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Research Article

Hermite-Hadamard and Simpson-Like Type Inequalities for Differentiable (α, m)-Convex Mappings

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The author establish several Hermite-Hadamard and Simpson-like type inequalities for mappings whose first derivative in absolute value aroused to the qth $(q \geq 1)$ power are (α, m)-convex. Some applications to special means of positive real numbers are also given.

1. Introduction

Recall that, for some fixed $\alpha \in (0,1]$ and $m \in [0,1]$, a mapping $f : \mathbb{I} \subseteq [0,\infty) \to \mathbb{R}$ is said to be (α, m)-convex on an interval $\mathbb{I}$ if the inequality

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y) \tag{1.1}$$

holds for $x, y \in \mathbb{I}$, and $t \in [0,1]$.

Denote by $K_m^{\alpha}(\mathbb{I})$ the set of all (α, m)-convex mappings on $\mathbb{I}$. For recent results and generalizations concerning m-convex and (α, m)-convex mappings, see [1–4].

For the simplicities of notations, for $f \in K_m^{\alpha}(\mathbb{I})$, let us denote

$$(a - mb) S_a^b(f)(\alpha, m, r) = \frac{1}{r}\left\{ f(a) + (r-2) f\left(\frac{a+mb}{2}\right) + f(mb) \right\} - \frac{1}{mb-a} \int_a^{mb} f(x)\,dx. \tag{1.2}$$

In [1, 3], Klaričić Bakula and Özdemir et al., proved the following Hadamard's inequalities for mappings whose second derivative in absolute value aroused to the q-th ($q \geq 1$) power are (α, m)-convex.

Theorem 1.1. *Let $f : \mathbb{I} \subseteq [0, b^*] \to \mathbb{R}$ be a twice differentiable mapping on the interior $\mathbb{I}^0$ of an interval $\mathbb{I}$ such that $f'' \in L_1([a,b])$, where $a, b \in \mathbb{I}$ with $a < b$ and $b^* > 0$. If $|f''|^q$ is (α, m)-convex on $[a,b]$ for $(\alpha, m) \in [0,1]^2$ and $q \geq 1$ with $1/p + 1/q = 1$, then the following inequality holds:*

$$\text{(a)}\ \left|S_a^b(f)(\alpha, m, 2)\right| \leq \frac{mb-a}{2}\left(\frac{1}{6}\right)^{1/p}\left\{\mu\left|f''(a)\right|^q + m\nu\left|f''(b)\right|^q\right\}^{1/q}, \tag{1.3}$$

where

$$\mu = \frac{1}{(\alpha+2)(\alpha+3)}, \qquad \nu = \frac{\alpha^2+5\alpha}{6(\alpha+2)(\alpha+3)}, \tag{1.4}$$

$$\text{(b)}\ \left|S_a^b(f)(\alpha, m, 6)\right| \leq \frac{mb-a}{2}\left(\frac{1}{2}\right)^{1/p}\left\{\mu_0\left|f''(a)\right|^q + m\nu_0\left|f''(b)\right|^q\right\}^{1/q}, \tag{1.5}$$

where

$$\begin{gathered} \mu_0 = \left(\frac{q}{q+\alpha+2}\right)\frac{\Gamma(\alpha+2)\Gamma(q)}{\Gamma(q+\alpha+1)}, \\ \nu_0 = \frac{1}{(q+1)(q+2)} - \left(\frac{q}{q+\alpha+2}\right)\frac{\Gamma(\alpha+2)\Gamma(q)}{\Gamma(q+\alpha+1)}, \end{gathered} \tag{1.6}$$

where

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > 0. \tag{1.7}$$

Theorem 1.2. *Under the same notations in Theorem 2.2, if $|f''|^q$ is (α, m)-convex on $[a,b]$ for $(\alpha, m) \in [0,1]^2$ and $q > 1$ with $1/p + 1/q = 1$, then the following inequality holds:*

$$\left|S_a^b(f)(\alpha, m, 2)\right| \leq \frac{mb-a}{8}\left\{\frac{\Gamma(p+1)}{\Gamma(p+3/2)}\right\}^{1/p}\left\{\left|f''(a)\right|^q\frac{1}{\alpha+1} + m\left|f''(b)\right|^q\left(\frac{\alpha}{\alpha+1}\right)\right\}^{1/q}. \tag{1.8}$$

Note that for $(\alpha, m) \in \{(0,0), (\alpha,0), (1,0), (1,m), (1,1), (\alpha,1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex, and α-convex. For the definitions and elementary properties of these classes, see [4–8].

