Research Article

On Solving Systems of Autonomous Ordinary Differential Equations by Reduction to a Variable of an Algebra

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A new technique for solving a certain class of systems of autonomous ordinary differential equations over $\mathbb{K}^n$ is introduced ($\mathbb{K}$ being the real or complex field). The technique is based on two observations: (1), if $\mathbb{K}^n$ has the structure of certain normed, associative, commutative, and with a unit, algebras $\mathbb{A}$ over $\mathbb{K}$, then there is a scheme for reducing the system of differential equations to an autonomous ordinary differential equation on one variable of the algebra; (2) a technique, previously introduced for solving differential equations over $\mathbb{C}$, is shown to work on the class mentioned in the previous paragraph. In particular it is shown that the algebras in question include algebras linearly equivalent to the tensor product of matrix algebras with certain normal forms.

1. Introduction

Throughout this work, $\mathbb{K}$ will stand for a field, usually the real $\mathbb{R}$ or the complex field $\mathbb{C}$.

Consider the autonomous ordinary differential equation

$$\frac{dx}{dt} = f(x), \quad x \in \Omega \subset \mathbb{K}^n, t \in \mathbb{R}, \tag{1.1}$$

with $f : \Omega \subset \mathbb{K}^n \to \mathbb{K}^n$ having certain regularity conditions.

In general the solution is not easy to obtain since this is usually a system of coupled differential equations. There is a vast literature regarding the solution of ordinary differential equations by different means and in particular by techniques utilizing *generalized analytic functions,* see for instance [1–14]. These include applications to the three-dimensional Stokes problem, solutions of planar elliptic vector fields with degeneracies, the Dirichlet problem, multidimensional stationary Schrödinger equation, among others [3, 5, 6, 12, 13]. In particular, the technique that we present is of interest for people working on vector fields with singularities. For instance, in order to gain insight into the behaviour of analytic vector fields, correct visualization of vector fields in the vicinity of their singular set is required, in the case of visualization of two-dimensional complex analytic vector fields with essential singularities the usual methods only provide partial results (see [15–17]), whilst the technique which we promote provides accurate and correct solutions [18, 19]. These questions arise naturally in discrete and continuous dynamical systems (see [20–22]).

As a first step in obtaining a solution to (1.1), we notice that, if $\mathbb{K}^n$ can be given the structure of a certain algebra $\mathbb{A}$, it is possible to reduce this system to a single autonomous differential equation

$$\frac{d\zeta}{dt} = \widetilde{g}(\zeta), \quad \zeta \in \Omega' \subset \mathbb{A}, \, t \in \mathbb{R}, \tag{1.2}$$

with $\widetilde{g}$ being a function $\mathbb{A}$-differentiable with respect to the variable ζ in the algebra $\mathbb{A}$.

Having done so, we proceed to show that it is possible to solve (1.2) by extending a geometric technique introduced in [18, 19] in the context of complex analytic vector fields related to Newton vector fields which are first studied by [23]. This technique is based upon the construction of two functions, which, are respectively, constant and linear on the trajectories which are the solutions of (1.2).

The paper is organized as follows. In Section 2 the algebras in question are introduced, in particular we introduce the notion of normed, associative, commutative, and with a unit, finite dimensional algebra $\mathbb{A}$ over $\mathbb{K}$, showing in Section 2.1 that these have a *first fundamental representation* into the algebra of $n \times n$ matrices over $\mathbb{K}$, $M(n, \mathbb{K})$. Furthermore in Section 2.2 *normal algebras* are defined and their corresponding tensor products are constructed. This is standard material which can be found in [24–26] but is presented here for completeness.

In Section 3 we give the definition of $\mathbb{A}$-differentiability, and proceed to show that if the family of matrices $\{Jf(x) : x \in \Omega\}$ is linearly equivalent to a subset of an algebra $\mathbb{B}$ in $M(n, \mathbb{K})$, that is, the image of the first fundamental representation of an algebra $\mathbb{A}$ with respect to the canonical basis of $\mathbb{K}^n$, that is, $\mathbb{B} = R(\mathbb{A})$, then in fact f is $\mathbb{A}$-differentiable on Ω. The problem of determining if a map $f : \Omega \subset \mathbb{K}^n \to \mathbb{K}^n$ is $\mathbb{A}$-differentiable for some algebra $\mathbb{A}$ is treated indirectly in [27], where the conditions that are given ensure the existence of an algebra $\mathbb{A}$ such that the set of relations are the generalized Cauchy-Riemann equations for $\mathbb{A}$, ([27, 28] show that these equations give a criterion for $\mathbb{A}$-differentiability). Furthermore [29] considers the case when $\mathbb{K} = \mathbb{C}$, where he proves that every analytic map f (in the usual sense) which is $\mathbb{A}$-differentiable has an expansion in power series (see [29, pp. 646 and 653]).

In Section 4 we show that for normal algebras it is possible to express the function $f(x)$ in terms of a single variable ζ in the algebra, and hence there is a function $\widetilde{g}(\zeta)$ that represents $f(x)$. The analogy being the algebra of complex numbers, where $z = x + iy$ is the variable and the $\mathbb{A}$-differentiable functions being the analytic functions. We also show that there is a differentiable operator $\partial/\partial\zeta$ which has the property that $\partial\widetilde{g}/\partial\zeta = \widetilde{g}'$, where $\widetilde{g}'$ is the Lorch derivative of $\widetilde{g}$ (see [26]), hence providing a framework for the usual calculus of one variable.

In Section 5 we start by showing that in this context the differential equation (1.1) takes the form

$$\frac{d\zeta}{dt} = \widetilde{g}(\zeta), \quad \zeta \in \Omega' \setminus \mathcal{S} \subset \mathbb{A}, t \in \mathbb{R}, \tag{1.3}$$

with $\widetilde{g}$ being a function $\mathbb{A}$-differentiable with respect to the variable ζ in the algebra $\mathbb{A}$, and $\mathcal{S}$ being a certain singular set, where the solutions are not defined.

We then proceed to show that the geometric technique, introduced in [18, 19], of finding two functions h_1 and h_2 which are constant and linear on the trajectories $\zeta(t)$ which are solutions of the differential equation, can be extended to the case of (1.3). We end the section and the paper with an example.

2. Algebras

We introduce $\mathbb{K}$-algebras (see, e.g., [24]).

Definition 2.1. A $\mathbb{K}$-*algebra* (or algebra over $\mathbb{K}$) is a finite dimensional $\mathbb{K}$-linear space $\mathbb{A}$ on which is defined a bilinear map $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$ that is associative and commutative, and there is a unit element $e = e_{\mathbb{A}}$ in $\mathbb{A}$ that satisfies $ex = xe = x$ for all $x \in A$.

An element $a \in \mathbb{A}$ is called *regular* if there exist a unique element in $\mathbb{A}$ denoted by $a^{-1} \in \mathbb{A}$ called *inverse* of a such that $a^{-1}a = aa^{-1} = e$. An element $a \in \mathbb{A}$ which is not regular is called *singular*. If $a, b \in \mathbb{A}$ and b is regular, the quotient a/b will mean ab^{-1}.

