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*Research Article*

# A Fixed Point Result for Boyd-Wong Cyclic Contractions in Partial Metric Spaces

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A fixed point theorem involving Boyd-Wong-type cyclic contractions in partial metric spaces is proved. We also provide examples to support the concepts and results presented herein.

## 1. Introduction and Preliminaries

Partial metric spaces were introduced by Matthews [1] to the study of denotational semantics of data networks. In particular, he proved a partial metric version of the Banach contraction principle [2]. Subsequently, many fixed points results in partial metric spaces appeared (see, e.g., [1, 3–19] for more details).

Throughout this paper, the letters $\mathbb{R}$ and $\mathbb{N}^*$ will denote the sets of all real numbers and positive integers, respectively. We recall some basic definitions and fixed point results of partial metric spaces.

*Definition 1.1.* A partial metric on a nonempty set $X$ is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$

(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p2) $p(x, x) \leq p(x, y)$,

(p3) $p(x, y) = p(y, x)$,

(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair $(X, p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$.

If $p$ is a partial metric on $X$, then the function $d_p : X \times X \to [0, \infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.1}$$

is a metric on $X$.

*Example 1.2* (see, e.g., [1, 3, 11, 12]). Consider $X = [0, \infty)$ with $p(x, y) = \max\{x, y\}$. Then, $(X, p)$ is a partial metric space. It is clear that $p$ is not a (usual) metric. Note that in this case $d_p(x, y) = |x - y|$.

*Example 1.3* (see, e.g., [1]). Let $X = \{[a, b] : a, b, \in \mathbb{R}, a \leq b\}$, and define $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then, $(X, p)$ is a partial metric space.

*Example 1.4* (see, e.g., [1, 20]). Let $X := [0, 1] \cup [2, 3]$, and define $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \begin{cases} \max\{x, y\} & \text{if } \{x, y\} \cap [2, 3] \neq \emptyset, \\ |x - y| & \text{if } \{x, y\} \subset [0, 1]. \end{cases} \tag{1.2}$$

Then, $(X, p)$ is a complete partial metric space.

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$, which has as a base the family of open $p$-balls $\{B_p(x, \varepsilon), x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

*Definition 1.5.* Let $(X, p)$ be a partial metric space and $\{x_n\}$ a sequence in $X$. Then,

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to +\infty} p(x, x_n)$,

(ii) $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m \to +\infty} p(x_n, x_m)$ exists and is finite.

*Definition 1.6.* A partial metric space $(X, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$, such that $p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m)$.

**Lemma 1.7** (see, e.g., [3, 11, 12]). *Let $(X, p)$ be a partial metric space. Then,*

(a) *$\{x_n\}$ is a Cauchy sequence in $(X, p)$ if and only if it is a Cauchy sequence in the metric space $(X, d_p)$,*

(b) *$(X, p)$ is complete if and only if the metric space $(X, d_p)$ is complete. Furthermore, $\lim_{n \to +\infty} d_p(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n \to +\infty} p(x_n, x) = \lim_{n,m \to +\infty} p(x_n, x_m). \tag{1.3}$$

**Lemma 1.8** (see, e.g., [3, 11, 12]). *Let $(X, p)$ be a partial metric space. Then,*

(a) *if $p(x, y) = 0$, then $x = y$,*

(b) *if $x \neq y$, then $p(x, y) > 0$.*

*Remark 1.9.* If $x = y$, $p(x, y)$ may not be 0.

**Lemma 1.10** (see, e.g., [3, 11, 12]). *Let $x_n \to z$ as $n \to \infty$ in a partial metric space $(X, p)$ where $p(z, z) = 0$. Then, $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Let $\Phi$ be the set of functions $\phi : [0, \infty) \to [0, \infty)$ such that

(i) $\phi$ is upper semicontinuous (i.e., for any sequence $\{t_n\}$ in $[0, \infty)$ such that $t_n \to t$ as $n \to \infty$, we have $\limsup_{n\to\infty} \phi(t_n) \leq \phi(t)$),

(ii) $\phi(t) < t$ for each $t > 0$.

