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*Research Article*

# The Numerical Class of a Surface on a Toric Manifold

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In this paper, we give a method to describe the numerical class of a torus invariant surface on a projective toric manifold. As applications, we can classify toric 2-Fano manifolds of the Picard number 2 or of dimension at most 4.

## 1. Introduction

The classification of smooth toric Fano $d$-folds is an important and interesting problem. They are classified for $d = 3$ by [1, 2], for $d = 4$ by [3, 4], and for $d = 5$ by [5]. In Øbro's recent excellent paper [6], an algorithm which classifies all the smooth toric Fano $d$-folds for any given natural number $d$ was constructed. So, we can say that the classification of smooth toric Fano varieties is completed.

On the other hand, de Jong and Starr defined a special class of Fano manifolds called 2-*Fano manifolds* in [7] (see Definition 4.2). So, we consider the problem of the classification of toric 2-Fano manifolds as a next step. For this classification, we give a method to describe the numerical class of a 2-cycle on projective toric manifolds (see Section 3). This method makes calculations of intersection numbers much easier. As results, we obtain the classification of toric 2-Fano manifolds for the case of the Picard number $\rho(X) = 2$ and for the case of $\dim(X) \leq 4$. We remark that Nobili classified smooth toric 2-Fano 4-folds in [8] by using a Maple program.

The contents of this paper are as follows. In Section 2, we define the basic notation such as nef 2-cocycle and 2-*Mori cone* for our theory. In Section 3, we define a polynomial $I_{Y/X}$ for a torus invariant subvariety $Y \subset X$. This polynomial has all the information of intersection numbers of $Y$ on $X$. So, we can consider this polynomial as the numerical class of $Y$. For a some special surface $S$, $I_{S/X}$ has a good property to calculate intersection numbers (see Theorems 3.4 and 3.5). As applications, we classify toric 2-Fano manifolds under some assumptions in Section 4.

*Notation.* We will work over an algebraically closed field $k$ throughout this paper. We denote a projective toric $d$-fold by $X = X_\Sigma$, where $\Sigma$ is the associated fan in $N := \mathbb{Z}^d$. $G(\Sigma) \subset N$ is the set of the primitive generators for the 1-dimensional cones in $\Sigma$.

## 2. Preliminaries

In this section, we explain the notation and some basic facts of the toric geometry and the birational geometry used in this paper. See [9–11] for the details.

Let $X$ be a smooth projective toric $d$-fold. Put $Z^2(X)$ to be the free $\mathbb{Z}$-module of 2-cocycles on $X$ and $Z_2(X)$ the free $\mathbb{Z}$-module of 2-cycles on $X$. We define the *numerical equivalence* "$\equiv$" on $Z^2(X)$ and $Z_2(X)$. A 2-cocycle $E \in Z^2(X)$ is *numerically equivalent to* 0; that is, $E \equiv 0$ if the intersection number $(E \cdot S) = 0$ for any 2-cycle $S \in Z_2(X)$, while a 2-cycle $S \in Z_2(X)$ is *numerically equivalent* to 0; that is, $S \equiv 0$ if the intersection number $(E \cdot S) = 0$ for any 2-cocycle $E \in Z^2(X)$. We define $N^2(X) := (Z^2(X)/\equiv) \otimes \mathbb{R}$ and $N_2(X) := (Z_2(X)/\equiv) \otimes \mathbb{R}$.

The following definitions are similar to the case of divisors and curves.

*Definition 2.1.* A 2-cocycle $E \in Z^2(X)$ is a nef 2-cocycle if $(E \cdot S) \geq 0$ for any effective 2-cycle $S \in Z_2(X)$.

*Definition 2.2.* For a projective toric manifold $X$, let $NE_2(X) \subset N_2(X)$ be the cone of effective 2-cycles; namely,

$$NE_2(X) := \left\{ \left[ \sum_i a_i S_i \right] \in N_2(X) \mid a_i \geq 0 \right\}. \tag{2.1}$$

One calls $NE_2(X) \subset N_2(X)$ the 2-*Mori cone* of $X$.

We should remark that $N^l(X)$, $N_l(X)$, and $NE_l(X)$ can be defined for any $1 \leq l \leq d$ similarly.

The following is an immediate consequence of the *projectivity* of $X$.

