*(z^*)$ is the complex potential associated with the problem:

$$\begin{gathered}
\operatorname{div}\left(\lambda^* \operatorname{grad} u^*\right) = 0 \quad \text{on } \Omega^* \setminus \partial B^*, \\
u^* = g^* \quad \text{on } \partial\Omega^*, \\
\frac{\lambda^* \partial u^*}{\partial n} \text{is continuous through } \partial B^*.
\end{gathered} \tag{1.4}$$

Here, $\Omega^* = f(\Omega)$ and $B^* = f(B)$, and $h^* : D \to \mathbb{R}$ denotes the function $h^*(x^*, y^*) = h(x, y)$ whenever $h : D \to \mathbb{R}$. Of symmetry reasons, the auxiliary problem (1.4) is usually much easier to solve. Once this problem is solved, we obtain the solution of our original problem (1.2) by setting

$$u(x, y) = u^*(x^*, y^*) = u^*\left(\operatorname{Re} f(x + iy), \operatorname{Im} f(x + iy)\right). \tag{1.5}$$

In this paper, we discuss problems of this type when the boundary $\partial\Omega$ may be a little more general than that above. In Section 2, we recall the simplest possible example and present some regularity results for the solution. Moreover, we consider criteria which ensure the single valueness of the corresponding complex potential. An interesting example referred to as the Hashin-Shtrikman problem is discussed in Section 3. In Section 4, we describe methods, involving the so-called PQ-algorithm, for generating all integer values of $(x_0 - R)^{-1}$ and $(x_0 + R)^{-1}$ making x_0, R, and b rational. Finally, in Section 5, we briefly discuss alternative ways of calculating the heat flux.

2. Single-Valued Complex Potentials

The simplest example of a problem of the form (1.2) is when $\Omega = D \setminus B$ and

$$g = \begin{cases} u_0 & \text{on } \partial D, \\ u_1 & \text{on } \partial B, \end{cases} \tag{2.1}$$

for some constants u_0 and u_1. In this case, it is easily seen that $u(x, y) = \operatorname{Re} f(z) = \operatorname{Re} f^*(f(z))$, where the complex potential associated with the auxiliary problem F^* is given by

$$F^*(z^*) = \frac{u_1 - u_0}{\ln R^*} \log z^* + u_0. \tag{2.2}$$

In this case, Ω is multiple connected and the harmonic conjugate of u (given by $v(x, y) = \operatorname{Re} f(z)$) is clearly multiple valued. The following results show in particular that this is not the case when Ω is simply connected.

Proposition 2.1. *Assume that the boundary $\partial\Omega$ is Lipschitz continuous and let u be the solution of the weak formulation corresponding to* (1.2)*. Then u is harmonic in the interior of each region where λ is constant. Moreover, if Ω is simply connected, then the harmonic conjugate v of u is single valued in these regions.*

Proof. Let u be a weak solution of (1.2), that is, u belongs to the Sobolev space $W^1(\Omega)$ with trace $u|_{\partial\Omega} = g$ satisfying

$$\int_\Omega \lambda(\operatorname{grad} u) \cdot \operatorname{grad} \varphi \, dx \, dy = 0, \tag{2.3}$$

for all $\varphi \in C_0^\infty(\Omega)$. Assume that λ is constant in a disk O with centre at some point (x_0, y_0). Then, (2.3) gives that

$$\int_O \left(\frac{\partial u}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \varphi}{\partial y} \right) dx \, dy = 0, \tag{2.4}$$

for all $\varphi \in W_0^{1,2}(O)$, that is, u is (by definition) a generalized solution of the Laplace equation in O. Hence, by standard regularity results for elliptic partial differential equations, this gives that u is a solution of the Laplace equation in O in the classical sense (this is, e.g., a consequence of the regularity result stated in [1] Theorem 8.8; see also [2]). Hence, we conclude that u is harmonic in the interior of each region where λ is constant.