Recently, Romaguera [21] obtained the following fixed point theorem of Boyd-Wong type [22].

**Theorem 1.11.** *Let $(X, p)$ be a complete partial metric space, and let $T : X \to X$ be a map such that for all $x, y \in X$*

$$p(Tx, Ty) \leq \phi(M(x, y)), \tag{1.4}$$

*where $\phi \in \Phi$ and*

$$M(x, y) = \max\left\{p(x, y),\ p(x, Tx),\ p(y, Ty), \frac{1}{2}[p(x, Ty) + p(y, Tx)]\right\}. \tag{1.5}$$

*Then, $T$ has a unique fixed point.*

In 2003, Kirk et al. [23] introduced the following definition.

*Definition 1.12* (see [23]). Let $X$ be a nonempty set, $m$ a positive integer, and $T : X \to X$ a mapping. $X = \bigcup_{i=1}^{m} A_i$ is said to be a cyclic representation of $X$ with respect to $T$ if

(i) $A_i$, $i = 1, 2, \ldots, m$ are nonempty closed sets,

(ii) $T(A_1) \subset A_2, \ldots, T(A_{m-1}) \subset A_m, T(A_m) \subset A_1$.

Recently, fixed point theorems involving a cyclic representation of $X$ with respect to a self-mapping $T$ have appeared in many papers (see, e.g., [24–28]).

Very recently, Abbas et al. [24] extended Theorem 1.11 to a class of cyclic mappings and proved the following result, but with $\phi \in \Phi$ being a continuous map.

**Theorem 1.13.** *Let $(X, p)$ be a complete partial metric space. Let $m$ be a positive integer, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of $(X, d_p)$, and $Y = \bigcup_{i=1}^{m} A_i$. Let $T : Y \to Y$ be a mapping such that*

(i) *$Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of $Y$ with respect to $T$,*

(ii) *there exists* $\phi : [0,\infty) \to [0,\infty)$ *such that* $\phi$ *is continuous and* $\phi(t) < t$ *for each* $t > 0$, *satisfying*

$$p(Tx,Ty) \leq \phi(M(x,y)), \tag{1.6}$$

*for any* $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,m$, *where* $A_{m+1} = A_1$ *and* $M(x,y)$ *is defined by* (1.5).

*Then,* $T$ *has a unique fixed point* $z \in \bigcap_{i=1}^{m} A_i$.

In the following example, $\phi \in \Phi$, but it is not continuous.

*Example 1.14.* Define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t/2$ for all $t \in [0,1)$ and $\phi(t) = n(n+1)/(n+2)$ for $t \in [n,n+1)$, $n \in \mathbb{N}^*$. Then, $\phi$ is upper semicontinuous on $[0,\infty)$ with $\phi(t) < t$ for all $t > 0$. However, it is not continuous at $t = n$ for all $n \in \mathbb{N}$.

Following Example 1.14, the main aim of this paper is to present the analog of Theorem 1.13 for a weaker hypothesis on $\phi$, that is, with $\phi \in \Phi$. Our proof is simpler than that in [24]. Also, some examples are given.

## 2. Main Results

Our main result is the following.

**Theorem 2.1.** *Let* $(X,p)$ *be a complete partial metric space. Let* $m$ *be a positive integer,* $A_1, A_2, \ldots, A_m$ *nonempty closed subsets of* $(X,d_p)$, *and* $Y = \bigcup_{i=1}^{m} A_i$. *Let* $T : Y \to Y$ *be a mapping such that*

(1) $Y = \bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* $Y$ *with respect to* $T$,

(2) *there exists* $\phi \in \Phi$ *such that*

$$p(Tx,Ty) \leq \phi(M(x,y)), \tag{2.1}$$

*for any* $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,m$, *where* $A_{m+1} = A_1$ *and* $M(x,y)$ *is defined by* (1.5).

*Then,* $T$ *has a unique fixed point* $z \in \bigcap_{i=1}^{m} A_i$.