**Proposition 2.3.** *$NE_2(X)$ is a strongly convex cone.*

*Proof.* Let $D$ be an ample divisor on $X$. Then, for any $S \in NE_2(X) \setminus \{0\}$, we have $(D^2 \cdot S) > 0$; namely, $NE_2(X)$ is strongly convex. □

On the other hand, for the toric case, the following is obvious.

**Proposition 2.4.** *Let $X$ be a smooth projective toric $d$-fold. Then, $NE_2(X)$ is a polyhedral cone.*

Thus, $NE_2(X)$ is a strongly convex polyhedral rational cone similarly as $NE(X)$.

We end this section by giving the following simple examples.

*Example 2.5.* (1) If $X = \mathbb{P}^d$, then

$$\mathrm{NE}_2(X) = \mathbb{R}_{\geq 0}[S], \tag{2.2}$$

where $S$ is a plane in $X$.

(2) If $X = \mathbb{P}^1 \times \mathbb{P}^3$, then

$$\mathrm{NE}_2(X) = \mathbb{R}_{\geq 0}\left[(\text{a point}) \times \mathbb{P}^2\right] + \mathbb{R}_{\geq 0}\left[\mathbb{P}^1 \times \mathbb{P}^1\right]. \tag{2.3}$$

(3) If $X = \mathbb{P}^2 \times \mathbb{P}^2$, then

$$\mathrm{NE}_2(X) = \mathbb{R}_{\geq 0}\left[(\text{a point}) \times \mathbb{P}^2\right] + \mathbb{R}_{\geq 0}\left[\mathbb{P}^1 \times \mathbb{P}^1\right] + \mathbb{R}_{\geq 0}\left[\mathbb{P}^2 \times (\text{a point})\right]. \tag{2.4}$$

## 3. Combinatorial Descriptions

In this section, we establish a method to describe the numerical class of a torus invariant subvariety. We assume that $X = X_\Sigma$ is a *smooth* projective toric variety.

Let $Y = Y_\sigma \subset X$ be a torus invariant subvariety of $\dim Y = l$ associated to a cone $\sigma \in \Sigma$ and $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_m\}$. Put

$$\begin{aligned} I_{Y/X} = I_{Y/X}(X_1, \ldots, X_m) &:= \sum_{1 \leq i_1, \ldots, i_l \leq m} \left(D_{x_{i_1}} \cdots D_{x_{i_l}} \cdot Y\right) X_{i_1} \cdots X_{i_l} \\ &\in \mathbb{Z}[X_1, \ldots, X_m], \end{aligned} \tag{3.1}$$

where $D_{x_i}$ is the torus invariant prime divisor corresponding to $x_i$, while $X_i$ is defined to be the independent variable corresponding to $x_i$. We will use this notation throughout this paper.

*Remark 3.1.* $I_{Y/X}$ has all the informations of intersection numbers of $Y$ on $X$. So, we can consider $I_{Y/X}$ as the numerical class of $Y \in N_l(X)$.

*Example 3.2.* Let $C = C_\tau \subset X$ be a torus invariant curve, where $\tau$ is a $(d-1)$-dimensional cone, that is, a wall in $\Sigma$. In this case,

$$I_{C/X} = \sum_i (D_i \cdot C) X_i \tag{3.2}$$

is a polynomial of degree 1. On the other hand,

$$\sum_i (D_i \cdot C) x_i = 0 \tag{3.3}$$

is the so-called *Reid's wall relation* associated to the wall $\tau$ (see [12]); namely, $I_{C/X}$ is calculated from the wall relation immediately.

*Example 3.3.* When $Y = X$, $I_{X/X}$ sometimes becomes a simple shape as follows.

(1) *Projective spaces.* Let $X$ be the $d$-dimensional projective space $\mathbb{P}^d$ and $G(\Sigma) = \{x_1 := e_1, \ldots, x_d := e_d,\ x_{d+1} := -(e_1 + \cdots + e_d)\}$. Then,

$$I_{X/X} = (X_1 + \cdots + X_{d+1})^d. \tag{3.4}$$

(2) *Hirzebruch surfaces*. Let $X$ be the Hirzebruch surface $F_\alpha$ of degree $\alpha$ and $G(\Sigma) = \{x_1 := e_1,\ x_2 := e_2,\ x_3 := -e_1 + \alpha e_2,\ x_4 = -e_2\}$. Then,

$$I_{X/X} = \alpha(X_2 + X_4)^2 + 2(X_2 + X_4)(X_1 + X_3 - \alpha X_2). \tag{3.5}$$

Let $X$ be a smooth projective toric variety and $S \subset X$ a torus invariant *surface*. For some special cases, $I_{S/X}$ is simply calculated as follows. These are the main theorems of this paper.