2.1. Algebras and Their Fundamental Representations

We define the first fundamental representation.

If $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ is an ordered basis of an algebra $\mathbb{A}$, the product between the elements of $\mathcal{B}$ is given by

$$\beta_i \beta_j = \sum_{k=1}^{n} c_{ijk} \beta_k, \tag{2.1}$$

where $c_{ijk} \in \mathbb{K}$ for $i, j, k \in \{1, 2, \ldots, n\}$ are called the *structure constants* of $\mathbb{A}$. The *first fundamental representation* of $\mathbb{A}$ associated to $\mathcal{B}$ is the isomorphism $R : \mathbb{A} \to M(n, \mathbb{K})$ defined by

$$R(x_1 \beta_1 + x_2 \beta_2 + \cdots + x_n \beta_n) = x_1 R_1 + x_2 R_2 + \cdots + x_n R_n, \tag{2.2}$$

where R_i is the matrix whose entry (j, k) is c_{ijk} for $i = 1, 2, \ldots, n$. The commutativity and the associativity of $\mathbb{A}$ are equivalent to the identities

(1) $c_{ijk} = c_{jik}$, for all $i, j, k \in \{1, 2 \ldots, n\}$ and

(2) $\sum_l c_{ijl} c_{lks} = \sum_l c_{jkl} c_{ils}$, for all $i, j, k, s \in \{1, 2 \ldots, n\}$,

respectively.

Using R we assign to $\mathbb{A}$ the norm induced from the operator norm in $M(n, \mathbb{K})$ (see [24]). In this way each algebra is a normed algebra, that is, there exists a norm $\| \cdot \| : \mathbb{A} \to \mathbb{R}$ satisfying $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathbb{A}$ and $\|e\| = 1$.

Example 2.2. Let $\mathbb{A}$ be the linear space $\mathbb{R}^3$ with the product between the elements of the standard basis $\mathcal{B} = \{e_1, e_2, e_3\}$ given in the following equation:

$$
\begin{array}{c|ccc}
 & e_1 & e_2 & e_3 \\
\hline
e_1 & e_1 & e_2 & e_3 \\
e_2 & e_2 & -2e_1 + 2e_2 + e_3 & -e_1 + e_2 + e_3 \\
e_3 & e_3 & -e_1 + e_2 + e_3 & -e_1 + 2e_3
\end{array}
\tag{2.3}
$$

that extends to the product in $\mathbb{R}^3$ given by

$$
\begin{aligned}
(x, y, z)(u, v, w) &= (e_1 x, e_2 y, e_3 z)(e_1 u, e_2 v, e_3 w) \\
&= xu e_1 e_1 + xv e_1 e_2 + xw e_1 e_3 + yu e_1 e_2 + yv e_2 e_2 \\
&\quad + yw e_2 e_3 + zu e_1 e_3 + zv e_2 e_3 + xw e_3 e_3 \\
&= (xu - 2yv - yw - zv - zw)e_1 \\
&\quad + (xv - yu - zyv - yw + zv)e_2 \\
&\quad + (zw + zu + yv + yw + zv + zw)e_3.
\end{aligned}
\tag{2.4}
$$

The product between the elements of $\mathcal{B}$ define the *structure constants* c_{ijk} for $i, j, k \in \{1, 2, 3\}$. So, $\mathbb{R}^3$ with the product given is an algebra $\mathbb{A}$.

2.2. Normal Algebras and Their Tensor Products

Let $B_1, B_2, \ldots, B_l$ be matrices $B_i \in M(k_i, \mathbb{K})$, each B_i of one the following four types:

$$
(r), \quad
\begin{pmatrix}
s & 0 & 0 & \cdots & 0 \\
1 & s & 0 & & 0 \\
0 & 1 & s & & 0 \\
\vdots & & & \ddots & \\
0 & 0 & 0 & \cdots & s
\end{pmatrix},
\quad
\begin{pmatrix}
a & -b \\
b & a
\end{pmatrix},
\quad
\begin{pmatrix}
C & 0 & 0 & \cdots & 0 \\
I_2 & C & 0 & & 0 \\
0 & I_2 & C & & 0 \\
\vdots & & & \ddots & \\
0 & 0 & 0 & \cdots & C
\end{pmatrix},
\tag{2.5}
$$

where

$$
C = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}, \qquad
I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\tag{2.6}
$$

$r, s, a, b, u, v \in \mathbb{K}$ and $k_1 + \cdots + k_l = n$. In this case we will say that the matrix B given by

$$
B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & B_l \end{pmatrix},
\tag{2.7}
$$

is in its *normal form*. We will associate to B an algebra of matrices of dimension n over $\mathbb{K}$ which contains B and use the following nomenclature: The first block will be called *real simple block*, the second *real Jordan block*, the third *simple complex block,* and the fourth *complex Jordan block.*

For $i = 1, 2, \ldots, l$, let $\sigma_i : M(k_i, \mathbb{K}) \rightarrow M(n, \mathbb{K})$ be the linear sections defined by substituting the matrix $M \in M(k_i, \mathbb{K})$ in the block B_i of the matrix B and taking the other entries as zero. The real Jordan block may be written in the following way $B_i = a_i D_i + N_i$, where D_i is the identity and N_i is a nilpotent matrix of order k_i. The simple complex block may be written in the form $B_i = a_i D_i + b_i J_i$, where D_i is a diagonal matrix and J_i is a matrix with $J_i^2 = -D_i$. The complex Jordan blocks may be written in the form $B_i = a_i D_i + b_i J_i + N_i$, where D_i is the identity and J_i is a matrix with $J_i^2 = -D_i$ and N_i is a nilpotent matrix of order $k_i/2$.

We define the matrices $\{\beta_{i,j} : 1 \leq i \leq l, \ 1 \leq j \leq k_i\}$ whose entries are in the block B_i as follows.

(i) If B_i is a real simple block, $\beta_{i,1} := \sigma_i(1)$, in this case $k_i = 1$.

(ii) If B_i is a real Jordan block, $\beta_{i,1} := \sigma_i(D_i)$ and $\beta_{i,j} := \sigma_i(N_i^{j-1})$ for $j = 2, \ldots, k_i$.

(iii) If B_i is a simple complex block, $\beta_{i,1} := \sigma_i(D_i)$ and $\beta_{i,2} := \sigma_i(J_i)$.

(iv) If B_i is a complex Jordan block, $\beta_{i,1} := \sigma_i(D_i)$, $\beta_{i,2} := \sigma_i(J_i)$, $\beta_{i,2j-1} := \sigma_i(N_i^{2j-1})$, and $\beta_{i,2j} := \beta_{i,2}\sigma_i(N_i^{2j-1})$ for $j = 2, \ldots, k_i/2$.

Observe that the product of the matrices β_{i_1,j_1} and β_{i_2,j_2} is the zero matrix if $i_1 \neq i_2$. The products of β_{i,j_1} and β_{i,j_2} for $i = 1, \ldots, l$ are as follows.

(i) If B_i is a real simple block, $\beta_{i,1}\beta_{i,1} = \beta_{i,1}$.