For recent years, many authors present some new results about Simpson's inequality for (α, m)-convex mappings and have established error estimations for the Simpson's inequality: for refinements, counterparts, generalizations, and new Simpson's type inequalities, see [1–3, 6].

In [9], Dragomir et al. proved the following theorem.

Theorem 1.3. *Let $f : \mathbb{I} \subset [0,\infty) \to \mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ such that $f' \in L_p([a,b])$, where $a,b \in \mathbb{I}$ with $a < b$. Then the following inequality holds:*

$$\left|S_a^b(f)(1,1,6)\right| \leq \frac{(b-a)^{-1/p}}{6}\left\{\frac{2^{q+1}+1}{3(q+1)}\right\}^{1/q}\|f'\|_p. \tag{1.9}$$

The readers can estimate the $error\ (f)$ in the generalized Simpson's formula without going through its higher derivatives which may not exist, not be bounded, or may be hard to find.

In this paper, the author establishes some generalizations of Hermite-Hadamard and Simpson-like type inequalities based on differentiable (α,m)-convex mappings by using the following new identity in Lemma 2.1 and by using these results, obtain some applications to special means of positive real numbers.

2. Generalizations of Simpson-Like Type Inequalities on $K_m^\alpha(\mathbb{I})$

In order to generalize the classical Simpson-like type inequalities and prove them, we need the following lemma [6].

Lemma 2.1. *Let $f : \mathbb{I} \subseteq [0,b^*] \to \mathbb{R}$ be a differentiable mapping on the interior $\mathbb{I}^0$ of an interval $\mathbb{I}$, where $a,b \in \mathbb{I}$ with $0 \leq a < b$ and $b^* > 0$. If $f' \in L^1([a,b])$, then, for $r \geq 2$ and $h \in (0,1)$ with $1/r \leq h \leq (r-1)/r$, the following equality holds:*

$$S_a^b(f)(\alpha,m,r) = \int_0^1 p(r,t)f'(ta+m(1-t)b)dt \tag{2.1}$$

for $f \in K_m^\alpha([a,b])$ and each $t \in [0,1]$, where

$$p(r,t) = \begin{cases} t-\dfrac{1}{r} & t \in \left[0,\dfrac{1}{2}\right], \\ t-\dfrac{r-1}{r} & t \in \left(\dfrac{1}{2},1\right]. \end{cases} \tag{2.2}$$

By the similar way as Theorems 1.1–1.3, we obtain the following theorems.

Theorem 2.2. *Let $f : \mathbb{I} \subset [0,b^*] \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^0$ such that $f' \in L([a,b])$, where $a,b \in \mathbb{I}$ with $0 \leq a < b < \infty$ and $b^* > 0$. If $|f'| \in K_m^\alpha([a,b])$, for some $(\alpha,m) \in (0,1]^2$ and $mb > a$, then, for any $r \geq 2$, the following inequality holds:*

$$\left|S_a^b(f)(\alpha,m,r)\right| \leq \left\{\mu_1^1+\mu_1^2+\mu_1^3+\mu_1^4\right\}|f'(a)| + \left\{\nu_1^1+\nu_1^2+\nu_1^3+\nu_1^4\right\}m|f'(b)|, \tag{2.3}$$

where

$$\begin{aligned}
\mu_1^1 &= \frac{1}{(\alpha+1)(\alpha+2)r^{\alpha+2}},\\
\mu_1^2 &= \frac{1}{(\alpha+1)(\alpha+2)r^{\alpha+2}} + \frac{(\alpha+1)r-2(\alpha+2)}{2^{\alpha+2}(\alpha+1)(\alpha+2)r},\\
\mu_1^3 &= \frac{(r-1)^{\alpha+2}}{(\alpha+1)(\alpha+2)r^{\alpha+2}} + \frac{2(\alpha+2)-3r}{2^{\alpha+2}(\alpha+1)(\alpha+2)r},\\
\mu_1^4 &= \frac{(r-1)^{\alpha+2}}{(\alpha+1)(\alpha+2)r^{\alpha+2}} + \frac{(\alpha+2)-r}{(\alpha+1)(\alpha+2)r},\\
\nu_1^1 &= \frac{1}{2r^2} - \mu_1^1,\\
\nu_1^2 &= \frac{(r-2)^2}{8r^2} - \mu_1^2,\\
\nu_1^3 &= \frac{(r-2)^2}{8r^2} - \mu_1^3,\\
\nu_1^4 &= \frac{1}{2r^2} - \mu_1^4.
\end{aligned} \tag{2.4}$$