**Theorem 3.4.** *Suppose $S \cong \mathbb{P}^2$. Let $C \subset S$ be a torus invariant curve. Then, $I_{S/X} = (I_{C/X})^2$.*

*Proof.* Let $\tau = \mathbb{R}_{\geq 0}\, x_1 + \cdots + \mathbb{R}_{\geq 0}\, x_{d-2} \in \Sigma$ be the $(d-2)$-dimensional cone associated to $S = S_\tau$, where $\tau \cap G(\Sigma) = \{x_1, \ldots, x_{d-2}\}$. Then, there exist exactly three maximal cones $\tau + \mathbb{R}_{\geq 0}\, y_1$, $\tau + \mathbb{R}_{\geq 0}\, y_2$, and $\tau + \mathbb{R}_{\geq 0}\, y_3 \in \Sigma$ which contain $\tau$. Put

$$y_1 + y_2 + y_3 + a_1 x_1 + \cdots + a_{d-2} x_{d-2} = 0 \tag{3.6}$$

to be the wall relation corresponding to $C$. For the proof, it is sufficient to show that

$$D_z D_w S = a_z a_w, \tag{3.7}$$

for any $z, w \in G(\Sigma)$, where $D_z$ is the prime torus invariant divisor corresponding to $z$, while $a_z$ is the coefficient of $z$ in the above wall relation.

Suppose that $z$ or $w \notin \{x_1, \ldots, x_{d-2}, y_1, y_2, y_3\}$; namely, $a_z = 0$ or $a_w = 0$. In this case, trivially, $D_z S = 0$ or $D_w S = 0$. So, $D_z D_w S = a_z a_w = 0$.

For any $1 \leq i,\ j \leq 3$,

$$D_{y_i} D_{y_j} S = \left(D_{y_i}|_S\right)\left(D_{y_j}|_S\right) = C^2 = 1. \tag{3.8}$$

So, the remaining case is $z$ or $w \in \{x_1, \ldots, x_{d-2}\}$. By calculating the rational functions associated to a $\mathbb{Z}$-basis $\{x_1, \ldots, x_{d-2}, y_1, y_2\}$ for $N$, we have the relations

$$\begin{gathered} D_{x_1} - a_1 D_{y_3} + E_1 = 0, \ldots, D_{x_{d-2}} - a_{d-2} D_{y_3} + E_{d-2} = 0, \\ D_{y_1} - D_{y_3} + E_{d-1} = 0, \qquad D_{y_1} - D_{y_3} + E_d = 0 \end{gathered} \tag{3.9}$$

in Pic$X$, where $E_1, \ldots, E_d$ are torus invariant divisors such that Supp $E_i \cap S = \emptyset$ for any $1 \le i \le d$. Therefore, we have

$$D_{x_1}S = a_1 D_{y_3}S, \ldots, D_{x_{d-2}}S = a_{d-2}D_{y_3}S. \tag{3.10}$$

By these relations, the equality $D_z D_w S = a_z a_w$ is obvious. □

**Theorem 3.5.** *Suppose $S \cong F_\alpha$, that is, a Hirzebruch surface of degree $\alpha$. Let $C_{\text{fib}} \subset S$ be a fiber of the projection $S = F_\alpha \to \mathbb{P}^1$, while let $C_{\text{neg}}$ be the negative section of $S$. Then, $I_{S/X} = \alpha(I_{C_{\text{fib}}/X})^2 + 2I_{C_{\text{fib}}/X}I_{C_{\text{neg}}/X}$.*

*Proof.* Let $\tau = \mathbb{R}_{\ge 0}\, x_1 + \cdots + \mathbb{R}_{\ge 0}\, x_{d-2} \in \Sigma$ be the $(d-2)$-dimensional cone associated to $S = S_\tau$, where $\tau \cap \mathrm{G}(\Sigma) = \{x_1, \ldots, x_{d-2}\}$. Then, there exist exactly four maximal cones $\tau + \mathbb{R}_{\ge 0}\, y_1,\ \tau + \mathbb{R}_{\ge 0}\, y_2,\ \tau + \mathbb{R}_{\ge 0}\, y_3$, and $\tau + \mathbb{R}_{\ge 0}\, y_4 \in \Sigma$ which contain $\tau$. Put