By (2.3),

$$p = \lambda(\operatorname{grad} u) \in L^2_{\mathrm{sol}}(\Omega), \tag{2.5}$$

where

$$L^2_{\mathrm{sol}}(\Omega) = \left\{ q \in \mathbf{L}^2(\Omega) : \int_\Omega q \cdot \operatorname{grad} \varphi \, dx \, dy = 0 \, \forall \varphi \in C_0^\infty(\Omega) \right\}. \tag{2.6}$$

When Ω is simply connected, it holds that (see e.g., [3, page 467])

$$L^2_{\mathrm{sol}}(\Omega) = \left\{ \operatorname{curl} \varphi : \varphi \in W^{1,2}(\Omega) \right\}, \tag{2.7}$$

where $\operatorname{curl} \varphi$ denotes the vector-function defined by $\operatorname{curl} \varphi = (-\partial\varphi/\partial y, \partial\varphi/\partial x)$. Hence, $p = \operatorname{curl} \psi$ for some function $\psi \in W^1(\Omega)$. Thus,

$$\lambda^{-1}(\operatorname{curl} \psi) = \lambda^{-1} p = \operatorname{grad} u, \tag{2.8}$$

that is, putting $v = -\lambda^{-1}\psi$, which clearly is single valued, we obtain that

$$-\operatorname{curl} v = \operatorname{grad} u, \tag{2.9}$$

on every domain where λ is constant. This shows that v is a harmonic conjugate of u on these domains, since (2.9) is nothing but the Cauchy-Riemann equations. □

3. On the Hashin-Shtrikman Problem

In the case when $x_0 = 0$,

$$\Omega = D, \qquad g(x, y) = (\xi_1 x + \xi_2 y), \tag{3.1}$$

where ξ_1 and ξ_2 are constant real numbers, the problem (1.2) becomes identical with that used in the proof of the attainability of the famous Hashin and Shtrikman bounds [4] and is, therefore, called *the Hashin-Shtrikman problem.* These bounds are important in the homogenization theory for composite structures. For more general information, we refer to the literature, see, for example, [5–10]. We also like to mention that an elementary introduction to the homogenization method can be found in the book of Persson et al. [11].

It will be clear from the arguements below that the solution of the Hashin-Shtrikman problem is given by

$$u(x, y) = w(x, y) + \xi_1 x + \xi_2 y, \tag{3.2}$$

where

$$w(x,y)=\begin{cases} l_1(\xi_1 x+\xi_2 y) & \text{if } (x,y)\in B,\\ l\left(\xi_1 x+\xi_2 y-\dfrac{\xi_1 x+\xi_2 y}{x^2+y^2}\right) & \text{if } (x,y)\in D\setminus B\end{cases} \tag{3.3}$$

(see, e.g., [12]). The constants l_1 and l will determined below. It is interesting to note that u is the real part of the complex potential:

$$F(z)=\begin{cases} (l_1+1)\overline{\xi}z & \text{if } z\in B,\\ l|\xi|h\left(\overline{\xi}|\xi|^{-1}z\right)+\overline{\xi}z & \text{if } z\in D\setminus B,\end{cases} \tag{3.4}$$

where

$$h(z)=z-\frac{1}{z}, \tag{3.5}$$

and $\xi=\xi_1+i\xi_2$. This follows from the fact that

$$g(z)=|\xi|h\left(\overline{\xi}|\xi|^{-1}z\right)=l\left(\overline{\xi}z-\frac{\xi\overline{z}}{|z|^2}\right). \tag{3.6}$$

Note also that $h(z)=-id(iz)$, where d is the well-known Joukowsky transformation:

$$d(z)=z+\frac{1}{z}. \tag{3.7}$$

The fact that $F(z)$ is analytic in the regions B and $D\setminus B$ shows that $u=\operatorname{Re}F(z)$ is harmonic in these regions. Hence, the first condition of (1.2) $\operatorname{div}(\lambda \operatorname{grad} u)=0$ is satisfied. The unit normal vector $n=(n_1,n_2)$ on the boundaries ∂B and ∂D can be represented by the complex number (still denoted n):

$$n=n_1+in_2=\frac{z}{|z|}. \tag{3.8}$$

Using the fact that

$$F'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \tag{3.9}$$

(by Cauchy.Riemann equations), we find that

$$\frac{\partial u}{\partial n}=\operatorname{grad} u\cdot n=\frac{\partial u}{\partial x}n_1+\frac{\partial u}{\partial y}n_2=\operatorname{Re}\left((n_1+in_2)\left(\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\right)\right)=\operatorname{Re}\left(nF'(z)\right), \tag{3.10}$$

that is,

$$\frac{\partial u}{\partial n} = \operatorname{Re}\left(nF'(z)\right). \tag{3.11}$$