(ii) If B_i is a real Jordan block, then

$$
\begin{array}{c|cc}
& \beta_{i,1} & \beta_{i,j_1} \\
\hline
\beta_{i,1} & \beta_{i,1} & \beta_{i,j_1} \\
\beta_{i,j_2} & \beta_{i,j_1} & \beta_{i,j_1+j_2}
\end{array},
\tag{2.8}
$$

for $2 \leq j_1, j_2 \leq k_i$, where $\beta_{i,j_1} = 0$ when $j_1 \geq k_i + 1$.

(iii) If B_i is a complex simple block, then

$$
\begin{array}{c|cc}
& \beta_{i,1} & \beta_{i,2} \\
\hline
\beta_{i,1} & \beta_{i,1} & \beta_{i,2} \\
\beta_{i,2} & \beta_{i,2} & -\beta_{i,1}
\end{array}
\tag{2.9}
$$

(iv) If B_i is a complex Jordan block, then

$$
\begin{array}{c|cccc}
 & \beta_{i,1} & \beta_{i,2} & \beta_{i,2j_1-1} & \beta_{i,2j_1} \\
\hline
\beta_{i,1} & \beta_{i,1} & \beta_{i,2} & \beta_{i,2j_1-1} & \beta_{i,2j_1} \\
\beta_{i,2} & \beta_{i,2} & -\beta_{i,1} & \beta_{i,2j_1} & -\beta_{i,2j_1-1} \\
\beta_{i,2j_2-1} & \beta_{i,2j_2-1} & \beta_{i,2j_2} & \beta_{i,2(j_1+j_2-1)-1} & \beta_{i,2(j_1+j_2-1)} \\
\beta_{i,2j_2} & \beta_{i,2j_2} & -\beta_{i,2j_2-1} & \beta_{i,2(j_1+j_2-1)} & -\beta_{i,2(j_1+j_2-1)-1}
\end{array}, \tag{2.10}
$$

for $2 \le j_1, j_2 \le k_i/2$, where $\beta_{i,j} = 0$ when $j \ge 2k_i + 1$.

The commutativity of the elements in the set

$$
\mathcal{B} = \{\beta_{1,1}, \ldots, \beta_{1,k_1}, \ldots, \beta_{l,1}, \ldots, \beta_{l,k_l}\}, \tag{2.11}
$$

with respect to the matrix product, follows from the well-known result: For a Jordan canonical form $D + N$ the diagonal matrix D commutes with the nilpotent matrix N.

Moreover, the $\mathbb{K}$-linear space spanned by $\mathcal{B}$ is an $\mathbb{K}$-algebra, as is claimed in the following proposition.

Proposition 2.3. *The set $\mathcal{B} := \{\beta_{1,1}, \ldots, \beta_{1,k_1}, \ldots, \beta_{l,1}, \ldots, \beta_{l,k_l}\}$ is a base for an n-dimensional linear space which is an algebra $\mathbb{A}$ with respect to the matrix product, and its first fundamental representation R with respect to $\mathcal{B}$ is the identity isomorphism.*

Proof. Let R be the first fundamental representation of $\mathbb{A}$ associated to $\mathcal{B}$. As the algebra $\mathbb{A}$ is generated by $\beta_{i,1}$, $\beta_{i,2}$ (considered only when this exists, i.e., $k_i \ge 2$), and $\beta_{i,3}$ (considered only in the case when B_i is a Jordan complex block), in order to prove that R is the identity, we only need to prove that $R(\beta_{i,j}) = \beta_{i,j}$, but for these three cases the equality is trivial. So, R is the identity isomorphism. $\qquad\square$

Definition 2.4. Given a matrix $B \in M(n, \mathbb{K})$ in its normal form, one will call the algebra in $M(n, \mathbb{K})$, as constructed above, an $\mathbb{K}$-*normal algebra (containing B)*.

Definition 2.5. Two matrix algebras $\mathbb{A}_1$ and $\mathbb{A}_2$ in $M(n, \mathbb{K})$ are *linearly equivalent* if there exists an invertible matrix $B \in M(n, \mathbb{K})$ such that $\mathbb{A}_1 = \{BAB^{-1} : A \in \mathbb{A}_2\}$.

For a proof of the following result, see [30].

Proposition 2.6. *Let $\mathbb{A}$ and $\mathbb{B}$ be algebras. There exists a product in $\mathbb{A} \otimes \mathbb{B}$ satisfying*

$$
(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2, \tag{2.12}
$$

where $a_1 a_2$ and $b_1 b_2$ denote the products in $\mathbb{A}$ and $\mathbb{B}$, respectively. The product is associative and $e_{\mathbb{A}} \otimes e_{\mathbb{B}} = e_{\mathbb{A} \otimes \mathbb{B}}$.

Therefore, the finite tensor product of algebras is an algebra.

Definition 2.7. The *complexification* of an $\mathbb{R}$-algebra $\mathbb{A}$ is the $\mathbb{C}$-algebra $\mathbb{A} \otimes \mathbb{C}$. One calls an algebra which is the complexification of an $\mathbb{R}$-normal algebra a $\mathbb{C}$-*normal algebra*.

As usual, if the context is clear, one will drop the $\mathbb{C}$ from the name and refer to the $\mathbb{C}$-normal algebra just as a normal algebra. The following proposition and its corollary follow from a straightforward calculation.

Proposition 2.8. *Let $\mathbb{A}$ and $\mathbb{B}$ be p- and q-dimensional matrix algebras in $M(n, \mathbb{K})$ and $M(m, \mathbb{K})$, respectively, and $P \in M(n, \mathbb{K})$ and $Q \in M(m, \mathbb{K})$ be invertible matrices. Then, one has*

$$P\mathbb{A}P^{-1} \otimes Q\mathbb{B}Q^{-1} = [P \otimes Q][\mathbb{A} \otimes \mathbb{B}][P \otimes Q]^{-1}. \tag{2.13}$$

Corollary 2.9. *The tensor product of matrix algebras linearly equivalent to normal algebras is algebras which are linearly equivalent to the tensor product of normal algebras.*

So, by Corollary 2.9 the algebras linearly equivalent to the tensor product of normal algebras are closed under the tensor product.

The following result shows that the first fundamental representation, with respect to an appropriate base, of a tensor product of normal algebras is the inclusion of the algebra in the corresponding matrix space.

