Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 496731, 13 pages
doi:10.1155/2012/496731

Research Article

On a Subclass of Harmonic Convex Functions of Complex Order

N. Magesh[1] and S. Mayilvaganan[2]

[1] *Department of Mathematics, Government Arts College, Krishnagiri 635 001, India*
[2] *Department of Mathematics, Adhiyamaan College of Engineering (Autonomous), Hosur 635 109, India*

Correspondence should be addressed to N. Magesh, nmagi_2000@yahoo.co.in

Received 21 December 2011; Accepted 30 March 2012

Academic Editor: Aloys Krieg

We introduce and study a subclass of harmonic convex functions of complex order. Coefficient bounds, extreme points, distortion bounds, convolution conditions, and convex combination are determined for functions in this class. Further, we obtain the closure property of this class under integral operator.

1. Introduction

A continuous function $f = u+iv$ is a complex-valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply connected domain $D \subset \Omega$, we can write $f = h+\overline{g}$, where h and g are analytic in D. We call h the analytic part and g the coanalytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [1]).

Denote by $\mathcal{S}_{\mathcal{H}}$ the family of functions $f = h + \overline{g}$, which are harmonic, univalent, and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, the functions h and g analytic in $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad (|b_1| < 1), \tag{1.1}$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \quad (|b_1| < 1). \tag{1.2}$$

We note that the family $\mathcal{S}_{\mathcal{H}}$ of orientation preserving, normalized harmonic univalent functions reduces to the well-known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of f is identically zero ($g \equiv 0$).

Also, we denote by $T\mathcal{S}_{\mathcal{H}}$ the subfamily of $\mathcal{S}_{\mathcal{H}}$ consisting of harmonic functions of the form $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \qquad g(z) = \sum_{k=1}^{\infty} |b_k| z^k. \tag{1.3}$$

In [1], Clunie and Sheil-Small investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as its geometric subclasses and its properties. Since then, there have been several studies related to the class $\mathcal{S}_{\mathcal{H}}$ and its subclasses. Following Clunie and Sheil-Small [1], Frasin [2], Jahangiri et al. [3–6], Silverman [7], Silverman and Silvia [8], Yalçin and Öztürk [9], and others have investigated various subclasses of $\mathcal{S}_{\mathcal{H}}$ and its properties. In particular, Avcı and Złotkiewicz [10] proved that the coefficient condition

$$\sum_{k=2}^{\infty} k^2 (|a_k| + |b_k|) \leq 1 \quad (b_1 = 0) \tag{1.4}$$

is sufficient for functions $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ to be harmonic convex. Also, Silverman [7] studied that this coefficient condition is also necessary if a_k and b_k $(k = 2, 3, \ldots)$ in (1.2) are negative. Further, Jahangiri [3] showed that if $f = h + \overline{g}$ is given by (1.2) and if

$$\sum_{k=1}^{\infty} \left(\frac{k(k-\gamma)}{1-\gamma} |a_k| + \frac{k(k+\gamma)}{1-\gamma} |b_k| \right) \leq 2 \quad (0 \leq \gamma < 1,\ a_1 = 1), \tag{1.5}$$

then, f is harmonic, univalent, and convex of order γ in $\mathcal{U}$. This condition is proved to be also necessary if h and g are of the form (1.3). Furthermore, Yalçin and Öztürk [11] have considered a class $T\mathcal{S}^*_{\mathcal{H}}(\gamma)$ of harmonic starlike functions of complex order based on a corresponding study of Nasr and Aouf [12] for the analytic case.

Motivated by the earlier works given in the literature [9, 11] now we define the class of harmonic convex functions of complex order as follows.