If $z \in D \setminus B$, then

$$\begin{aligned} F'(z) &= l\bar{\xi}h'\left(\bar{\xi}|\xi|^{-1}z\right) + \bar{\xi} = l\bar{\xi}\left(1 + \frac{1}{\left(\bar{\xi}|\xi|^{-1}z\right)^2}\right) + \bar{\xi} \\ &= l\bar{\xi}\left(1 + \frac{\xi\bar{z}\xi\bar{z}}{\left(\xi\bar{\xi}|\xi|^{-1}z\bar{z}\right)^2}\right) + \bar{\xi} = l\bar{\xi}\left(1 + \frac{\xi\bar{z}\xi\bar{z}}{|\xi|^2|z|^4}\right) + \bar{\xi}. \end{aligned} \tag{3.12}$$

Hence, multiplying with $n = z/|z|$ and using the formula $|q|^2 = q\bar{q}$ yield

$$nF'(z) = l\left(\bar{\xi}n + \frac{z}{|z|}\frac{\bar{\xi}\xi\bar{z}\xi\bar{z}}{|\xi|^2|z|^4}\right) + n\bar{\xi} = l\left(n\bar{\xi} + \frac{\bar{z}}{|z|}\frac{\bar{\xi}\xi z\bar{z}}{\bar{\xi}\xi|z|^4}\xi\right) + n\bar{\xi} = l\left(n\bar{\xi} + \frac{1}{|z|^2}\bar{n}\xi\right) + n\bar{\xi}, \tag{3.13}$$

which gives that

$$\frac{\partial u}{\partial n} = \operatorname{Re}\left(nF'(z)\right) = (n_1\xi_1 + n_2\xi_2)\left(l\left(1 + \frac{1}{|z|^2}\right) + 1\right). \tag{3.14}$$

If $z \in B$, then clearly

$$\frac{\partial u}{\partial n} = \operatorname{grad} u \cdot n = (l_1 + 1)(n_1\xi_1 + n_2\xi_2). \tag{3.15}$$

Thus, in order to fulfill the continuity of $\lambda\partial u/\partial n$, we obtain from (3.14) and (3.15) that

$$l\left(1 + \frac{1}{R^2}\right) + 1 = k(l_1 + 1). \tag{3.16}$$

Moreover, the continuity of the temperature u on the circle $x^2 + y^2 = R$ gives that (c.f. (3.3))

$$l\left(1 - \frac{1}{R^2}\right) + 1 = (l_1 + 1). \tag{3.17}$$

Hence,

$$l = \frac{R^2(k-1)}{2+(1-R^2)(k-1)}, \qquad l_1 = \frac{(k-1)\left(R^2-1\right)}{2+(1-R^2)(k-1)}. \tag{3.18}$$

In particular, this gives that

$$\lambda \frac{\partial u}{\partial n} = \widehat{k}(n_1\xi_1 + n_2\xi_2) = \widehat{k} n \cdot \xi \tag{3.19}$$

at the boundary ∂D, where $\widehat{k}$ is the so-called Hashin-Shtrikman bound given by

$$\widehat{k} = \left(\frac{R^2(k-1)}{2+(1-R^2)(k-1)} + 1\right). \tag{3.20}$$

Moreover, noting that $u = \xi_1 x + \xi_2 y = n_1\xi_1 + n_2\xi_2 = n \cdot \xi$ on on the unit circle ∂D, Greens formula,

$$\int_D u \underbrace{\operatorname{div}(\lambda \operatorname{grad} u)}_{=0} dx\, dy = -\int_D \lambda \left|\operatorname{grad} u\right|^2 dx\, dy + \int_{\partial D} u\lambda \frac{\partial u}{\partial n} ds, \tag{3.21}$$

gives that

$$\begin{aligned} \frac{1}{|D|}\int_D \lambda\left|\operatorname{grad} u\right|^2 dx\, dy &= \frac{1}{|D|}\int_{\partial D} u\lambda\frac{\partial u}{\partial n} ds = \frac{1}{|D|}\int_{\partial D} \widehat{k}(n\cdot\xi)^2 ds \\ &= \frac{1}{|D|}\int_0^{2\pi} \widehat{k}(|\xi|\cos\theta)^2 d\theta = \widehat{k}|\xi|^2. \end{aligned} \tag{3.22}$$

Here, θ is the angle between n and ξ.