Proposition 2.10. *Let $\mathbb{A}$ and $\mathbb{B}$ be p- and q-dimensional $\mathbb{K}$-algebras, and $R_1 : \mathbb{A} \rightarrow M(p, \mathbb{K})$ and $R_2 : \mathbb{B} \rightarrow M(q, \mathbb{K})$ be first fundamental representations associated to the basis $\mathcal{A} = \{\alpha_1, \ldots, \alpha_p\}$ and $\mathcal{B} = \{\beta_1, \ldots, \beta_q\}$, respectively. Then, $R : \mathbb{A} \otimes \mathbb{B} \rightarrow M(pq, \mathbb{K})$ defined by $R = R_1 \otimes R_2$ is the first fundamental representation of $\mathbb{A} \otimes \mathbb{B}$ associated to the base $\{\alpha_i \otimes \beta_j : 1 \leq j \leq p,\ 1 \leq j \leq q\}$.*

Proof. Let $\{\alpha_1, \ldots, \alpha_p\}$ be a base of $\mathbb{A}$ and let $\{\beta_1, \ldots, \beta_q\}$ be a base of $\mathbb{B}$. We use the notations C_i and G_j for the matrices $R_1(\alpha_i)$ and $R_2(\beta_j)$, respectively, for $1 \leq i \leq p, 1 \leq j \leq q$. The set

$$\{\alpha_i \otimes \beta_j : 1 \leq i \leq p,\ 1 \leq j \leq q\} \tag{2.14}$$

is a base for $\mathbb{A} \otimes \mathbb{B}$ (see [30]). In order to find the structure constants of $\mathbb{A} \otimes \mathbb{B}$ we take the products

$$\begin{aligned}
[\alpha_i \otimes \beta_l][\alpha_j \otimes \beta_s] &= \alpha_i \alpha_j \otimes \beta_l \beta_s \\
&= \left(\sum_{k=1}^{p} c_{ijk} \alpha_k\right) \otimes \left(\sum_{t=1}^{q} d_{lst} \beta_t\right) \\
&= \sum_{k=1}^{p}\sum_{t=1}^{q} c_{ijk} d_{lst}(\alpha_k \otimes \beta_t),
\end{aligned} \tag{2.15}$$

from which we obtain a first fundamental representation R of $A \otimes B$, where $R(\alpha_i \otimes \beta_l)$ is the matrix H_{il} whose entry (kl, js) is given by $h_{il,kt,js} := c_{ijk} d_{lst}$, where c_{ijk} and d_{lst} are the entries (k, j) and (t, s) of C_i and G_l for $1 \leq i \leq p$ and $1 \leq l \leq q$, respectively.

On the other hand we have that $R_1(\alpha_i) \otimes R_2(\beta_l) = C_i \otimes G_l$. The tensor product of C_i and G_l is given by

$$
C_i \otimes G_l =
\begin{pmatrix}
c_{i11}\begin{pmatrix} d_{i11} & d_{i21} & \dots & d_{iq1} \\ & \vdots & & \\ d_{i1q} & d_{i2q} & \dots & d_{iqq} \end{pmatrix} & \cdots & c_{ip1}\begin{pmatrix} d_{i11} & d_{i21} & \dots & d_{iq1} \\ & \vdots & & \\ d_{i1q} & d_{i2q} & \dots & d_{iqq} \end{pmatrix} \\
\vdots & & \\
c_{i1p}\begin{pmatrix} d_{i11} & d_{i21} & \dots & d_{iq1} \\ & \vdots & & \\ d_{i1q} & d_{i2q} & \dots & d_{iqq} \end{pmatrix} & \cdots & c_{ipp}\begin{pmatrix} d_{i11} & d_{i21} & \dots & d_{iq1} \\ & \vdots & & \\ d_{i1q} & d_{i2q} & \dots & d_{iqq} \end{pmatrix}
\end{pmatrix},
\tag{2.16}
$$

from which we see that in the position (kt, js) the element $c_{ijk}d_{lst}$ appears. Thus, we have the equalities of matrices $H_{il} = C_i \otimes G_l$, for $1 \leq i, l \leq n$. Therefore, $R = R_1 \otimes R_2$. $\square$

Corollary 2.11. *If $\mathbb{A}$ is a matrix algebra in $M(n, \mathbb{K})$ which is a tensor product of $\mathbb{K}$-normal algebras, then there exists a base $\mathcal{B}$ of $\mathbb{A}$ in which the corresponding first fundamental representation $R : \mathbb{A} \to M(n, \mathbb{K})$ is the identity isomorphism, that is, $R(x) = x$ for all $x \in \mathbb{A}$.*

Proof. We have that $\mathbb{A} = \mathbb{B}_1 \otimes \cdots \otimes \mathbb{B}_m$, where $\mathbb{B}_i$ is a $\mathbb{K}$-normal algebra for every $i \in \{1, \dots, m\}$. Obviously, for every $i \in \{1, \dots, m\}$ we can consider a base for $\mathbb{B}_i$ as that given in Proposition 2.3, and taking the corresponding tensor products of these basis we obtain a base $\mathcal{B}$ for $\mathbb{A}$. By Propositions 2.3 and 2.10 we have that the first fundamental representation of $\mathbb{A}$ associated to $\mathcal{B}$ is the identity. $\square$

3. Differentiability on Algebras

In a paper published in 1893, Sheffers laid a foundation for a theory of analytic functions on algebras, see [27, 29] and references therein.

Differentiability on algebras is a stronger concept than the usual differentiability over $\mathbb{K}^n$. If $f : U \subset \mathbb{K}^n \to \mathbb{K}^n$ is a differentiable map in the open set U, we denote by $Jf(x)$ its Jacobian matrix at the point x, in the standard base of $\mathbb{K}^n$. We also use the notation $J_{\mathcal{B}}f(x)$ for the Jacobian matrix at the point x of f with respect to a base $\mathcal{B}$.

The following definition was introduced in [26].

Definition 3.1. Let $\mathbb{A}$ be an algebra and let $f : \Omega \subset \mathbb{A} \to \mathbb{A}$ be a map defined in the open set Ω. One says that f is $\mathbb{A}$-*differentiable* at $x_0 \in \mathbb{A}$ if there exists an element $f'(x_0) \in \mathbb{A}$, which one calls the $\mathbb{A}$-*derivative of* f *at* x_0, satisfying

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - f'(x_0)h\|}{\|h\|} = 0, \tag{3.1}$$

where $f'(x_0)h$ denotes the product in $\mathbb{A}$ of $f'(x_0)$ with h. If f is $\mathbb{A}$-differentiable at all the points of Ω, one says that f is $\mathbb{A}$-*differentiable on* Ω and one calls the map f' assigning $f'(x)$ to the point $x \in \Omega$ the $\mathbb{A}$-*derivative of* f, or *Lorch derivative of* f.

It follows (see, e.g., [26–29]) that a map $f : \Omega \subset \mathbb{A} \to \mathbb{A}$ is $\mathbb{A}$-differentiable at x_0 if and only if $J_{\mathcal{B}} f(x_0) \in R(\mathbb{A})$ and is continuous as a function of x_0, where $\mathcal{B} := \{e_1, \ldots, e_n\}$ is a base of $\mathbb{A}$ and $R : \mathbb{A} \to M(n, \mathbb{K})$ is the first fundamental representation of the algebra $\mathbb{A}$ associated to $\mathcal{B}$.

In fact, in this case, $J_{\mathcal{B}} f(x)$ is the image of $f'(x)$ under the map R, that is, if f is $\mathbb{A}$-differentiable, then

$$J_{\mathcal{B}} f(x) = \sum_{i=1}^{n} u_i(x) R_i, \tag{3.2}$$

where $R_i = R(e_i)$ and $u_i : \Omega \to \mathbb{K}$ for $i = 1, 2, \ldots, n$.