Definition 1.1. For $0 \le \gamma < 1$, $0 \le \lambda \le \gamma/(1+\gamma)$ or $\lambda \ge 1/(1+\gamma)$, and $b \in \mathbb{C} \setminus \{0\}$ with $|b| < 1$, let $\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$ denote the family of harmonic functions $f \in \mathcal{S}_{\mathcal{H}}$ of the form (1.2) that satisfy the following condition:

$$\Re\left\{1 + \frac{1}{b}\left(\frac{\lambda\left(z^3h'''(z) - \overline{z^3g'''(z)}\right) + (2\lambda+1)z^2h''(z)}{\lambda\left(z^2h''(z) + \overline{z^2g''(z)}\right) + zh'(z) + (2\lambda-1)\overline{zg'(z)}} + \frac{(1-4\lambda)\overline{z^2g''(z)} + zh'(z) + (1-2\lambda)\overline{zg'(z)}}{\lambda\left(z^2h''(z) + \overline{z^2g''(z)}\right) + zh'(z) + (2\lambda-1)\overline{zg'(z)}} - 1\right)\right\} \ge \gamma, \tag{1.6}$$

for $z \in \mathcal{U}$. Further, we define the subclass $T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$ of $\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$ consisting of functions $f = h + \overline{g}$ of the form (1.3).

We observe that for $b = 1$ the class $\mathcal{SC}_{\mathcal{H}}(1,\gamma,\lambda) = \mathcal{SC}_{\mathcal{H}}(\gamma,\lambda)$ was introduced and studied by Yalçin and Öztürk [9], the class $\mathcal{SC}_{\mathcal{H}}(1,\gamma,0) = \mathcal{SC}_{\mathcal{H}}(\gamma)$ is given in [3, 4], and the class $\mathcal{SC}_{\mathcal{H}}(1,0,0) = \mathcal{SC}_{\mathcal{H}}$ is studied in [10].

In this paper, we investigate coefficient conditions, extreme points, and distortion bounds for the function class $T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$. We also examine their convolution and convex combination properties and the closure property of this class under integral operator. We remark that the results obtained for these general families can be viewed as extensions and generalizations for various subclasses of $\mathcal{S}_{\mathcal{H}}$ as listed previously in this section.

2. Coefficient Inequalities

Our first theorem gives a sufficient condition for functions in $\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be so that h and g are given by* (1.2)*. If*

$$\sum_{k=1}^{\infty}\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_k| \le 2, \tag{2.1}$$

where $a_1 = 1$, $0 \le \gamma < 1$, and b ($|b| < 1$) is a nonzero complex number, $0 \le \lambda \le \gamma/(1+\gamma)$ or $\lambda \ge 1/(1+\gamma)$. Then, $f \in \mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$ and f is sense preserving, univalent, and harmonic in $\mathcal{U}$.

Proof. We show that $f \in \mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$. We only need to show that if (2.1) holds, then condition (1.6) is satisfied. In view of (1.2), condition (1.6) takes the form

$$\begin{aligned}
&\Re\left(\frac{(1-\gamma)+\sum_{k=2}^{\infty}\left(\left(k(k\lambda-\lambda+1)\left[(k-1)+b(1-\gamma)\right]\right)/b\right)|a_k|\left(z^k/z\right)}{1+\sum_{k=2}^{\infty}k(k\lambda-\lambda+1)|a_k|\left(z^k/z\right)+\sum_{k=1}^{\infty}k(k\lambda+\lambda-1)|b_k|\left(\overline{z}^k/z\right)}\right.\\
&\qquad\left.-\frac{\sum_{k=1}^{\infty}\left(\left(k(k\lambda+\lambda-1)\left[(k+1)-b(1-\gamma)\right]\right)/b\right)|b_k|\left(\overline{z}^k/z\right)}{1+\sum_{k=2}^{\infty}k(k\lambda-\lambda+1)|a_k|\left(z^k/z\right)+\sum_{k=1}^{\infty}k(k\lambda+\lambda-1)|b_k|\left(\overline{z}^k/z\right)}\right) \qquad (2.2)\\
&=\Re\frac{1+A(z)}{1+B(z)}.
\end{aligned}$$