We can transform the solution on the unit-disk D to a general disk $\widetilde{D}$ with center at $\mathbf{c}_0$ and radius ε where the conductivity $\widetilde{\lambda}$ is given by $\widetilde{\lambda}(\mathbf{x}) = \lambda((\mathbf{x}-\mathbf{c}_0)/\varepsilon)$, $\mathbf{x} = (x, y)$. It is easy to see that the solution $\widetilde{u}$ of the problem,

$$\begin{gathered} \operatorname{div}\left(\widetilde{\lambda} \operatorname{grad} \widetilde{u}\right) = 0 \quad \text{on } \widetilde{D}, \\ \widetilde{u}(\mathbf{x}) = \xi\mathbf{x} \quad \text{on } \partial\widetilde{D}, \\ \widetilde{u} \text{ and } \frac{\lambda\partial\widetilde{u}}{\partial n} \text{ are continuous through } \partial B \text{ and } \partial D, \end{gathered} \tag{3.23}$$

is given by

$$\tilde{u}(\mathbf{x}) = \varepsilon w\left(\frac{\mathbf{x}-\mathbf{c}_0}{\varepsilon}\right) + \xi\mathbf{x}, \tag{3.24}$$

where w is given by (3.3). Moreover, similarly as above, we find that

$$\frac{1}{|\tilde{D}|}\int_{\tilde{D}} \lambda|\operatorname{grad}\tilde{u}|^2 dx\,dy = \hat{k}|\xi|^2. \tag{3.25}$$

Now, let $\xi = (0,1)$ and consider the interior of the square $\square = [-1,1]^2$ centered at 0 whose boundary consists of the four line segments Γ_1, Γ_2, Γ_3, and Γ_4. We can extend the conductivity function λ to $\square$ by setting $\lambda(x) = \hat{k}$ on $\square \setminus D$. Thanks to the above results, it is now easy to see that the solution u of the Dirichlet/Neumann problem,

$$\begin{gathered}
\operatorname{div}(\lambda \operatorname{grad} u) = 0 \quad \text{on } \square \setminus (\partial B \cup \partial D), \\
\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_1, \Gamma_3, \\
u = -1 \quad \text{on } \Gamma_2, \qquad u = 1 \quad \text{on } \Gamma_4, \\
u, \frac{\lambda \partial u}{\partial n} \text{ are continuous through } \partial B \text{ and } \partial D,
\end{gathered} \tag{3.26}$$

is the real part of the complex potential $F^*(z)$ given by (see Figures 2 and 3)

$$F^*(z) = \begin{cases} (l_1+1)\bar{\xi}z & \text{if } z \in B, \\ l|\xi|h\left(\bar{\xi}|\xi|^{-1}z\right) + \bar{\xi}z & \text{if } z \in D \setminus B, \\ \bar{\xi}z & \text{if } z \in \square \setminus D. \end{cases} \tag{3.27}$$

The inverse of the Möbius transformation,

$$w = f(z) = \frac{z-b}{bz-1}, \tag{3.28}$$

is of the same type, namely,

$$z = f^{-1}(w) = \frac{w-b}{bw-1}. \tag{3.29}$$

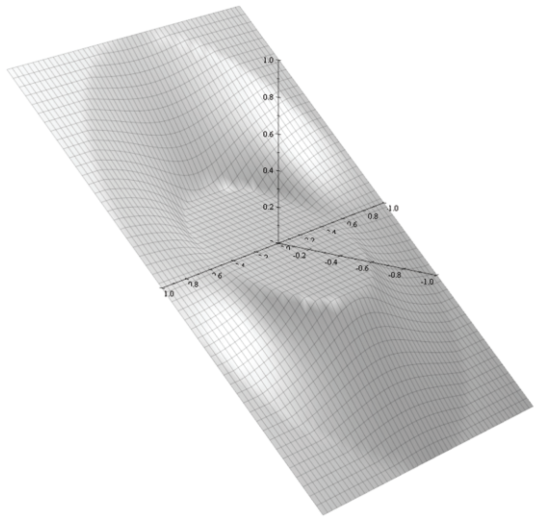
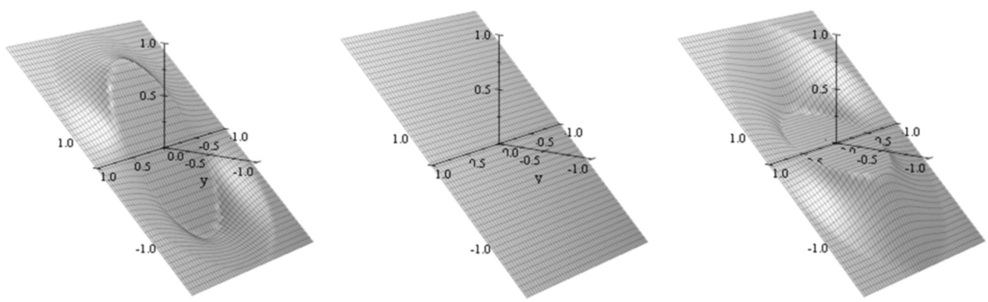