If $\Omega' := R(\Omega)$, $\mathbb{B} := R(\mathbb{A})$, and $g : \Omega' \subset \mathbb{B} \to \mathbb{B}$ is defined by

$$g(y) = R \circ f \circ R^{-1}(y), \tag{3.3}$$

then g is $\mathbb{B}$-differentiable and its differential at y is given by $g'(y) = \sum_{i=1}^{n} u_i(R^{-1}(y)) R_i$, thus, the relation between the Jacobian matrix of f and the $\mathbb{B}$-differential of g is $J_{\mathcal{B}} f(x) = g'(R(x))$.

The matrix equation (3.2) is equivalent to the n^2 equations

$$\frac{\partial f_i}{\partial x_j} = \sum_{l=1}^{n} u_l c_{lji}, \tag{3.4}$$

for all $i, j \in \{1, 2, \ldots, n\}$. Using (3.4) and the associativity of the algebra, we can obtain

$$\sum_{i=1}^{n} c_{iks} \frac{\partial f_i}{\partial x_j} = \sum_{i=1}^{n} c_{ijs} \frac{\partial f_i}{\partial x_k}, \tag{3.5}$$

appearing in [29, p. 646].

Remark 3.2. It should be noted that in the context of algebras equations (3.5) play the same role as the Cauchy-Riemann equations in the case of one complex variable and thus serve as a criterion for analyticity, see [27, 28, 31–33].

Suppose we consider a basis $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ of $\mathbb{A}$, where $\alpha_i = \sum_{j=1}^n s_{ji}e_j$ for $i = 1, \ldots, n$, $s_{ji} \in \mathbb{K}$, it can be proved that $J_{\mathcal{A}}f = S^{-1}(Jf)S$, where $S = (s_{ij})$. If we denote by S_i the image of α_i under the first fundamental representation of $\mathbb{A}$ associated to $\mathcal{A}$, we have for $i = 1, \ldots, n$, that $S_i = S^{-1}(\sum_{j=1}^n s_{ji}R_j)S$.

Remark 3.3. For the differentiation of algebras the usual properties of differentiation of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ remain true. Furthermore, the usual rules of differentiation of functions of one variable are satisfied in the case of algebras, therefore polynomial functions, rational functions, and those expressed by means of convergent power series as the exponential, trigonometric, and other usual functions are differentiable in algebras.

The following theorem gives conditions that ensure the existence of an algebra $\mathbb{A}$ in which f is $\mathbb{A}$-differentiable.

Theorem 3.4. *Let $f : \Omega \subset \mathbb{K}^n \to \mathbb{K}^n$ be a C^1 map defined in the open set Ω. f is $\mathbb{A}$-differentiable for an algebra $\mathbb{A}$ if and only if the set of matrices*

$$\{Jf(x) : x \in \Omega\} \tag{3.6}$$

is a subset of an algebra $\mathbb{B}$ which is linearly equivalent to an algebra $\mathbb{T}$ which is a finite tensor product of normal algebras in $M(n, \mathbb{K})$. Moreover, $\mathbb{A}$ is a $\mathbb{K}$-linear space $\mathbb{K}^n$ and has a base $\mathcal{B}$ such that the image of the first fundamental representation of $\mathbb{A}$ associated to $\mathcal{B}$ is $R(\mathbb{A}) = \mathbb{T}$.

Proof. Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ be a base for $\mathbb{T}$ as given in Corollary 2.11. Then $\mathcal{B} = \{B\alpha_1 B^{-1}, \ldots, B\alpha_n B^{-1}\}$ is a base for $\mathbb{B}$, where B is a matrix such that $\mathbb{B} = B\mathbb{T}B^{-1}$. Because $Jf(x) \in \mathbb{B}$ for every $x \in \Omega$, we have $Jf(x) = \sum_{i=1}^n u_i(x)\beta_i$, where $u_i : \Omega \to \mathbb{K}$ are functions and $\beta_i = B\alpha_i B^{-1}$.

Now consider the base $\mathcal{G} = \{\gamma_1 \ldots, \gamma_n\}$ of $\mathbb{K}^n$ defined by $\gamma_i = \sum_{j=1}^n b_{ji}e_j$, where $B = (b_{ij})$ and $\{e_1, \ldots, e_n\}$ is the standard base of $\mathbb{K}^n$. Then, we have that $J_{\mathcal{G}}f = B^{-1}(Jf)B$. Thus,

$$J_{\mathcal{G}}f(x) = B^{-1}\left(\sum_{i=1}^n u_i(x)\beta_i\right)B = B^{-1}\left(\sum_{i=1}^n u_i(x)\left(B\alpha_1 B^{-1}\right)\right)B$$

$$= \sum_{i=1}^n u_i(x)\alpha_i, \tag{3.7}$$

in other words $J_{\mathcal{G}}f(x) \in \mathbb{T}$, which means that if we define a product between the elements of $\mathcal{G}$ using the structure constants of the products of the elements of $\mathcal{A}$, we have that $\mathbb{K}^n$ is an algebra $\mathbb{A}$ such that its first fundamental representation associated to $\mathcal{G}$ is that given by $R(\gamma_i) = \alpha_i$ for $i = 1, \ldots, n$. Therefore f is $\mathbb{A}$-differentiable. $\qquad\square$

4. Reduction to a Variable in the Algebra

Sometimes $f = (f_1, \ldots, f_n) : \Omega \subset \mathbb{K}^n \to \mathbb{K}^n$ can be expressed as a function of a variable $\zeta = \sum_{i=1}^n x_i e_i$ of an algebra $\mathbb{A}$ whose image under the first fundamental representation is the tensor product of normal algebras (as in the previous sections). In this section we show some necessary conditions for this to be true and also allow for the expression of f in terms of ζ.

Proposition 4.1. *With the same hypothesis as Theorem 3.4, there exists a basis $\mathfrak{D} = \{\delta_1, \ldots, \delta_n\}$ of invertible elements of the algebra $\mathbb{A}$ such that the following diagram commutes*

$$
\begin{array}{ccc}
\Omega \subset \mathbb{K}^n & \overset{f}{\longrightarrow} & \mathbb{K}^n \\
A \downarrow & & \downarrow A \\
\Omega' \subset \mathbb{A} & \overset{\widetilde{g}}{\longrightarrow} & \mathbb{A},
\end{array}
\tag{4.1}
$$

where A is the matrix associated to the change of basis $\mathcal{E} \to \mathfrak{D}$, where $\mathcal{E} = \{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{K}^n$.