*Proof.* Let $x_0 \in Y = \bigcup_{i=1}^{m} A_i$. Consider the Picard iteration $\{x_n\}$ given by $Tx_n = x_{n+1}$ for $n = 0,1,2,\ldots$. If there exists $n_0$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0+1} = Tx_{n_0} = x_{n_0}$ and the existence of the fixed point is proved.

Assume that $x_n \neq x_{n+1}$, for each $n \geq 0$. Having in mind that $Y = \bigcup_{i=1}^{m} A_i$, so for each $n \geq 0$, there exists $i_n \in \{1,2,\ldots,m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} = Tx_n \in T(A_{i_n}) \subseteq A_{i_n+1}$. Then, by (2.1)

$$p(x_{n+1},x_{n+2}) = p(Tx_n,Tx_{n+1}) \leq \phi(M(x_n,x_{n+1})), \tag{2.2}$$

where

$$\begin{aligned}
M(x_n, x_{n+1}) &= \max\left\{p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Tx_{n+1}), \right.\\
&\qquad \left.\frac{p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n)}{2}\right\}\\
&= \max\left\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{p(x_n, x_{n+2}) + p(x_{n+1}, x_{n+1})}{2}\right\}\\
&= \max\left\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})}{2}\right\}\\
&= \max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\}.
\end{aligned} \tag{2.3}$$

Therefore,

$$M(x_n, x_{n+1}) = \max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} \quad \forall n \geq 0. \tag{2.4}$$

If for some $k \in \mathbb{N}$, we have $M(x_k, x_{k+1}) = p(x_{k+1}, x_{k+2})$, so by (2.2)

$$0 < p(x_{k+1}, x_{k+2}) \leq \phi(p(x_{k+1}, x_{k+2})) < p(x_{k+1}, x_{k+2}), \tag{2.5}$$

which is a contradiction. It follows that

$$M(x_n, x_{n+1}) = p(x_n, x_{n+1}) \quad \forall n \geq 0. \tag{2.6}$$

Thus, from (2.2), we get that

$$0 < p(x_{n+1}, x_{n+2}) \leq \phi(p(x_{n+1}, x_{n+2})) < p(x_{n+1}, x_{n+2}). \tag{2.7}$$

Hence, $\{p(x_n, x_{n+1})\}$ is a decreasing sequence of positive real numbers. Consequently, there exists $\gamma \geq 0$ such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = \gamma$. Assume that $\gamma > 0$. Letting $n \to \infty$ in the above inequality, we get using the upper semicontinuity of $\phi$

$$0 < \gamma \leq \limsup_{n\to\infty} \phi(p(x_{n+1}, x_{n+2})) \leq \phi(\gamma) < \gamma, \tag{2.8}$$

which is a contradiction, so that $\gamma = 0$, that is,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.9}$$

By (1.1), we have $d_p(x, y) \leq 2p(x, y)$ for all $x, y \in X$, and then from (2.9)

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0. \tag{2.10}$$

Also, by (p2),

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{2.11}$$

In the sequel, we will prove that $\{x_n\}$ is a Cauchy sequence in the partial metric space $(Y = \bigcup_{i=1}^m A_i, p)$. By Lemma 1.7, it suffices to prove that $\{x_n\}$ is Cauchy sequence in the metric space $(Y, d_p)$. We argue by contradiction. Assume that $\{x_n\}$ is not a Cauchy sequence in $(Y, d_p)$. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \geq k$ such that

$$d_p\left(x_{n(k)}, x_{m(k)}\right) \geq \varepsilon. \tag{2.12}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.12). Then,

$$d_p\left(x_{n(k)-1}, x_{m(k)}\right) < \varepsilon. \tag{2.13}$$

We use (2.13) and the triangular inequality

$$\begin{aligned} \varepsilon \leq d_p\left(x_{n(k)}, x_{m(k)}\right) &\leq d_p\left(x_{n(k)}, x_{n(k)-1}\right) + d_p\left(x_{n(k)-1}, x_{m(k)}\right) \\ &< \varepsilon + d_p\left(x_{n(k)}, x_{n(k)-1}\right). \end{aligned} \tag{2.14}$$