Setting

$$\frac{1+A(z)}{1+B(z)}=\frac{1+\omega(z)}{1-\omega(z)}, \qquad (2.3)$$

we will have $\Re((1+A(z))/(1+B(z)))>0$ if $|\omega(z)|<1$,

$$\begin{aligned}
\omega(z)&=\frac{A(z)-B(z)}{2+A(z)+B(z)}\\
&=\frac{-\gamma+\sum_{k=2}^{\infty}k(k\lambda-\lambda+1)\left[\frac{\left[(k-1)+b(1-\gamma)\right]}{b}-1\right]|a_k|z^{k-1} - \sum_{k=1}^{\infty}k(k\lambda+\lambda-1)\left[\left(\left[(k+1)-b(1-\gamma)\right]/b\right)+1\right]|b_k|\left(\overline{z}^k/z\right)}{2-\gamma+\sum_{k=2}^{\infty}k(k\lambda-\lambda+1)\left[\left(\left[(k-1)+b(1-\gamma)\right]/b\right)+1\right]|a_k|z^{k-1} - \sum_{k=1}^{\infty}k(k\lambda+\lambda-1)\left[\frac{\left[(k+1)-b(1-\gamma)\right]}{b}-1\right]|b_k|\frac{\overline{z}^k}{z}},
\end{aligned} \qquad (2.4)$$

so that

$$|\omega(z)|<\frac{\gamma+\sum_{k=2}^{\infty}k(k\lambda-\lambda+1)[(\mathfrak{A}/|b|)-1]|a_k|+\sum_{k=1}^{\infty}k(k\lambda+\lambda-1)[(\mathfrak{B}/|b|)+1]|b_k|}{2-\gamma-\sum_{k=2}^{\infty}k(k\lambda-\lambda+1)[(\mathfrak{A}/|b|)+1]|a_k|-\sum_{k=1}^{\infty}k(k\lambda+\lambda-1)[(\mathfrak{B}/|b|)-1]|b_k|}, \qquad (2.5)$$

where $\mathfrak{A}$ denotes $[(k-1)+|b|(1-\gamma)]$ and $\mathfrak{B}$ denotes $[(k+1)-|b|(1-\gamma)]$.This last expression is bounded above by 1 if and only if

$$\sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{|b|}|a_k|+\sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{|b|}|b_k|\leq(1-\gamma). \qquad (2.6)$$

Or, equivalently

$$\sum_{k=1}^{\infty}\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}|a_k|+\sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_k|\leq 2. \tag{2.7}$$

In order to show that f is univalent in $\mathcal{U}$, we show that $f(z_1)\neq f(z_2)$ whenever $z_1\neq z_2$. Since $\mathcal{U}$ is simply connected and convex, we have $z(t)=(1-t)z_1+tz_2\in\mathcal{U}$, where $0\leq t\leq 1$, and if $z_1,z_2\in\mathcal{U}$ so that $z_1\neq z_2$. Then, we write

$$f(z_2)-f(z_1)=\int_0^1\left[(z_2-z_1)h'(z(t))+\overline{(z_2-z_1)g'(z(t))}\right]dt. \tag{2.8}$$

Dividing the above equation by $z_2-z_1\neq 0$ and taking the real part, we obtain

$$\begin{aligned}\Re\left\{\frac{f(z_2)-f(z_1)}{z_2-z_1}\right\}&=\int_0^1\Re\left[h'(z(t))+\frac{\overline{z_2-z_1}}{z_2-z_1}\overline{g'(z(t))}\right]dt\\&>\int_0^1\left[\Re h'(z(t))-\left|g'(z(t))\right|\right]dt.\end{aligned} \tag{2.9}$$

On the other hand, for $|b|<1$, $\lambda\geq 1/(1+\gamma)$ or $0\leq\lambda\leq\gamma/(1+\gamma)$, we have

$$\begin{aligned}\Re h'(z)-\left|g'(z)\right|&\geq\Re h'(z)-\sum_{k=1}^{\infty}k|b_k|\\&\geq 1-\sum_{k=2}^{\infty}k|a_k|-\sum_{k=1}^{\infty}k|b_k|\\&\geq 1-\sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}|a_k|\\&\quad-\sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_k|\\&\geq 0,\ \text{using (2.1)}.\end{aligned} \tag{2.10}$$