Proof. By a change of basis one has the following commutative diagram

$$
\begin{array}{ccc}
\Omega \subset \mathbb{K}^n & \overset{f}{\longrightarrow} & \mathbb{K}^n \\
B \downarrow & & \downarrow B \\
\Omega'' \subset \mathbb{A} & \overset{g}{\longrightarrow} & \mathbb{A},
\end{array}
\tag{4.2}
$$

where B is the matrix associated to the change of basis from $\mathcal{E}$ to $\mathcal{B}$, which was used in the previous theorem, and $g = B \circ f \circ B^{-1}$. Now consider a base $\mathfrak{D} = \{\delta_1, \ldots, \delta_n\}$ of regular elements of $\mathbb{A}$, where without loss of generality $\delta_1 = e$. Then if A is the matrix associated to the change of basis from $\mathcal{E}$ to $\mathfrak{D}$, let $\widetilde{g} = A \circ f \circ A^{-1}$. Note that the domain of $\widetilde{g}$ is $\Omega' = A(\Omega)$. $\square$

Theorem 4.2. *Let $f : \Omega \subset \mathbb{K}^n \to \mathbb{K}^n$ be a $\mathbb{A}$-differentiable map over an algebra $\mathbb{A}$ with first fundamental representation $R(\mathbb{A})$, then $\widetilde{g} = A \circ f \circ A^{-1}$ can be expressed in a variable $\zeta = \sum_{j=1}^n y_j \delta_j$, for some regular basis $\mathfrak{D} = \{\delta_1, \ldots, \delta_n\}$. Moreover there is a partial differential operator $\partial/\partial\zeta$ such that $\widetilde{g}'(\zeta) = (\partial\widetilde{g}/\partial\zeta)(\zeta)$.*

Proof. By Proposition 4.1 there is a basis $\mathfrak{D} = \{\delta_1, \ldots, \delta_n\}$ of regular elements of the algebra $\mathbb{A}$, with $\delta_1 = e$, without loss of generality we may assume that $\|\delta_j\| = 1$. Introducing the variable ζ in the base $\mathfrak{D}$ as $\zeta = \sum_{i=1}^n y_i \delta_i = \sum_{i=1}^n x_i e_i$, we then have $f(x_1, \ldots, x_n) = \widetilde{g}(\zeta)$, with $\widetilde{g} = A \circ f \circ A^{-1}$, where A is the matrix associated to the change of basis $\mathcal{E}$ to $\mathfrak{D}$.

Denote by $\overline{\zeta_k}$ the *k-conjugate* of ζ related to $\mathfrak{D}$, which is defined for $k = 2, \ldots, n$ by

$$
\begin{aligned}
\overline{\zeta_k} &= y_1\delta_1 - y_k\delta_k + \sum_{\substack{j=2 \\ j \neq k}}^n y_j\delta_j \\
&= -y_k\delta_k + \sum_{\substack{j=1 \\ j \neq k}}^n y_j\delta_j = \zeta - 2y_k\delta_k.
\end{aligned}
\tag{4.3}
$$

Then we have

$$
y_k e = \frac{\zeta - \overline{\zeta_k}}{2}\delta_k^{-1}, \quad \text{for } k = 2, \ldots, n,
\tag{4.4}
$$

and since

$$y_1 \delta_1 = \zeta - \sum_{j=2}^{n} y_j \delta_j = \zeta - \sum_{j=2}^{n} \frac{\zeta - \overline{\zeta_j}}{2}, \tag{4.5}$$

then,

$$y_1 e = \frac{1}{2} \left((3-n)\zeta + \sum_{j=2}^{n} \overline{\zeta_j} \right) \delta_1^{-1}. \tag{4.6}$$

Furthermore one can define the differential operators

$$\frac{\partial}{\partial \zeta} = \frac{1}{n} \sum_{j=1}^{n} \delta_j^{-1} \frac{\partial}{\partial y_j}, \tag{4.7}$$

$$\frac{\partial}{\partial \overline{\zeta_k}} = \frac{1}{2} \left\{ \delta_1^{-1} \frac{\partial}{\partial y_1} - \delta_k^{-1} \frac{\partial}{\partial y_k} \right\}, \quad \text{for } k = 2, \dots, n, \tag{4.8}$$

which satisfy

$$\frac{\partial \zeta}{\partial \zeta} = 1 \qquad \frac{\partial \overline{\zeta_k}}{\partial \zeta} = \frac{n-2}{n} \quad \text{for } k = 2, \dots, n,$$

$$\frac{\partial \overline{\zeta_k}}{\partial \overline{\zeta_k}} = 1 \qquad \frac{\partial \zeta}{\partial \overline{\zeta_k}} = \frac{\partial \overline{\zeta_j}}{\partial \overline{\zeta_k}} = 0 \quad \text{for } k = 2, \dots, n, \; k \neq j. \tag{4.9}$$

If f is $\mathbb{A}$-differentiable, $\widetilde{g}$ is also $\mathbb{A}$-differentiable, hence

$$\widetilde{g}'(\zeta) = \frac{d\widetilde{g}}{dt}(\zeta + t\delta_k)|_{t=0} = \frac{d\widetilde{g}}{dt}(\zeta + t\delta_j)|_{t=0}, \quad \forall j, k, \tag{4.10}$$

and since

$$\frac{d\widetilde{g}}{dt}(\zeta + t\delta_k)|_{t=0} = \delta_k^{-1} \frac{\partial \widetilde{g}}{\partial y_k}(\zeta), \tag{4.11}$$

then for $k = 2, \dots, n$

$$\frac{\partial \widetilde{g}}{\partial \overline{\zeta_k}}(\zeta) = \frac{1}{2} \sum_{i=1}^{n} \left\{ \delta_1^{-1} \frac{\partial g_i}{\partial y_1}(\zeta) - \delta_k^{-1} \frac{\partial g_i}{\partial y_k}(\zeta) \right\} \delta_i = 0, \tag{4.12}$$

so the right hand expression of $\widetilde{g}(\zeta) = \sum_{i=1}^{n} g_i(\zeta)\delta_i$ does not depend on the conjugated variables. In this way we obtain an expression $\widetilde{g}(\zeta)$ depending only on ζ, and not on $\overline{\zeta_k}$ for $k = 2, \dots, n$.

Moreover, by (4.7), (4.10), and (4.11) one has that

$$\widetilde{g}'(\zeta) = \frac{\partial\widetilde{g}}{\partial y_1}(\zeta) = \frac{\partial\widetilde{g}}{\partial\zeta}(\zeta). \tag{4.13}$$

$\square$

In the following example we show how we can reduce the variables of a map by substituting for variables in an algebra.

Example 4.3. Let $f(x_1, x_2, x_3) = (x_1^2, 2x_1x_2, x_2^2 + 2x_1x_3)$. Then

$$Jf = \begin{pmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 2x_3 & 2x_2 & 2x_1 \end{pmatrix}, \tag{4.14}$$

the Jf is in normal form, hence f is $\mathbb{A}$-differentiable. If

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.15}$$

and I is the identity, the multiplication in the associated algebra $R(\mathbb{A})$ is given by multiplication of matrices I, N, and N^2 representing, respectively, $R(e_1), R(e_2)$, and $R(e_3)$, where $\mathcal{E} = \{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$. It is easy to see that e_2 and e_3 do not have inverse in the algebra $\mathbb{A}$. Consider the basis of regular elements

$$\mathfrak{D} = \{\delta_1 = e_1,\ \delta_2 = e_1 + e_2,\ \delta_3 = e_1 + e_3\}, \tag{4.16}$$

we see that the matrix associated to the change of basis from $\mathcal{E}$ to $\mathfrak{D}$ is

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.17}$$

so f is transformed to $\widetilde{g} = A \circ f \circ A^{-1}$,

$$\widetilde{g}(y_1\delta_1 + y_2\delta_2 + y_3\delta_3) = \left(y_1^2 - 2y_2^2 - 2y_2y_3 - y_3^2\right)\delta_1$$
$$+ 2y_2(y_1 + y_2 + y_3)\delta_2 \tag{4.18}$$
$$+ \left(y_2^2 + 2y_3(y_1 + y_2 + y_3)\right)\delta_3.$$