Proof. From Lemma 2.1 and using the properties of the modulus, we have the following:

$$\begin{aligned}
\left|S_a^b(f)(\alpha,m,r)\right| \le &\int_0^{1/r}\left(\frac{1}{r}-t\right)\left|f'(ta+m(1-t)b)\right|dt\\
&+\int_{1/r}^{1/2}\left(t-\frac{1}{r}\right)\left|f'(ta+m(1-t)b)\right|dt\\
&+\int_{1/2}^{(r-1)/r}\left(\frac{r-1}{r}-t\right)\left|f'(ta+m(1-t)b)\right|dt\\
&+\int_{(r-1)/r}^{1}\left(t-\frac{r-1}{r}\right)\left|f'(ta+m(1-t)b)\right|dt.
\end{aligned} \tag{2.5}$$

Since $|f'|$ is (α,m)-convex on $[a,b]$, we know that for any $t\in[0,1]$

$$\left|f'(ta+m(1-t)b)\right| \le t^{\alpha}\left|f'(a)\right| + m(1-t^{\alpha})\left|f'(b)\right|. \tag{2.6}$$

By (2.5) and (2.6), we get the following:

$$\begin{aligned}
\left|S_a^b(f)(\alpha,m,r)\right| \le &\int_0^{1/r}\left(\frac{1}{r}-t\right)\left\{t^{\alpha}\left|f'(a)\right| + m(1-t^{\alpha})\left|f'(b)\right|\right\}dt\\
&+\int_{1/r}^{1/2}\left(t-\frac{1}{r}\right)\left\{t^{\alpha}\left|f'(a)\right| + m(1-t^{\alpha})\left|f'(b)\right|\right\}dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_{1/2}^{(r-1)/r}\left(\frac{r-1}{r}-t\right)\{t^{\alpha}|f'(a)|+m(1-t^{\alpha})|f'(b)|\}dt\\
&+\int_{(r-1)/r}^{1}\left(t-\frac{r-1}{r}\right)\{t^{\alpha}|f'(a)|+m(1-t^{\alpha})|f'(b)|\}dt\\
&\leq\left\{\int_0^{1/2}\left|\frac{1}{r}-t\right|t^{\alpha}dt+\int_{1/2}^{1}\left|\frac{r-1}{r}-t\right|t^{\alpha}dt\right\}|f'(a)|\\
&+\left\{\int_0^{1/2}\left|\frac{1}{r}-t\right|(1-t^{\alpha})dt+\int_{1/2}^{1}\left|\frac{r-1}{r}-t\right|(1-t^{\alpha})dt\right\}m|f'(b)|\\
&=\left\{\mu_1^1+\mu_1^2+\mu_1^3+\mu_1^4\right\}|f'(a)|+\left\{\nu_1^1+\nu_1^2+\nu_1^3+\nu_1^4\right\}m|f'(b)|,
\end{aligned}
\tag{2.7}
$$

which completes the proof. □

Corollary 2.3. *In Theorem 2.2,* (i) *if we choose $\alpha = 1$ and $r = 2$, then we have the following:*

$$
\begin{aligned}
\left|(mb-a)S_a^b(f)(1,m,2)\right| &= \left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|\\
&\leq\frac{mb-a}{8}\{|f'(a)|+m|f'(b)|\},
\end{aligned}
\tag{2.8}
$$

and (ii) *if we choose $\alpha = 1$ and $r = 6$, then we have the following*

$$
\begin{aligned}
\left|(mb-a)S_a^b(f)(1,m,6)\right| &= \left|\frac{1}{6}\left\{f(a)+4f\left(\frac{a+mb}{2}\right)+f(mb)\right\}-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|\\
&\leq\frac{5}{72}(mb-a)\{|f'(a)|+m|f'(b)|\}.
\end{aligned}
\tag{2.9}
$$