Letting $k \to \infty$ in (2.14) and using (2.10), we find

$$\lim_{k\to\infty} d_p\left(x_{n(k)}, x_{m(k)}\right) = \varepsilon. \tag{2.15}$$

On the other hand

$$\begin{aligned} d_p\left(x_{n(k)}, x_{m(k)}\right) &\leq d_p\left(x_{n(k)}, x_{n(k)+1}\right) + d_p\left(x_{n(k)+1}, x_{m(k)+1}\right) + d_p\left(x_{m(k)+1}, x_{m(k)}\right), \\ d_p\left(x_{n(k)+1}, x_{m(k)+1}\right) &\leq d_p\left(x_{n(k)+1}, x_{n(k)}\right) + d_p\left(x_{n(k)}, x_{m(k)}\right) + d_p\left(x_{m(k)}, x_{m(k)+1}\right). \end{aligned} \tag{2.16}$$

Letting $k \to +\infty$ in the two above inequalities and using (2.10) and (2.15),

$$\lim_{k\to\infty} d_p\left(x_{n(k)+1}, x_{m(k)+1}\right) = \varepsilon. \tag{2.17}$$

Similarly, we have

$$\lim_{k\to\infty} d_p\left(x_{n(k)}, x_{m(k)+1}\right) = \lim_{k\to+\infty} d_p\left(x_{m(k)}, x_{n(k)+1}\right) = \varepsilon. \tag{2.18}$$

Also, by (1.1), (2.11), and (2.15)–(2.18), we may find

$$\begin{aligned}\lim_{k\to\infty} p\left(x_{n(k)}, x_{m(k)}\right) = \lim_{k\to\infty} p\left(x_{n(k)}, x_{m(k)+1}\right) = \frac{\epsilon}{2}, \\ \lim_{k\to\infty} p\left(x_{n(k)+1}, x_{m(k)+1}\right) = \lim_{k\to\infty} p\left(x_{m(k)}, x_{n(k)+1}\right) = \frac{\epsilon}{2}.\end{aligned} \tag{2.19}$$

On the other hand, for all $k$, there exists $j(k)$, $0 \le j(k) \le p$, such that $n(k) - m(k) + j(k) \equiv 1(p)$. Then, $x_{m(k)-j(k)}$ (for $k$ large enough, $m(k) > j(k)$) and $x_{n(k)}$ lie in different adjacently labeled sets $A_i$ and $A_{i+1}$ for certain $i = 1, \dots, p$. Using the contractive condition (2.1), we get

$$\begin{aligned}p\left(x_{n(k)+1}, x_{m(k)-j(k)+1}\right) &= p\left(Tx_{n(k)}, Tx_{m(k)-j(k)}\right) \\ &\le \phi\left(M\left(x_{n(k)}, x_{m(k)-j(k)}\right)\right),\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}M\left(x_{n(k)}, x_{m(k)-j(k)}\right) = \max\Bigg\{ & p\left(x_{n(k)}, x_{m(k)-j(k)}\right), p\left(x_{n(k)}, Tx_{n(k)}\right), p\left(x_{m(k)-j(k)}, Tx_{m(k)-j(k)}\right), \\ & \frac{p\left(x_{n(k)}, Tx_{m(k)-j(k)}\right) + p\left(x_{m(k)-j(k)}, Tx_{n(k)}\right)}{2} \Bigg\} \\ = \max\Bigg\{ & p\left(x_{n(k)}, x_{m(k)-j(k)}\right), p\left(x_{n(k)}, x_{n(k)+1}\right), p\left(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}\right), \\ & \frac{p\left(x_{n(k)}, x_{m(k)-j(k)+1}\right) + p\left(x_{m(k)-j(k)}, x_{n(k)+1}\right)}{2} \Bigg\}.\end{aligned} \tag{2.21}$$