$$y_1 + y_3 - \alpha y_2 + a_1 x_1 + \cdots + a_{d-2}x_{d-2} = 0 \tag{3.11}$$

to be the wall relation corresponding to $C_{\text{neg}}$, while

$$y_2 + y_4 + b_1 x_1 + \cdots + b_{d-2}x_{d-2} = 0 \tag{3.12}$$

to be the wall relation corresponding to $C_{\text{fib}}$. As in the proof of Theorem 3.4, by calculating the rational functions associated to a $\mathbb{Z}$-basis $\{x_1, \ldots, x_{d-2}, y_1, y_2\}$ for $N$, we have the relations

$$\begin{gathered} D_{x_1}S = a_1 D_{y_3}S + b_1 D_{y_4}S, \ldots, D_{x_{d-2}}S = a_{d-2}D_{y_3}S + b_{d-2}D_{y_4}S, \\ D_{y_1}S = D_{y_3}S, \qquad D_{y_2} = -\alpha D_{y_3}S + D_{y_4}S. \end{gathered} \tag{3.13}$$

First, we remark that, for any $1 \le i,\ j \le 4$,

$$D_{y_i}D_{y_j}S = (D_{y_i}|_S)\left(D_{y_j}|_S\right) \tag{3.14}$$

on $S$. So, these intersection numbers can be recovered from $I_{S/S}$ (see Example 3.3).

The above relations say that, for any $1 \le i,\ j \le d-2$,

$$D_{x_i}D_{x_j}S = \alpha b_i b_j + a_i b_j + a_j b_i, \tag{3.15}$$

while for any $1 \le i \le d-2$,

$$D_{y_1}D_{x_i} = b_i, \qquad D_{y_2}D_{x_i} = a_i, \qquad D_{y_3}D_{x_i} = b_i, \qquad D_{y_4}D_{x_i} = a_i + \alpha b_i. \tag{3.16}$$

On the other hand, put $f_1 = f_1(X_1, \dots, X_{d-2}) := a_1 X_1 + \cdots + a_{d-2} X_{d-2}$ and $f_2 = f_2(X_1, \dots, X_{d-2}) := b_1 X_1 + \cdots + b_{d-2} X_{d-2}$. Then,

$$\begin{aligned}\alpha(I_{C_{\rm fib}/X})^2 + 2I_{C_{\rm fib}/X} I_{C_{\rm neg}/X} &= \alpha(Y_2 + Y_4 + f_1)^2 + 2(Y_2 + Y_4 + f_1)(Y_1 + Y_3 - \alpha Y_2 + f_2) \\ &= I_{S/S}(Y_1, Y_2, Y_3, Y_4) + \alpha f_2^2 + 2 f_1 f_2 \\ &\quad + 2Y_1 f_2 + 2Y_2 f_1 + 2Y_3 f_2 + Y_4(2f_1 + 2\alpha f_2).\end{aligned} \tag{3.17}$$

This coincides with $I_{S/X}$ by the above calculations. □

## 4. 2-Fano Manifolds

As an application of Section 3, we study on *toric* 2-*Fano manifolds* in this section. The notion of 2-Fano manifolds was introduced in [7].

*Definition 4.1.* A smooth projecive algebraic variety $X$ is a *Fano manifold* if its first Chern class$c_1(X) = -K_X$ is an ample divisor.

*Definition 4.2* (see [7]). A Fano manifold $X$ is a 2-*Fano manifold* if its second Chern character $\mathrm{ch}_2(X) = (1/2)(c_1(X)^2 - 2c_2(X))$ is a nef 2-cocycle.

*Remark 4.3.* Since a 2-Fano manifold is a Fano manifold by the definition, for the classification of toric 2-Fano manifolds, all we have to do is to check the list of toric Fano manifolds. The classification of toric Fano manifolds can be done by the algorithm of Øbro [6] for any dimension.

For a projective toric manifold $X$, one can easily see that $\mathrm{ch}_2(X) = (1/2)\sum_{i=1}^{m} D_i^2$, where $D_1, \dots, D_m$ are the torus invariant prime divisors. So, the following is immediate.