Figure 2: The solution of (3.26) for the case when the conductivity of the disk B is $k = 5$.



$k = 0$ $k = 1$ $k = 5$

Figure 3: The solution of (3.26) for the cases when the conductivity of the disk B is $k = 0$, $k = 1$, and $k = 5$, respectively.

For any fixed real value b, $|b| < 1$, we can transform the above problem on $\square$ to a corresponding problem on the domain $\square' = f^{-1}(\square)$ as follows (see Figure 4):

$$\begin{gathered}
\operatorname{div}(\lambda \operatorname{grad} u) = 0 \quad \text{on } \square' \setminus (\partial B' \cup \partial D), \\
\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_1', \Gamma_3', \\
u = -1 \quad \text{on } \Gamma_2', \quad u = 1 \quad \text{on } \Gamma_4', \\
u, \frac{\lambda \partial u}{\partial n} \text{ are continuous through } \partial B' \text{ and } \partial D.
\end{gathered} \tag{3.30}$$

Here, for an arbitrary set $A \subset \square$, A' denotes the set $A' = f^{-1}(A)$

We obtain a more complex structure than that described on $\square$ if we cover $\square$ by an infinite sequence of nonintersecting discs $\{\widetilde{D}_i\}$, $\widetilde{D}_i \subset \square$ and define the conductivity λ on each

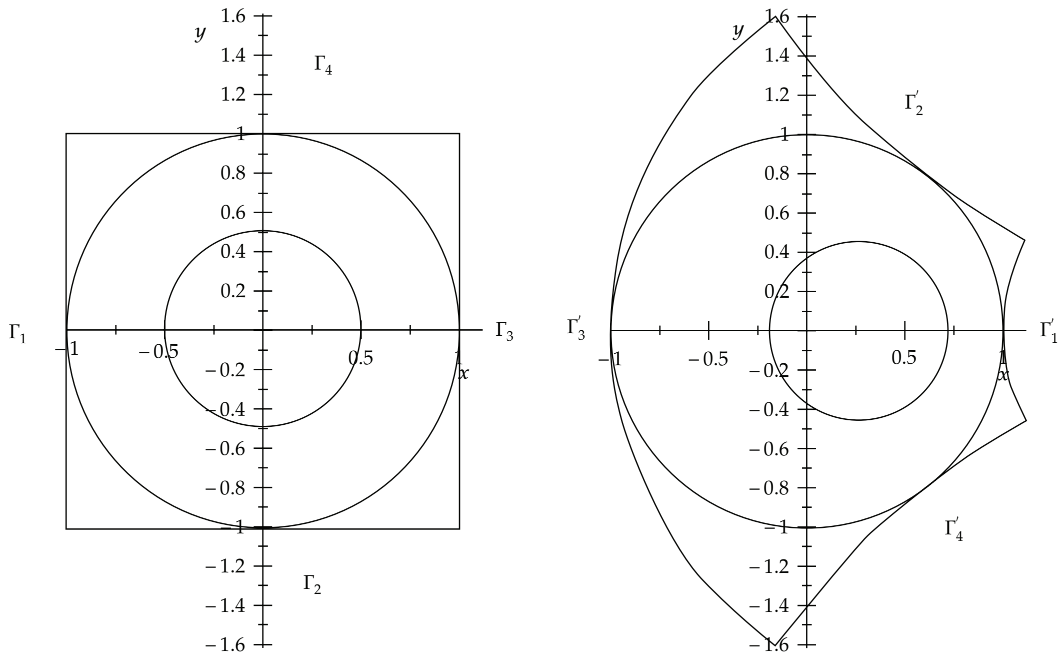


Figure 4: Dirichlet/Neumann problem on □ and the domain $\square' = f^{-1}(\square)$.