As (2.19), using (2.9), we may get

$$\lim_{k\to\infty} p\left(x_{n(k)}, x_{m(k)-j(k)}\right) = \lim_{k\to\infty} p\left(x_{n(k)+1}, x_{m(k)-j(k)+1}\right) = \frac{\epsilon}{2}, \tag{2.22}$$

$$\lim_{k\to\infty} p\left(x_{n(k)}, x_{m(k)-j(k)+1}\right) = \lim_{k\to\infty} p\left(x_{n(k)+1}, x_{m(k)-j(k)}\right) = \frac{\epsilon}{2}. \tag{2.23}$$

By (2.22) and (2.23), we get that

$$\lim_{k\to\infty} M\left(x_{n(k)}, x_{m(k)-j(k)}\right) = \frac{\epsilon}{2}. \tag{2.24}$$

Letting $n \to \infty$ in (2.20), we get using (2.22), (2.24), and the upper semicontinuity of $\phi$

$$0 < \frac{\epsilon}{2} \leq \limsup_{k\to\infty} \phi\left(M\left(x_{n(k)}, x_{m(k)-j(k)}\right)\right) \leq \phi\left(\frac{\epsilon}{2}\right) < \frac{\epsilon}{2}, \tag{2.25}$$

which is a contradiction.

This shows that $\{x_n\}$ is a Cauchy sequence in the complete subspace $Y = \bigcup_{i=1}^{m} A_i$ equipped with the metric $d_p$. Thus, there exists $u = \lim_{n\to\infty} x_n \in (Y, d_p)$. Notice that the sequence $\{x_n\}_{n\in\mathbb{N}}$ has an infinite number of terms in each $A_i$, $i = 1, \ldots, m$, so since $(Y, d_p)$ is complete, from each $A_i$, $i = 1, \ldots, m$ one can extract a subsequence of $\{x_n\}$ that converges to $u$. Because $A_i$, $i = 1, \ldots, m$ are closed in $(Y, d_p)$, it follows that

$$u \in \bigcap_{i=1}^{m} A_i. \tag{2.26}$$

Thus, $\bigcap_{i=1}^{m} A_i \neq \emptyset$.

For simplicity, set $A = \bigcap_{i=1}^{m} A_i$. Clearly, $A$ is also closed in $(Y, d_p)$, so it is a complete subspace of $(Y, d_p)$ and then $(A, p)$ is a complete partial metric space. Consider the restriction of $T$ on $A$, that is, $T_{/A} : A \to A$. Then, $T_{/A}$ satisfies the assumptions of Theorem 1.11, and thus $T_{/A}$ has a unique fixed point in $Z$. □

## 3. Examples

We give some examples illustrating our results.

*Example 3.1.* Let $X = \mathbb{R}$ and $p(x, y) = \max\{|x|, |y|\}$. It is obvious that $(X, p)$ is a complete partial metric space.

Set $A_1 = [-8, 0]$, $A_2 = [0, 8]$, and $Y = A_1 \cup A_2 = [-8, 8]$. Define $T : T \to Y$ by

$$Tx = \begin{cases} \dfrac{-x}{4} & \text{if } \ x \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Notice that $T([-8, -1)) = 0$ and $T([-1, 0]) = [0, 1/4]$, and hence $T(A_1) \subseteq A_2$. Analogously, $T((1, 8]) = 0$ and $T([0, 1]) = [-1/4, 0]$, and hence $T(A_2) \subseteq A_1$.

Take

$$\phi(t) = \begin{cases} \dfrac{t}{3} & \text{if } t \in [0, 1), \\ \dfrac{n^2}{n^2 + 1} & \text{if } t \in [n, n+1), \ n \in \mathbb{N}^*. \end{cases} \tag{3.2}$$

Clearly, $T$ satisfies condition (2.1). Indeed, we have the following cases.

*Case 1.* ($x \in [-8,-1)$ and $y \in (1,8]$). Inequality (2.1) turns into

$$p(Tx,Ty) = \max\{0,0\} = 0 \leq \phi(M(x,y)), \tag{3.3}$$

which is necessarily true.