Theorem 2.4. *Under the same notations in Theorem 2.2, if $|f'|^q \in K_m^{\alpha}([a,b])$, for some $(\alpha,m) \in (0,1]^2$, $mb > a$ and $q > 1$ with $1/p + 1/q = 1$, then, for any $r \geq 2$, the following inequality holds:*

$$
\begin{aligned}
&\left|S_a^b(f)(\alpha,m,r)\right|\\
&\quad\leq\left\{\frac{1}{2r^2}\right\}^{1/p}\left\{\left(\mu_1^1|f'(a)|^q+\nu_1^1m|f'(b)|^q\right)^{1/q}+\left(\mu_1^4|f'(a)|^q+\nu_1^4m|f'(b)|^q\right)^{1/q}\right\}\\
&\qquad+\left\{\frac{1}{8}\left(\frac{r-2}{r}\right)^2\right\}^{1/p}\left\{\left(\mu_1^2|f'(a)|^q+\nu_1^2m|f'(b)|^q\right)^{1/q}+\left(\mu_1^3|f'(a)|^q+\nu_1^3m|f'(b)|^q\right)^{1/q}\right\}.
\end{aligned}
\tag{2.10}
$$

Proof. From Lemma 2.1 and using the properties of modulus, we have the following:

$$\begin{aligned}\left|S_a^b(f)(\alpha,m,r)\right| \le &\int_0^{1/r}\left(\frac{1}{r}-t\right)\left|f'(ta+m(1-t)b)\right|dt\\&+\int_{1/r}^{1/2}\left(t-\frac{1}{r}\right)\left|f'(ta+m(1-t)b)\right|dt\\&+\int_{1/2}^{(r-1)/r}\left(\frac{r-1}{r}-t\right)\left|f'(ta+m(1-t)b)\right|dt\\&+\int_{(r-1)/r}^{1}\left(t-\frac{r-1}{r}\right)\left|f'(ta+m(1-t)b)\right|dt.\end{aligned} \tag{2.11}$$

Using the power-mean integral inequality and (α,m)-convexity of $|f'|^q$ for any $t\in[0,1]$, we have the following

(a)

$$\int_0^{1/r}\left(\frac{1}{r}-t\right)\left|f'(ta+m(1-t)b)\right|^q dt \le \mu_1^1\left|f'(a)\right|^q+\nu_1^1 m\left|f'(b)\right|^q, \tag{2.12}$$

(b)

$$\int_{1/r}^{1/2}\left(t-\frac{1}{r}\right)\left|f(ta+m(1-t)b)\right|^q dt \le \mu_1^2\left|f'(a)\right|^q+\nu_1^2 m\left|f'(b)\right|^q, \tag{2.13}$$

(c)

$$\int_{1/2}^{(r-1)/r}\left(\frac{r-1}{r}-t\right)\left|f(ta+m(1-t)b)\right|^q dt \le \mu_1^3\left|f'(a)\right|^q+\nu_1^3 m\left|f'(b)\right|^q, \tag{2.14}$$

(d)

$$\int_{(r-1)/r}^{1}\left(t-\frac{r-1}{r}\right)\left|f(ta+m(1-t)b)\right|^q dt \le \mu_1^4\left|f'(a)\right|^q+\nu_1^4 m\left|f'(b)\right|^q. \tag{2.15}$$

By the similar way as the above inequalities (a)–(d), we have the following:

(a′)

$$\int_0^{1/r}\left(\frac{1}{r}-t\right)\left|f'(ta+m(1-t)b)\right|dt \le \left\{\frac{1}{2r^2}\right\}^{1/p}\left\{\mu_1^1|f'(a)|^q+\nu_1^1 m\left|f'(b)\right|^q\right\}^{1/q}, \tag{2.16}$$

(b′)

$$\int_{1/r}^{1/2}\left(t-\frac{1}{r}\right)\left|f'(ta+m(1-t)b)\right|dt \le \left\{\frac{1}{8}\left(\frac{r-2}{r}\right)^2\right\}^{1/p}\left\{\mu_1^2\left|f'(a)\right|^q+\nu_1^2 m\left|f'(b)\right|\right\}^{1/q}, \tag{2.17}$$