Thus

$$\zeta = y_1\delta_1 + y_2\delta_2 + y_3\delta_3 \tag{4.19}$$

is the variable in the algebra $\mathbb{A}$. Now consider the conjugates

$$\overline{\zeta_2} = y_1\delta_1 - y_2\delta_2 + y_3\delta_3, \qquad \overline{\zeta_3} = y_1\delta_1 + y_2\delta_2 - y_3\delta_3. \tag{4.20}$$

Then

$$y_1 e = \frac{1}{2}\left(\overline{\zeta_2} + \overline{\zeta_3}\right)\delta_1^{-1},$$

$$y_2 e = \frac{1}{2}\left(\zeta - \overline{\zeta_2}\right)\delta_2^{-1}, \tag{4.21}$$

$$y_3 e = \frac{1}{2}\left(\zeta - \overline{\zeta_3}\right)\delta_3^{-1},$$

so by substituting this in $\widetilde{g}$ and simplifying we obtain

$$\widetilde{g}(\zeta) = \zeta^2, \tag{4.22}$$

the function in the variable of the algebra.

5. Solving Systems of Ordinary Differential Equations by Reduction to One Variable on an Algebra

By following [18, 19], we obtain, as direct corollaries of Theorem 3.4, Proposition 4.1, and Theorem 4.2, the following results.

Corollary 5.1. *Let*

$$f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

$$\widetilde{g} : \Omega' \subset \mathbb{A} \longrightarrow \mathbb{A}, \tag{5.1}$$

be as in Theorem 4.2. Then there exists $\Phi : \Omega' \subset \mathbb{A} \to \mathbb{A}$ which is $\mathbb{A}$-differentiable on $\Omega' \setminus \mathcal{S}$, where $\mathcal{S}$ is a singular set, such that

$$\widetilde{g}(\zeta) = -\frac{\Phi(\zeta)}{\Phi'(\zeta)}, $$

$$f(x) = -\left[J\phi(x)\right]^{-1}\phi(x), \tag{5.2}$$

with $\phi(x) = (\Phi \circ A)(x)$.

Proof. We need to show that there exists $\Phi : \Omega' \subset \mathbb{A} \to \mathbb{A}$ such that

$$\widetilde{g}(\zeta) = -\frac{\Phi(\zeta)}{\Phi'(\zeta)}. \tag{5.3}$$

Noticing that

$$\log(\Phi(\zeta))' = \frac{\Phi'(\zeta)}{\Phi(\zeta)} = -\frac{1}{\widetilde{g}(\zeta)}, \tag{5.4}$$

and by Remark 3.3, it follows that

$$\Phi(\zeta) = \exp\left[-\int^{\zeta} \frac{dz}{\widetilde{g}(z)}\right], \tag{5.5}$$

for $\zeta \in \Omega' \setminus \mathcal{S}$, where

$$\mathcal{S} = \left\{\zeta \in \Omega' : \widetilde{g}(\zeta) \notin \mathbb{A}^*\right\}, \tag{5.6}$$

with $\mathbb{A}^*$ being the regular elements of $\mathbb{A}$. $\qquad\square$

Remark 5.2. Note that $\mathcal{S}$ can be many things. For instance if $\widetilde{g}(\zeta) = c\zeta$ with $c \notin \mathbb{A}^*$, then $\mathcal{S} = \mathbb{A}$. On the other hand, if $\widetilde{g}(\zeta)$ is the polynomial function $g(\zeta) = c_n\zeta^n + \cdots + c_1\zeta + c_0$ with c_n singular, then $\widetilde{g}(\zeta)$ may be regular for all $\zeta \in \mathbb{A}$, (see [26, p. 418]), in which case $\mathcal{S} = \emptyset$.

Remark 5.3. As a special case one notices that if $\mathbb{A} = \mathbb{C}$ then $1/\widetilde{g}$ will be a complex analytic function on $\Omega' \setminus \mathcal{S}$ hence $\mathcal{S}$ consists of isolated points (the isolated singularities of $1/\widetilde{g}$). This has been studied in [18, 19].

Remark 5.4. In case that $\mathcal{S} \neq \Omega'$, then $\Omega' \setminus \mathcal{S}$ is an open dense set in Ω'. This is true since the set $\mathbb{A}^*$ is an open dense set in $\mathbb{A}$ hence for a continuous $\widetilde{g}$ one has that $\widetilde{g}^{-1}(\mathbb{A}^*)$ is an open dense subset of Ω'.

Remark 5.5. By the previous remark, in case that $\mathcal{S} \neq \Omega'$, then $\Phi(\zeta)$ exists for $\zeta \in \Omega' \setminus \mathcal{S}$ even though it might be a multivalued function defined on each component of $\zeta \in \Omega' \setminus \mathcal{S}$.

Remark 5.6. Further characterization and properties of $\mathcal{S}$ will be studied elsewhere. In what follows we assume that $\mathcal{S} \neq \Omega'$.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\}$, then consider the projection onto S^{n-1}

$$\prod(x) = \frac{x}{\|x\|} \in S^{n-1}. \tag{5.7}$$

Corollary 5.7. *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be $\mathbb{A}$-differentiable in an algebra with first fundamental representation $R(\mathbb{A})$. Then the solutions to the differential equation,*

$$\frac{dx}{dt} = f(x), \tag{5.8}$$

correspond to the level curves of the function

$$\widetilde{h_1}(x) = \Pi(\phi(x)), \tag{5.9}$$

moreover the real-valued function,

$$\widetilde{h_2}(x) = \log\|\phi(x)\|, \tag{5.10}$$

is a linear function of t along the solutions $x(t)$ of (5.8).

Proof. By Theorem 3.4, Proposition 4.1, and Theorem 4.2, the solutions $x(t)$ of (5.8) are in correspondence with the solutions $\zeta(t)$ of

$$\frac{d\zeta}{dt} = \widetilde{g}(\zeta). \tag{5.11}$$

On the other hand, Corollary 5.1 shows that

$$\Phi(\zeta(t)) = \Phi(\zeta(t_0)) \exp[-t - t_0], \tag{5.12}$$

the result follows immediately by applying $\Pi(\cdot)$ and $\log\|\cdot\|$ to (5.12). $\qquad\square$

Corollary 5.8. *In particular, to visualize the trajectory that passes through the point $x_0 \in \Omega \setminus A^{-1}(\mathcal{S})$ at time $t_0 \in \mathbb{R}$, one needs only to plot the level curve $\{x \in \Omega : \widetilde{h_1}(x) = \widetilde{h_1}(x_0)\}$. Moreover one can find explicitly the point $x(t_1)$, for $t_1 \in \mathbb{R}$, as the intersection of the level curve $\widetilde{h}_1(x) = \widetilde{h}_1(x_0)$ and the hypersurface $\widetilde{h}_2(x) = \widetilde{h}_2(x_0) - t_1 + t_0$.*