**Proposition 4.4.** *For a torus invariant surface $S \subset X$, put $I_{S/X} := \sum_{i,j} a_{ij} X_i X_j$. Then, $(ch_2(X) \cdot S) = (1/2)\sum_{i=1}^{m} a_{ii}$.*

First of all, we classify toric 2-Fano manifolds of Picard number 2. So, let $X$ be a complete toric manifold of $\rho(X) = 2$. In this case, the structure of X is very simple as follows.

**Theorem 4.5** (see [13]). *Every complete toric manifold of the Picard number 2 is a projective space bundle over a projective space.*

By Theorem 4.5, we can put

$$X = X_\Sigma = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O} \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{m-1})), \tag{4.1}$$

where $a_1 \geq \cdots \geq a_{m-1} \geq 0$, $m + n - 2 = d := \dim X$. Let

$$x_1 + \cdots + x_m = 0, \tag{4.2}$$

$$y_1 + \cdots + y_n = a_1 x_1 + \cdots + a_{m-1} x_{m-1} \tag{4.3}$$

be the wall relations of $\Sigma$ which correspond to the extremal rays of $\mathrm{NE}(X)$, where

$$G(\Sigma) = \{x_1, \dots, x_m, y_1, \dots, y_n\}. \tag{4.4}$$

Let $C_1$ and $C_2$ be the extremal torus invariant curves corresponding to the wall relations (4.2) and (4.3), respectively.

First, we determine the extremal rays of $\mathrm{NE}_2(X)$. By calculating the rational functions for a $\mathbb{Z}$-basis $\{x_1, \dots, x_{m-1}, y_1, \dots, y_{n-1}\}$, we have the relations

$$\begin{gathered} D_1 - D_m + a_1 E_n = 0, \dots, D_{m-1} - D_m + a_{m-1} E_n = 0, \\ E_1 - E_n = 0, \dots, E_{n-1} - E_n = 0 \end{gathered} \tag{4.5}$$

in $\mathrm{N}^1(X)$, where $D_1, \dots, D_m,\ E_1, \dots, E_n$ are torus invariant prime divisors corresponding to $x_1, \dots, x_m,\ y_1, \dots, y_n$. Therefore, for $1 \le i,\ j \le m-1$,

$$\begin{gathered} D_j = D_i + (a_i - a_j) E_n, \\ E_1 = E_2 = \cdots = E_n. \end{gathered} \tag{4.6}$$

On the other hand, every $(d-2)$-dimensional cone $\tau \in \Sigma$ is expressed as

$$\tau = \mathbb{R}_{\ge 0}\, x_{i_1} + \cdots + \mathbb{R}_{\ge 0}\, x_{i_k} + \mathbb{R}_{\ge 0}\, y_{j_1} + \cdots + \mathbb{R}_{\ge 0}\, y_{j_l}, \tag{4.7}$$

for some $1 \le i_1 < \cdots < i_k \le m$, $1 \le j_1 < \cdots < j_l \le n$ such that $k < m,\ l < n$, and $k + l = d - 2$. So, the corresponding torus invariant surface $S_\tau$ is expressed as

$$S_\tau = D_{i_1} \cdots D_{i_k} E_{j_1} \cdots E_{j_l} \in \mathrm{N}_2(X). \tag{4.8}$$

By using (4.6), any $S_\tau$ is expressed as a linear combination of 2-cycles:

$$D_1 \cdots D_p E^q \quad (p \le m-1,\ q \le n-1,\ p + q = d - 2), \tag{4.9}$$

whose coefficients are nonnegative, because $i < j$ implies $a_i - a_j \ge 0$. Moreover, since $D_1 \cdots D_m = E_1 \cdots E_n = 0$ by wall relations (4.2) and (4.3), the possibilities for the generators of $\mathrm{NE}_2(X)$ are

$$\begin{gathered} S_1 := D_1 \cdots D_{m-3} E^{n-1}, \qquad S_2 := D_1 \cdots D_{m-2} E^{n-2}, \quad \text{or} \\ S_3 := D_1 \cdots D_{m-1} E^{n-3}. \end{gathered} \tag{4.10}$$

In fact, the following hold:

$$\begin{gathered} \mathrm{NE}_2(X) = \mathbb{R}_{\ge 0}\, S_1 + \mathbb{R}_{\ge 0}\, S_2 + \mathbb{R}_{\ge 0}\, S_3 \quad \text{if } m \ge 3,\ n \ge 3. \\ \mathrm{NE}_2(X) = \mathbb{R}_{\ge 0}\, S_2 + \mathbb{R}_{\ge 0}\, S_3 \quad \text{if } m = 2,\ n \ge 3. \\ \mathrm{NE}_2(X) = \mathbb{R}_{\ge 0}\, S_1 + \mathbb{R}_{\ge 0}\, S_2 \quad \text{if } m \ge 3,\ n = 2. \end{gathered} \tag{4.11}$$