(c′)

$$\int_{1/2}^{(r-1)/r}\left(\frac{r-1}{r}-t\right)\left|f'(ta+m(1-t)b)\right|dt \le \left\{\frac{1}{8}\left(\frac{r-2}{r}\right)^2\right\}^{1/p}\left\{\mu_1^3\left|f'(a)\right|^q+\nu_1^3 m\left|f'(b)\right|^q\right\}^{1/q}, \tag{2.18}$$

(d′)

$$\int_{(r-1)/r}^{1}\left(t-\frac{r-1}{r}\right)\left|f'(ta+m(1-t)b)\right|dt \le \left\{\frac{1}{2r^2}\right\}^{1/p}\left\{\mu_1^4\left|f'(a)\right|^q+\nu_1^4 m\left|f'(b)\right|^q\right\}^{1/q}. \tag{2.19}$$

By (2.11) and (2.16)–(2.19) the assertion (2.10) holds. □

Corollary 2.5. *In Theorem 2.4,* (i) *if we choose $\alpha = 1$ and $r = 2$, then we have that*

$$\begin{aligned}&\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|\\ &\quad\le \frac{6^{-1/q}}{8}(mb-a)\left[\left\{\left|f'(a)\right|^q+5m\left|f'(b)\right|^q\right\}^{1/q}+\left\{5\left|f'(a)\right|^q+m\left|f'(b)\right|^q\right\}^{1/q}\right],\end{aligned} \tag{2.20}$$

and (ii) *if we choose $\alpha = 1$ and $r = 6$, then we have that*

$$\begin{aligned}&\left|\frac{1}{6}\left\{f(a)+4f\left(\frac{a+mb}{2}\right)+f(mb)\right\}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|\\ &\quad\le (mb-a)\left[\left(\frac{1}{72}\right)\left(\frac{1}{18}\right)^{1/q}\left\{\left(\left|f'(a)\right|^q+17m\left|f'(b)\right|^q\right)^{1/q}\right.\right.\\ &\qquad\left.+\left(17\left|f'(a)\right|^q+m\left|f'(b)\right|^q\right)^{1/q}\right\}+\left(\frac{25}{288}\right)\left(\frac{2}{5}\right)^{2/q}\\ &\qquad\left.\times\left\{\left(7\left|f'(a)\right|^q+11m\left|f'(b)\right|^q\right)^{1/q}+\left(11\left|f'(a)\right|^q+7m\left|f'(b)\right|^q\right)^{1/q}\right\}\right].\end{aligned} \tag{2.21}$$

Theorem 2.6. *Under the same notations in Theorem 2.2, if $|f'|^q \in K_m^\alpha([a,b])$, for some $(\alpha,m) \in (0,1]^2$, $mb > a$ and $q > 1$ with $1/p + 1/q = 1$, then, for any $r \geq 2$, the following inequality holds:*

$$\begin{aligned}
&\left|S_a^b(f)(\alpha,m,r)\right| \\
&\quad \leq \left\{\frac{1}{r^{p+1}}\right\}^{1/p}\left\{\left(\mu_2^1|f'(a)|^q + \nu_2^1 m|f'(b)|^q\right)^{1/q} + \left(\mu_2^4|f'(a)|^q + \nu_2^4 m|f'(b)|^q\right)^{1/q}\right\} \\
&\qquad + \left\{\frac{(r-2)^{p+1}}{2^{p+1}r^{p+1}}\right\}^{1/p}\left\{\left(\mu_2^2|f'(a)|^q + \nu_2^2 m|f'(b)|^q\right)^{1/q} + \left(\mu_2^3|f'(a)|^q + \nu_2^3 m|f'(b)|^q\right)^{1/q}\right\},
\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}
\mu_2^1 &= \frac{1}{r^{\alpha+1}(\alpha+1)}, \\
\mu_2^2 &= \frac{r^{\alpha+1} - 2^{\alpha+1}}{2^{\alpha+1}r^{\alpha+1}(\alpha+1)}, \\
\mu_2^3 &= \frac{2^{\alpha+1}(r-1)^{\alpha+1} - r^{\alpha+1}}{2^{\alpha+1}r^{\alpha+1}(\alpha+1)}, \\
\mu_2^4 &= \frac{r^{\alpha+1} - (r-1)^{\alpha+1}}{r^{\alpha+1}(\alpha+1)}, \\
\nu_2^1 &= \frac{1}{r} - \mu_2^1, \\
\nu_2^2 &= \frac{r-2}{2r} + \mu_2^2, \\
\nu_2^3 &= \frac{r-2}{2r} + \mu_2^3, \\
\nu_2^4 &= \frac{1}{r} - \mu_2^4.
\end{aligned} \tag{2.23}$$