Example 5.9. Consider the function

$$f(x,y) = (u(x,y), v(x,y)), \tag{5.13}$$

where

$$u(x,y) = \frac{1}{6}\left(2 + 6x + \frac{-22 + 20x + 7y}{4 + 5x^2 + 8x(-1+y) + y(-4+5y)} \right.$$
$$\left. + \frac{5(2 + 4x + 5y)}{4 + 5x^2 + 8x(1+y) + y(4+5y)} \right),$$

$$v(x,y) = \frac{1}{6}\left(-4 + 6y - \frac{5(-4 + 5x + 4y)}{4 + 5x^2 + 8x(-1+y) + y(-4+5y)} \right. \tag{5.14}$$
$$\left. + \frac{4 - 7x - 20y}{4 + 5x^2 + 8x(1+y) + y(4+5y)} \right).$$

Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \tag{5.15}$$

and notice that

$$A \cdot Jf\left(A^{-1}(x,y)\right) \cdot A^{-1} = \begin{pmatrix} a(x,y) & b(x,y) \\ -b(x,y) & a(x,y) \end{pmatrix}, \tag{5.16}$$

with

$$a(x,y) = 1 - \frac{2}{x^2 + (-2+y)^2} + \frac{x(-6+4x+3y)}{\left(x^2 + (-2+y)^2\right)^2}$$

$$+ \frac{x(4x - 3(2+y))}{\left(x^2 + (2+y)^2\right)^2} - \frac{2}{x^2 + (2+y)^2},$$

$$b(x,y) = \frac{4y}{\left(x^4 + (-4+y^2)^2 + 2x^2(4+y^2)\right)^2}$$

$$\times \left[48 - 96x - 24x^2 - 16x^3 - 9x^4 + 2x^5 + 2(-3+2x)\left(4+x^2\right)y^2 + (3+2x)y^4\right]. \tag{5.17}$$

Hence $A \cdot Jf(A^{-1}(x,y)) \cdot A^{-1}$ belongs to the normal algebra $\mathbb{C}$, so the previous results are available.

Thus

$$\tilde{g}(x,y) = \left(A \circ f \circ A^{-1}\right)(x,y) = (u_1(x,y), v_1(x,y)), \tag{5.18}$$

with

$$u_1(x,y) = -1 + x + \frac{6 - 4x - 3y}{2\left(x^2 + (-2+y)^2\right)} + \frac{6 - 4x + 3y}{2\left(x^2 + (2+y)^2\right)},$$

$$v_1(x,y) = \frac{x(-12 + 12x + x^3)y + 2(-2+x^2)y^3 + y^5}{x^4 + (-4+y^2)^2 + 2x^2(4+y^2)} \tag{5.19}$$

is just a (complex) analytic function, since

$$J\tilde{g}(x,y) = A \cdot Jf\left(A^{-1}(x,y)\right) \cdot A^{-1} \in R(\mathbb{C}), \tag{5.20}$$

so by letting $z = x + iy$, we have

$$\tilde{g}(z) = u_1\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv_1\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = \frac{2 - z^2 + z^3}{4 + z^2}. \tag{5.21}$$

By Corollary 5.1 one has

$$\Phi(z) = \exp\left[-\int^z \frac{d\zeta}{\tilde{g}(\zeta)}\right], \tag{5.22}$$

so that

$$\Phi(z) = \left(\frac{z-1-i}{z-1+i}\right)^i (z+1)^{-1} = \frac{e^{2\arctan(1-z)}}{z+1}. \tag{5.23}$$

Note that in $\mathbb{R}^2$ one has $\Pi(x,y) = \exp[i\mathrm{Arg}(x,y)]$ hence the level curves of $\Pi(\cdot)$ are in correspondence with the level curves of $\mathrm{Arg}(\cdot)$. So

$$\begin{aligned} h(z) = \mathrm{Arg}(\Phi(z)) = &-\arctan\left(\frac{y}{1+x}\right) \\ &+ \frac{1}{2}\log\left(1 - \frac{4y}{2 + (-2+x)x + y(2+y)}\right) \end{aligned} \tag{5.24}$$

is a *constant of motion* h associated to $\tilde{g}(z)$ and the one associated to $f(x,y)$ is

$$\begin{aligned} \tilde{h}(x,y) = (h \circ A)(x,y) = &-\arctan\left(\frac{2x+y}{1+x+2y}\right) \\ &+ \frac{1}{2}\log\left(1 - \frac{4(2x+y)}{2 + 5x^2 + y(-2+5y) + x(2+8y)}\right). \end{aligned} \tag{5.25}$$

So by Corollary 5.8 the trajectories associated to

$$(x'(t), y'(t)) = f(x(t), y(t)), \tag{5.26}$$

are the level curves of $\tilde{h}(x,y)$. These are presented in Figure 1.

Table 1: Coordinates of the intersection of the level curves $\tilde{h}(x,y) = -1.68195$ and $\tilde{h}_2(x,y) = 2.26841 - t$ for $t = 0, 1, 2, 3, 4$.

Time (t)	$x_t = (x(t), y(t))$
0	$(0.863, -0.783)$
1	$(0.4808, -0.0710)$
2	$(0.2002, 0.2548)$
3	$(0.0678, 0.4654)$
4	$(91.1109, -52.307)$
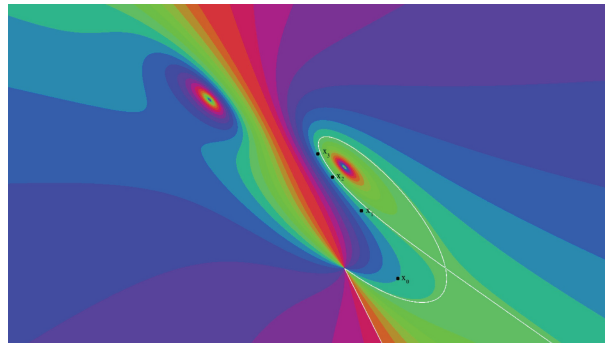
Figure 1: Visualization of the trajectories of (5.26). In this case we have plotted level bands of $\tilde{h}(x,y)$, so that the trajectories are in fact the boundaries between the colored bands. We have also plotted, in white, the trajectory that passes through a singular point corresponding to a pole of the (complex) analytic function $\tilde{g}(z)$. To exemplify the parametrization, we have plotted in black the points corresponding to $t = 0, 1, 2, 3$ (see Table 1).

In order to parametrize the solution we need to calculate $\tilde{h}_2(x,y)$ which turns out to be

$$
\begin{aligned}
\tilde{h}_2(x,y) = {} & -\arctan\left(\frac{-1 + x + 2y}{1 - 2x - y}\right) \\
& + \arctan\left(\frac{1 - x - 2y}{1 + 2x + y}\right) \\
& - \frac{1}{2}\log\left((2x + y)^2 + (1 + x + 2y)^2\right).
\end{aligned}
\tag{5.27}
$$

According to Corollary 5.8 we proceeded to calculate the intersection the level curves $\tilde{h}(x,y) = -1.68195$ and $\tilde{h}_2(x,y) = 2.26841 - t$ for $t = 0, 1, 2, 3, 4$. The results are shown in Table 1 and as black dots in Figure 1.

Acknowledgments

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