This along with inequality (2.9) leads to the univalence of f. Note that f is sense preserving in $\mathcal{U}$, for $0 \le \lambda \le \gamma/(1+\gamma)$ or $\lambda \ge 1/(1+\gamma)$. This is because

$$\begin{aligned}
\left|h'(z)\right| &\ge 1-\sum_{k=2}^{\infty}k|a_k|\,|z|^{k-1} > 1-\sum_{k=2}^{\infty}k|a_k| \\
&\ge 1-\sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}|a_k| \\
&\ge \sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_k| \\
&> \sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_k|\,|z|^{k-1} \\
&\ge \sum_{k=1}^{\infty}k|b_k|\,|z|^{k-1} \ge \left|g'(z)\right|.
\end{aligned} \tag{2.11}$$

The function

$$\begin{aligned}
f(z) &= z+\sum_{k=2}^{\infty}\frac{(1-\gamma)|b|}{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}x_k z^k \\
&\quad+\sum_{k=1}^{\infty}\frac{(1-\gamma)|b|}{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}\overline{y_k z^k},
\end{aligned} \tag{2.12}$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, shows that the coefficient bound given by (2.1) is sharp. The functions of the form (2.12) are in $\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$ because

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left(\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}|a_k|+\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_k|\right) \\
&\quad=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}\left|y_k\right|=2.
\end{aligned} \tag{2.13}$$

□

The next theorem shows that condition (2.1) is necessary for $f \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$.

Theorem 2.2. *Let $f=h+\overline{g}$ be so that h and g are given by (1.3). Then, $f \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}|a_k|+\sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_k| \le 2, \tag{2.14}$$

where $a_1=1$, $0\le\gamma<1$, $0\le\lambda\le\gamma/(1+\gamma)$ or $\lambda\ge 1/(1+\gamma)$, and $b\in\mathbb{C}\setminus\{0\}$.

Proof. The "if part" follows from Theorem 2.1 upon noting that $T\mathcal{S}C_{\mathcal{H}}(b,\gamma,\lambda) \subset \mathcal{S}C_{\mathcal{H}}(b,\gamma,\lambda)$. For the "only if" part, we show that $f \in T\mathcal{S}C_{\mathcal{H}}(b,\gamma,\lambda)$. Then, for $z = re^{i\theta}$ in $\mathcal{U}$, we obtain

$$\begin{aligned}
&\Re\left\{1+\frac{1}{b}\left(\frac{\lambda\left(z^3h'''(z)-\overline{z^3g'''(z)}\right)+(2\lambda+1)z^2h''(z)}{\lambda\left(z^2h''(z)+\overline{z^2g''(z)}\right)+zh'(z)+(2\lambda-1)\overline{zg'(z)}}\right.\right.\\
&\qquad\left.\left.+\frac{(1-4\lambda)\overline{z^2g''(z)}+zh'(z)+(1-2\lambda)\overline{zg'(z)}}{\lambda\left(z^2h''(z)+\overline{z^2g''(z)}\right)+zh'(z)+(2\lambda-1)\overline{zg'(z)}}-1\right)-\gamma\right\}\\
&=\Re\left\{\frac{(1-\gamma)z-\sum_{k=2}^{\infty}\left(\left(k(k\lambda-\lambda+1)\left[(k-1)+b(1-\gamma)\right]\right)/b\right)|a_k|z^k}{z-\sum_{k=2}^{\infty}k(k\lambda-\lambda+1)|a_k|z^k+\sum_{k=1}^{\infty}k(k\lambda+\lambda-1)|b_k|\overline{z}^k}\right.\\
&\qquad\left.-\frac{\sum_{k=1}^{\infty}\left(\left(k(k\lambda+\lambda-1)\left[(k+1)-b(1-\gamma)\right]\right)/b\right)|b_k|\overline{z}^k}{z-\sum_{k=2}^{\infty}k(k\lambda-\lambda+1)|a_k|z^k+\sum_{k=1}^{\infty}k(k\lambda+\lambda-1)|b_k|\overline{z}^k}\right\}\\
&\geq\frac{(1-\gamma)-\sum_{k=2}^{\infty}\left(\left(k(k\lambda-\lambda+1)\left[(k-1)+b(1-\gamma)\right]\right)/b\right)|a_k|r^{k-1}}{z-\sum_{k=2}^{\infty}k(k\lambda-\lambda+1)|a_k|r^{k-1}+\sum_{k=1}^{\infty}k(k\lambda+\lambda-1)|b_k|r^{k-1}}\\
&\qquad-\frac{\sum_{k=1}^{\infty}\left(\left(k(k\lambda+\lambda-1)\left[(k+1)-b(1-\gamma)\right]\right)/b\right)|b_k|r^{k-1}}{z-\sum_{k=2}^{\infty}k(k\lambda-\lambda+1)|a_k|r^{k-1}+\sum_{k=1}^{\infty}k(k\lambda+\lambda-1)|b_k|r^{k-1}}\\
&>0.
\end{aligned} \tag{2.15}$$