Proof. Suppose that $q > 1$. From Lemma 2.1, using the Hölder integral inequality, we get the following:

$$\begin{aligned}
\left|S_a^b(f)(\alpha,m,r)\right| &\leq \left\{\int_0^{1/r}\left(\frac{1}{r} - t\right)^p dt\right\}^{1/p}\left\{\int_0^{1/r}|f'(tb + m(1-t)a)|^q dt\right\}^{1/q} \\
&\quad + \left\{\int_{1/r}^{1/2}\left(t - \frac{1}{r}\right)^p dt\right\}^{1/p}\left\{\int_{1/r}^{1/2}|f'(tb + m(1-t)a)|^q dt\right\}^{1/q} \\
&\quad + \left\{\int_{1/2}^{(r-1)/r}\left(\frac{r-1}{r} - t\right)^p dt\right\}^{1/p}\left\{\int_{1/2}^{(r-1)/r}|f'(tb + m(1-t)a)|^q dt\right\}^{1/q} \\
&\quad + \left\{\int_{(r-1)/r}^{1}\left(t - \frac{r-1}{r}\right)^p dt\right\}^{1/p}\left\{\int_{(r-1)/r}^{1}|f'(tb + m(1-t)a)|^q dt\right\}^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\{\frac{1}{r^{p+1}}\right\}^{1/p}\left\{\int_0^{1/r}\left|f'(tb+m(1-t)a)\right|^q dt\right\}^{1/q} \\
&\quad+\left\{\frac{1}{2^{p+1}}\left(\frac{r-2}{r}\right)^{p+1}\right\}^{1/p}\left\{\int_{1/r}^{1/2}\left|f'(tb+m(1-t)a)\right|^q dt\right\}^{1/q} \\
&\quad+\left\{\frac{1}{2^{p+1}}\left(\frac{r-2}{r}\right)^{p+1}\right\}^{1/p}\left\{\int_{1/2}^{(r-1)/r}\left|f'(tb+m(1-t)a)\right|^q dt\right\}^{1/q} \\
&\quad+\left\{\frac{1}{r^{p+1}}\right\}^{1/p}\left\{\int_{(r-1)/r}^{1}\left|f'(tb+m(1-t)a)\right|^q dt\right\}^{1/q},
\end{aligned}
\tag{2.24}
$$

where we have used the fact that $1/2 < (1/(p+1))^{1/p} < 1$.

Since $|f'|^q \in K_\alpha^m([a,b])$ for some fixed $\alpha \in (0,1]$ and $m \in [0,1]$, we have the followings:

$$
\begin{aligned}
\int_0^{1/r}\left|f'(tb+(1-t)a)\right|^q dt &\leq \mu_2^1\left|f'(a)\right|^q+\nu_2^1 m\left|f'(b)\right|^q, \\
\int_{1/r}^{1/2}\left|f(tb+(1-t)a)\right|^q dt &\leq \mu_2^2\left|f'(a)\right|^q+\nu_2^2 m\left|f'(b)\right|^q, \\
\int_{1/2}^{(r-1)/r}\left|f'(tb+(1-t)a)\right|^q dt &\leq \mu_2^3\left|f'(a)\right|^q+\nu_2^3 m\left|f'(b)\right|^q, \\
\int_{(r-1)/r}^{1}\left|f(tb+(1-t)a)\right|^q dt &\leq \mu_2^4\left|f'(a)\right|^q+\nu_2^4 m\left|f'(b)\right|^q.
\end{aligned}
\tag{2.25}
$$

Hence, if we combine the inequalities in (2.24)-(2.25), we get the desired result. □