disk as we did in the definition of $\widetilde{\lambda}$ above. This function can also be extended by periodicity to the whole plane. The underlying structure is called the Hashin, coated cylinder assemblage. The Dirichlet/Neumann problem,

$$\begin{gathered}
\operatorname{div}(\lambda \operatorname{grad} u) = 0 \quad \text{on } \widetilde{D}_i, \\
\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_1, \Gamma_3, \\
u = -1 \quad \text{on } \Gamma_2, \qquad u = 1 \quad \text{on } \Gamma_4, \\
u, \frac{\lambda \partial u}{\partial n} \text{ are continuous in the } n\text{-direction,}
\end{gathered} \tag{3.31}$$

is easily solved by using the above results. By (3.25) and the fact that $|\square| = \sum_{i=1}^{\infty} |\widetilde{D}_i|$, we obtain that

$$\begin{aligned}
\frac{1}{|\square|} \int_{|\square|} \lambda \left|\operatorname{grad} u\right|^2 dx dy &= \frac{1}{|\square|} \sum_{i=1}^{\infty} \int_{D_i} \lambda \left|\operatorname{grad} u\right|^2 dx dy \\
&= \frac{1}{|\square|} \sum_{i=1}^{\infty} \left|\widetilde{D}_i\right| \widehat{k} |\xi|^2 = \widehat{k} |\xi|^2
\end{aligned} \tag{3.32}$$

for $\xi = (0, 1)$. Due to symmetry, we obtain the same result for $\xi = (1, 0)$ if we replace Γ_1 with Γ_2 and Γ_3 with Γ_4 in (3.31).

4. Rational Parameters

In order to find the value of R^* and b, we utilize the fact that $f(z)$ maps the real line onto itself. In particular, this gives that $|f(x_1)| = |f(x_2)| = R^*$ and $f(x_1) = -f(x_2)$, where $x_1 = x_0 - R$ and $x_2 = x_0 + R$. Hence,

$$\frac{x_1 - b}{bx_1 - 1} = -\frac{x_2 - b}{bx_2 - 1}. \tag{4.1}$$

Provided $x_0 \neq 0$, we find that

$$b = \frac{x_1 x_2 + 1 - \sqrt{(1 - x_1^2)(1 - x_2^2)}}{x_1 + x_2}, \tag{4.2}$$

since this solution satisfies the criteria $|b| < 1$ while the the other solution,

$$\frac{x_1 x_2 + 1 + \sqrt{(1 - x_1^2)(1 - x_2^2)}}{x_1 + x_2}, \tag{4.3}$$

does not. This follows by the following inequalities:

$$\begin{aligned} x_1 &= \frac{x_1 x_2 + 1 - (1 - x_1^2)}{x_1 + x_2} < \frac{x_1 x_2 + 1 - \sqrt{(1 - x_1^2)(1 - x_2^2)}}{x_1 + x_2} \\ &< \frac{x_1 x_2 + 1 - (1 - x_2^2)}{x_1 + x_2} = x_2. \end{aligned} \tag{4.4}$$

Hence,

$$-1 \leq x_1 < b < x_2 \leq 1. \tag{4.5}$$

Moreover, if $x_0 > 0$, then

$$\begin{aligned} &\frac{x_1 x_2 + 1 + \sqrt{(1 - x_1^2)(1 - x_2^2)}}{x_1 + x_2} > \frac{x_1 x_2 + 1 + (1 - x_2^2)}{x_1 + x_2} \\ &\quad = \frac{x_0^2 - R^2 + 2 - (x_0 + R)^2}{2x_0} = \frac{-2R^2 - 2x_0 R + 2}{2x_0} = \frac{1 - R^2 - x_0 R}{x_0} \\ &\quad = \frac{(1 - R)(1 + R) - x_0 R}{x_0} \geq \frac{x_0(1 + R) - x_0 R}{x_0} = 1. \end{aligned} \tag{4.6}$$

Similarly, we obtain that

$$\frac{x_1x_2 + 1 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 + x_2} < -1 \tag{4.7}$$

if $x_0 < 0$.