The above inequality must hold for all $z \in \mathcal{U}$. In particular, letting $z = r \to 1^-$ yields the required condition. □

As special cases of Theorem 2.2, we obtain the following two corollaries.

Corollary 2.3. *A function $f = h + \overline{g}$ belongs to $T\mathcal{S}C_{\mathcal{H}}(b,\gamma,0)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{k\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}|a_k|+\sum_{k=1}^{\infty}\frac{k\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_k|\leq 2. \tag{2.16}$$

Corollary 2.4. *A function $f = h + \overline{g}$ belongs to $T\mathcal{S}C_{\mathcal{H}}(b,\gamma,1)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{k^2\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}|a_k|+\sum_{k=1}^{\infty}\frac{k^2\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_k|\leq 2. \tag{2.17}$$

3. Extreme Points and Distortion Bounds

In this section, our first theorem gives the extreme points of the closed convex hulls of $T\mathcal{S}C_{\mathcal{H}}(b,\gamma,\lambda)$.

Theorem 3.1. *A function $f = h + \overline{g}$ belongs to $TSC_{\mathcal{H}}(b,\gamma,\lambda)$ if and only if f can be expressed as*

$$f(z) = \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_k(z)\right), \quad z \in \mathcal{U}, \tag{3.1}$$

where $h_1(z) = z$, $h_k(z) = z - ((1-\gamma)|b|/k(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)])z^k (k = 2,3,\ldots)$, and $g_k(z) = z + ((1-\gamma)|b|/k(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)])\overline{z}^k (k = 1,2,3,\ldots)$, $\sum_{k=1}^{\infty}(X_k + Y_k) = 1$, $X_k \geq 0$, $Y_k \geq 0$. In particular, the extreme points of $TSC_{\mathcal{H}}(b,\gamma,\lambda)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (3.1), we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_k(z)\right) \\ &= \sum_{k=1}^{\infty}(X_k + Y_k)z - \sum_{k=2}^{\infty}\frac{(1-\gamma)|b|}{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}X_k z^k \\ &\quad + \sum_{k=1}^{\infty}\frac{(1-\gamma)|b|}{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}Y_k \overline{z}^k. \end{aligned} \tag{3.2}$$

Then,

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}\left(\frac{(1-\gamma)|b|}{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}\right)X_k \\ &\qquad + \sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}\left(\frac{(1-\gamma)|b|}{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}\right)Y_k \\ &\quad = \sum_{k=2}^{\infty}X_k + \sum_{k=1}^{\infty}Y_k \\ &\quad = 1 - X_1 \leq 1, \end{aligned} \tag{3.3}$$

and so $f \in \operatorname{cl}\operatorname{co}TSC_{\mathcal{H}}(b,\gamma,\lambda)$.

Conversely, suppose that $f \in \operatorname{cl}\operatorname{co}TSC_{\mathcal{H}}(b,\gamma,\lambda)$. Letting

$$X_1 = 1 - \sum_{k=2}^{\infty}X_k - \sum_{k=1}^{\infty}Y_k, \tag{3.4}$$

where $X_k = (k(k\lambda - \lambda + 1)[(k-1) + |b|(1-\gamma)]/(1-\gamma)|b|)|a_k|$, $k = 2, 3, \ldots$, and $Y_k = (k(k\lambda + \lambda - 1)[(k+1) - |b|(1-\gamma)]/(1-\gamma)|b|)|b_k|$, $k = 1, 2, \ldots$, we obtain the required representation since