Corollary 2.7. *In Theorem 2.6,* (i) *if we choose $\alpha = 1$ and $r = 2$, then we have that*

$$
\begin{aligned}
&\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \\
&\quad\leq \left(\frac{1}{4}\right)^{1+1/q}(mb-a)\left\{\left(\left|f'(a)\right|^q+3m\left|f'(b)\right|^q\right)^{1/q}+\left(3\left|f'(a)\right|^q+m\left|f'(b)\right|^q\right)^{1/q}\right\},
\end{aligned}
\tag{2.26}
$$

and (ii) *if we choose $\alpha = 1$ and $r = 6$, then we have*

$$
\begin{aligned}
&\left|\frac{1}{6}\left\{f(a)+4f\left(\frac{a+mb}{2}\right)+f(mb)\right\}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \\
&\quad\leq (mb-a)\left[\left(\frac{1}{6}\right)^{1+2/q}\left\{\left(\left|f'(a)\right|^q+11m\left|f'(b)\right|^q\right)^{1/q}+\left(11\left|f'(a)\right|^q+m\left|f'(b)\right|^q\right)^{1/q}\right\}\right. \\
&\qquad\left.+\left(\frac{1}{3}\right)^{2+1/q}\left\{\left(\left|f'(a)\right|^q+2m\left|f'(b)\right|^q\right)^{1/q}+\left(2\left|f'(a)\right|^q+m\left|f'(b)\right|^q\right)^{1/q}\right\}\right],
\end{aligned}
\tag{2.27}
$$

where we have used the fact that $(1/2)^{1/q} < 1$.

3. Applications to Special Means

Now using the results of Section 2, we give some applications to the following special means of positive real numbers $a,\ b \in \mathbb{R}^+$ with $b \geq a$.

(1) The arithmetic mean: $A(a,b) = (a+b)/2$.

(2) The geometric mean: $G(a,b) = \sqrt{ab}$.

(3) The logarithmic mean: $L(a,b) = (b-a)/(\ln b - \ln a)$ for $a \neq b$.

(4) The harmonic mean: $H(a,b) = 2ab/(a+b)$.

(5) The power mean: $M_r(a,b) = ((a^r + b^r)/2)^{1/r}$, $r \geq 1,\ a,\ b \in \mathbb{R}$.

(6) The generalized logarithmic mean:

$$L_n(a,b) = \left[\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}\right]^{1/n}, \quad a \neq b. \tag{3.1}$$

(7) The identric mean:

$$I(a,b) = \begin{cases} a & a = b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)} & a \neq b. \end{cases} \tag{3.2}$$

Proposition 3.1. *For $n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ and $[a,b] \in [0, b^*]$ with $b^* > 0$, we have the following inequalities:*

$$\begin{aligned} &\text{(a) } \left|A(a^n, m^n b^n) - L_n^n(a, mb)\right| \leq \frac{mb-a}{8}|n| M_{n-1}^{n-1}\left(a, m^{1/(n-1)} b\right). \\ &\text{(b) } \left|\frac{1}{3}A(a^n, m^n b^n) + \frac{2}{3}A^n(a, mb) - L_n^n(a, mb)\right| \\ &\quad \leq \left(\frac{5}{72}\right)(mb-a)|n| M_{n-1}^{n-1}\left(a, m^{1/(n-1)} b\right). \end{aligned} \tag{3.3}$$

Proof. The assertions follow from Corollary 2.3 for $f(x) = x^n$. □

Proposition 3.2. *For $[a,b] \in [0, b^*]$, we have the following inequalities:*

$$\begin{aligned} &\text{(a) } \left|H^{-1}(a, mb) - L^{-1}(a, mb)\right| \leq \left(\frac{mb-a}{8a^2b^2}\right) M_2^2\left(m^{1/2} a, b\right), \\ &\text{(b) } \left|\frac{1}{3}H^{-1}(a, mb) + \frac{2}{3}A^{-1}(a, mb) - L^{-1}(a, mb)\right| \leq \left(\frac{5(mb-a)}{72a^2b^2}\right) M_2^2\left(m^{1/2} a, b\right). \end{aligned} \tag{3.4}$$