The R^*-value corresponding to b is given by

$$R^* = \left| \frac{1 - x_1x_2 - \sqrt{(1-x_1^2)(1-x_2^2)}}{x_2 - x_1} \right|. \tag{4.8}$$

Now, assume that $x_1 = 1/m$ and $x_2 = 1/n$ where m and n are integers. Then,

$$\begin{aligned} b &= \frac{1 + mn - \sqrt{(m^2-1)(n^2-1)}}{m+n}, \\ R &= \left| \frac{1}{m-n}\left(-\sqrt{(m^2-1)(n^2-1)} + mn - 1 \right) \right|. \end{aligned} \tag{4.9}$$

It is clear that the pairs (m, n) making b and R rational numbers are precisely those of the form:

$$m^2 - 1 = dr^2, \qquad n^2 - 1 = ds^2, \tag{4.10}$$

where d, r and s are integers. It is possible to show that the integer solutions of the generalized Pell's equation,

$$x^2 - dy^2 = 1, \tag{4.11}$$

for a given integer d which is not a square, are precisely those of the form $x = G_{kl-1}, y = B_{kl-1}$ where l is odd, k is even, and G_i, B_i are the integers obtained from the following algorithm (called the PQ-algorithm, see [13, pages 346–348 and page 358] and [14, pages 125–128]).

Set

$$\begin{aligned} B_{-2} &= 1, \qquad B_{-1} = 0, \\ G_{-2} &= 0, \qquad G_{-1} = 1, \\ P_0 &= 0, \qquad Q_0 = 1. \end{aligned} \tag{4.12}$$

For $i \geq 0$, set

$$\begin{aligned} a_i &= \frac{\left(P_i + \sqrt{d}\right)}{Q_i}, \\ B_i &= a_i B_{i-1} + B_{i-2}, \\ G_i &= a_i G_{i-1} + G_{i-2}. \end{aligned} \tag{4.13}$$

For $i \geq 1$, set

$$\begin{aligned} P_i &= a_{i-1} Q_{i-1} - P_{i-1}, \\ Q_i &= \frac{\left(d - P_i^2\right)}{Q_{i-1}}. \end{aligned} \tag{4.14}$$

Hence, by (4.10) all pairs (m, n) making b and r rational numbers are precisely those of the form $(m, n) = (G_{k_1 l_1 - 1}, G_{k_2 l_2 - 1})$ where l_1 and l_2 are odd, and k_1 and k_2 are even.

5. Calculating the Heat Flux

If C is a simple contour in $\overline{\Omega}$, the corresponding heat flux is given by

$$\int_C \lambda \frac{\partial u}{\partial n} ds = \int_C \lambda \operatorname{grad} u \cdot n ds. \tag{5.1}$$

If u is the real part of some complex potential $F(z)$, then

$$\begin{aligned} \int_C \lambda F'(z) dz &= \int_C \lambda (u_x - i u_y) dz = \int_C \lambda (u_x - i u_y)(dx + i dy) \\ &= \left(\int_C \lambda u_x dx + \int_C \lambda u_y dy \right) + i \left(\int_C \lambda u_x dy - \int_C \lambda u_y dx \right) \\ &= \int_C \lambda (u_x, u_y) \cdot t ds + i \int_C \lambda (u_x, u_y) \cdot n ds = \int_C \lambda \operatorname{grad} u \cdot t ds + i \int_C \lambda \operatorname{grad} u \cdot n ds, \end{aligned} \tag{5.2}$$

when t denotes the unit tangential vector. Hence,

$$\int_C \lambda \frac{\partial u}{\partial n} ds = \operatorname{Im} \int_C \lambda F'(z) dz. \tag{5.3}$$

As an example, consider the simplest case discussed at the beginning of Section 2 for which

$$F^*(z^*) = \frac{u_1 - u_0}{\ln R^*} \log z^* + u_0. \tag{5.4}$$

In this case, the derivative of the complex potential $F(z)$ is given by

$$F'(z) = f'(z)F^{*'}(f(z)) = \frac{(b^2-1)}{(bz-1)^2}\left(\frac{u_1-u_0}{\ln R^*}\frac{bz-1}{z-b}\right) = \frac{u_1-u_0}{\ln R^*}\frac{b^2-1}{(bz-1)(z-b)}. \tag{5.5}$$

By the residue theorem, we obtain that

$$\int_C F'(z)dz = 2\pi i\frac{u_1-u_0}{\ln R^*}, \tag{5.6}$$

which gives that

$$\int_C \lambda\frac{\partial u}{\partial n}ds = 2\pi\frac{u_1-u_0}{\ln R^*}. \tag{5.7}$$

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