For each case, $\dim \mathrm{N}_2(X) = 3$, $\dim \mathrm{N}_2(X) = 2$, and $\dim \mathrm{N}_2(X) = 2$, respectively. So, $\mathrm{NE}_2(X)$ is a *simplicial* cone for each case, and $S_1$, $S_2$, and $S_3$ are extremal surfaces.

Next, we will check when $X$ becomes a 2-Fano manifold.

So, let $C_2$ be the torus invariant curve which generates the extremal ray corresponding to the wall relation (4.3). Then,

$$(-K_X \cdot C_2) = n - (a_1 + \cdots + a_{m-1}). \tag{4.12}$$

Therefore, $X$ is a Fano manifold if and only if

$$n - (a_1 + \cdots + a_{m-1}) > 0. \tag{4.13}$$

Since $S_1 \cong S_3 \cong \mathbb{P}^2$, $(\mathrm{ch}_2(X) \cdot S_1) \geq 0$ and $(\mathrm{ch}_2(X) \cdot S_3) \geq 0$ are trivial by Theorem 3.4. On the other hand, we can easily check that $S_2 \cong F_{a_{m-1}}$. By Theorem 3.5, we have

$$\begin{aligned} I_{S_2} = a_{m-1}(I_{C_1})^2 + 2I_{C_1}I_{C_2} = a_{m-1}(X_1 + \cdots + X_m)^2 \\ + 2(X_1 + \cdots + X_m)(Y_1 + \cdots + Y_n - (a_1X_1 + \cdots + a_{m-1}X_{m-1})). \end{aligned} \tag{4.14}$$

So, we obtain

$$(\mathrm{ch}_2(X) \cdot S_2) = ma_{m-1} - 2(a_1 + \cdots + a_{m-1}). \tag{4.15}$$

In (4.15), suppose that $m \geq 3$ and $(\mathrm{ch}_2(X) \cdot S_2) \geq 0$. Then,

$$(\mathrm{ch}_2(X) \cdot S_2) = (m-2)a_{m-1} - 2(a_1 + \cdots + a_{m-2}). \tag{4.16}$$

The assumption $a_1 \geq \cdots \geq a_{m-1} \geq 0$ says that $a_1 = \cdots = a_{m-1} = 0$; that is, $X \cong \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$. On the other hand, suppose that $m = 2$ in (4.15). Then, $(\mathrm{ch}_2(X) \cdot S_2) = 0$; that is, $\mathrm{ch}_2(X)$ is nef.

By (4.13), we can summarize as follows.

**Theorem 4.6.** *If X is a toric 2-Fano manifold of the Picard number 2, then X is one of the following*:

(1) *a direct product of projective spaces,*

(2) $\mathbb{P}_{\mathbb{P}^{d-1}}(\mathcal{O} \oplus \mathcal{O}(a))$ $(1 \leq a \leq d-1)$.

*Remark 4.7.* This calculation shows that there exist infinitely many projective toric manifolds of fixed dimension d whose second Chern character is nef.

Next, we consider the classification of toric 2-Fano manifolds of a fixed dimension $d$. For $d \leq 4$, fortunately, these classifications can be done by only Theorems 3.4 and 3.5. Table 1 is the classification list (see [8] for the detail).

Since there exist 124 smooth toric Fano 4-folds, it is hard to check all the smooth toric Fano 4-folds. However, by using the following trivial Lemma 4.8, we can omit a large part of the calculations.

**Table** 1

| $d$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| # of toric Fano | 1 | 5 | 18 | 124 |
| # of toric 2-Fano | 1 | 3 | 8 | 25 |

**Lemma 4.8.** *Let X be a 4-dimensional toric 2-Fano manifold. Then,*

$$c_1^4(X) - 2c_1^2(X)c_2(X) \geq 0. \tag{4.17}$$

For any smooth toric Fano 4-fold $X$, $c_1^4(X)$ and $c_1^2(X)c_2(X)$ are calculated in [3]. One can see that for 52 smooth toric Fano 4-folds, they are not 2-Fano manifolds by Lemma 4.8.

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