*Case 2.* ($x \in [-8,-1)$ and $y \in [0,1]$). Inequality (2.1) becomes

$$\begin{aligned} p(Tx,Ty) &= \max\left\{0, \frac{|y|}{4}\right\} = \frac{y}{4} \leq \phi(M(x,y)) \\ &= \phi\left(\max\left\{p(x,y), p(x,Tx), p(Ty,y), \frac{1}{2}[p(x,Ty) + p(Tx,y)]\right\}\right) \\ &= \phi\left(\max\left\{|x|, |x|, |y|, \frac{1}{2}[|x| + |y|]\right\}\right) \\ &= \phi(|x|). \end{aligned} \tag{3.4}$$

It is clear that $1/2 \leq \phi(t) < 1$ for all $t > 1$. Hence, (3.4) holds.

*Case 3.* ($x \in [-1,0]$ and $y \in (1,8]$). Inequality (2.1) turns into

$$\begin{aligned} p(Tx,Ty) &= \max\left\{\frac{|x|}{4}, 0\right\} = \frac{|x|}{4} \leq \phi(M(x,y)) \\ &= \phi\left(\max\left\{p(x,y), p(x,Tx), p(Ty,y), \frac{1}{2}[p(x,Ty) + p(Tx,y)]\right\}\right) \\ &= \phi\left(\max\left\{|y|, |x|, |y|, \frac{1}{2}[|x| + |y|]\right\}\right) \\ &= \phi(|y|) = \phi(y), \end{aligned} \tag{3.5}$$

which is true again by the fact that $1/2 \leq \phi(t) < 1$ for all $t > 1$.

*Case 4.* ($x \in [-1,0]$ and $y \in [0,1]$). Inequality (2.1) becomes

$$\begin{aligned} p(Tx,Ty) &= \max\left\{\frac{|x|}{4}, \frac{|y|}{4}\right\} \leq \phi(M(x,y)) \\ &= \phi\left(\max\left\{p(x,y), p(x,Tx), p(Ty,y), \frac{1}{2}[p(x,Ty) + p(Tx,y)]\right\}\right) \\ &= \phi\left(\max\left\{\max\{|x|, |y|\}, |x|, |y|, \frac{1}{2}\left[\max\left\{\frac{|x|}{4}, |y|\right\} + \max\left\{|x|, \frac{|y|}{4}\right\}\right]\right\}\right). \end{aligned} \tag{3.6}$$

Let use examine all possibilities:

$$p(Tx,Ty) = \begin{cases} \dfrac{|x|}{4} & \text{if } |x| \geq |y|, \\ \dfrac{|y|}{4} & \text{if } \dfrac{|y|}{4} \leq |x| \leq |y|, \\ \dfrac{|y|}{4} & \text{if } |x| \leq \dfrac{|y|}{4}, \end{cases}$$
$$M(x,y) \leq \begin{cases} |x| & \text{if } |x| \geq |y|, \\ |y| & \text{if } \dfrac{|y|}{4} \leq |x| \leq |y|, \\ |y| & \text{if } |x| \leq \dfrac{|y|}{4}. \end{cases} \tag{3.7}$$

Thus, (2.1) holds for $\phi(t) = t/3$.

The rest of the assumptions of Theorem 2.1 are also satisfied. The function $T$ has 0 as a unique fixed point.

However, since $\phi$ is not a continuous function, we could not apply Theorem 1.13.

*Example 3.2.* Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$ for all $x,y \in X$. Then, $(X,p)$ is a complete partial metric space. Take $A_1 = \cdots = A_p = X$. Define $T : X \to X$ by $Tx = x/2$. Consider $\phi : [0,\infty) \to [0,\infty)$ given by Example 1.14.

For all $x,y \in X$, we have

$$p(Tx,Ty) = \max\left\{\frac{x}{2}, \frac{y}{2}\right\} = \phi(p(x,y) \leq \phi(M(x,y)). \tag{3.8}$$

Then all the assumptions of Theorem 2.1 are satisfied. The function $T$ has 0 as a unique fixed point.

Similarly, Theorem 1.13 is not applicable.

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