Proof. The assertions follow from Corollary 2.3 for $f(x) = 1/x$. □

Proposition 3.3. *For $n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ and $[a, b] \in [0, b^*]$, we have the following inequalities:*

(a) $|A(a^n, m^n b^n) - L_n^n(a, mb)|$

$$\leq \left(\frac{3^{-1/q}}{8}\right)(mb-a)|n| \times \left\{A^{1/q}\left(a^{(n-1)q}, 5mb^{(n-1)q}\right)^{1/q} + A^{1/q}\left(5a^{(n-1)q}, mb^{(n-1)q}\right)^{1/q}\right\},$$

(b) $\left|\frac{1}{3}A(a^n, m^n b^n) + \frac{2}{3}A^n(a, mb) - L_n^n(a, mb)\right|$

$$\begin{aligned}&\leq \left(\frac{1}{72}\right)\left(\frac{1}{9}\right)^{1/q}(mb-a)|n| \times \left\{A^{1/q}\left(a^{(n-1)q}, 17mb^{(n-1)q}\right) + A^{1/q}\left(17a^{(n-1)q}, mb^{(n-1)q}\right)\right\}\\ &\quad+ \left(\frac{25}{288}\right)\left(\frac{2}{25}\right)^{1/q}(mb-a)|n|\\ &\quad\times\left\{A^{1/q}\left(7a^{(n-1)q}, 11mb^{(n-1)q}\right) + A^{1/q}\left(11a^{(n-1)q}, 7mb^{(n-1)q}\right)\right\}.\end{aligned} \tag{3.5}$$

Proof. The assertions follow from Corollary 2.5 for $f(x) = x^n$. □

Proposition 3.4. *For $n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ and $[a, b] \in [0, b^*]$, we have the following inequalities:*

(a) $|A(a^n, m^n b^n) - L_n^n(a, b)| \leq \left(\frac{1}{2^{2+1/q}}\right)|n|(mb-a)$

$$\times\left\{A^{1/q}\left(a^{(n-1)q}, 3mb^{(n-1)q}\right) + A^{1/q}\left(3a^{(n-1)q}, mb^{(n-1)q}\right)\right\},$$

(b) $\left|\frac{1}{3}A(a^n, m^n b^n) + \frac{2}{3}A^n(a, mb) - L_n^n(a, mb)\right|$

$$\begin{aligned}&\leq \left(\frac{1}{6}\right)^{1+1/q}\left(\frac{1}{3}\right)^{1/q}|n|(mb-a) \times \left\{A^{1/q}\left(a^{(n-1)q}, 11mb^{(n-1)q}\right) + A^{1/q}\left(11a^{(n-1)q}, mb^{(n-1)q}\right)\right\}\\ &\quad+ \left(\frac{1}{3}\right)^{2+1/q}\left(2^{1/q}\right)|n|(mb-a) \times \left\{A^{1/q}\left(a^{(n-1)q}, 2mb^{(n-1)q}\right) + A^{1/q}\left(2a^{(n-1)q}, mb^{(n-1)q}\right)\right\}.\end{aligned} \tag{3.6}$$

Proof. The assertions follow from Corollary 2.7 for $f(x) = x^n$. □

Proposition 3.5. *For $[a, b] \in [0, b^*]$, we have the following inequalities:*

$$\begin{aligned}&\text{(a)}\ \left|\ln\frac{I(a,b)}{G(a,b)}\right| \leq \frac{b-a}{ab}\left(\frac{1}{4^{1+1/q}}\right)\left\{(a^q+3b^q)^{1/q} + (3a^q+b^q)^{1/q}\right\},\\ &\text{(b)}\ \left|\ln\frac{I(a,b)}{G^{1/3}(a,b)A^{2/3}(a,b)}\right| \leq \frac{b-a}{ab}\left[\left(\frac{1}{6}\right)^{1+2/q}\left\{(a^q+11b^q)^{1/q} + (11a^q+b^q)^{1/q}\right\}\right.\\ &\qquad\left.+\left(\frac{1}{3}\right)^{2+1/q}\left\{(a^q+2b^q)^{1/q} + (2a^q+b^q)^{1/q}\right\}\right].\end{aligned} \tag{3.7}$$

Proof. The assertions follow from Corollary 2.7 for $f(x) = -\ln x$ and $m = 1$. □

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