$$\begin{aligned} f(z) &= z - \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|\overline{z}^k \\ &= z - \sum_{k=2}^{\infty}\frac{(1-\gamma)|b|X_k}{k(k\lambda - \lambda + 1)[(k-1) + |b|(1-\gamma)]}z^k + \sum_{k=1}^{\infty}\frac{(1-\gamma)|b|Y_k}{k(k\lambda + \lambda - 1)[(k+1) - |b|(1-\gamma)]}\overline{z}^k \\ &= z - \sum_{k=2}^{\infty}(z - h_k(z))X_k - \sum_{k=1}^{\infty}(z - g_k(z))Y_k \\ &= \left(1 - \sum_{k=2}^{\infty}X_k - \sum_{k=1}^{\infty}Y_k\right)z + \sum_{k=2}^{\infty}h_k(z)X_k + \sum_{k=1}^{\infty}g_k(z)Y_k \\ &= \sum_{k=1}^{\infty}(X_k h_k(z) + Y_k g_k(z)). \end{aligned} \tag{3.5}$$

□

The following theorem gives the distortion bounds for functions in $T\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda)$, which yields a covering result for this family.

Theorem 3.2. *If $f \in T\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda)$ then for $z = re^{i\theta}$, one has*

$$\begin{aligned} |f(z)| &\leq (1 + |b_1|)r + r^2\left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1 + b(1-\gamma)]} - \frac{(2\lambda - 1)[2 - b(1-\gamma)]}{2(\lambda+1)[1 + b(1-\gamma)]}|b_1|\right), \\ |f(z)| &\geq (1 - |b_1|)r - r^2\left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1 + b(1-\gamma)]} - \frac{(2\lambda - 1)[2 - b(1-\gamma)]}{2(\lambda+1)[1 + b(1-\gamma)]}|b_1|\right). \end{aligned} \tag{3.6}$$

Proof. Let $f \in T\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda)$. Taking the absolute value of f and then by Theorem 2.2, we obtain

$$\begin{aligned} |f(z)| &\leq (1 + |b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k \\ &\leq (1 + |b_1|)r + \frac{(1-\gamma)|b|}{2(\lambda+1)[1 + b(1-\gamma)]}\left(\sum_{k=2}^{\infty}\frac{k(k\lambda - \lambda + 1)[(k-1) + |b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|\right. \\ &\qquad \left. + \frac{k(k\lambda + \lambda - 1)[(k+1) - |b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|\right)r^2 \\ &\leq (1 + |b_1|)r + \left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1 + b(1-\gamma)]} - \frac{(2\lambda - 1)[2 - b(1-\gamma)]}{2(\lambda+1)[1 + b(1-\gamma)]}|b_1|\right)r^2. \end{aligned} \tag{3.7}$$

Similarly,

$$\left|f(z)\right| \geq (1-|b_1|)r - \left(\frac{(1-\gamma)|b|}{2(\lambda+1)\left[1+b(1-\gamma)\right]} - \frac{(2\lambda-1)\left[2-b(1-\gamma)\right]}{2(\lambda+1)\left[1+b(1-\gamma)\right]}|b_1|\right)r^2. \tag{3.8}$$

The upper and lower bounds given in Theorem 3.2 are, respectively, attained for the following functions

$$\begin{aligned} f(z) &= z + |b_1|\overline{z} + \frac{1}{\Gamma(2)}\left(\frac{(1-\gamma)|b|}{2(\lambda+1)\left[1+b(1-\gamma)\right]} - \frac{(2\lambda-1)\left[2-b(1-\gamma)\right]}{2(\lambda+1)\left[1+b(1-\gamma)\right]}|b_1|\right)\overline{z}^2, \\ f(z) &= (1-|b_1|)z - \frac{1}{\Gamma(2)}\left(\frac{(1-\gamma)|b|}{2(\lambda+1)\left[1+b(1-\gamma)\right]} - \frac{(2\lambda-1)\left[2-b(1-\gamma)\right]}{2(\lambda+1)\left[1+b(1-\gamma)\right]}|b_1|\right)z^2. \end{aligned} \tag{3.9}$$

□

4. Convolution and Convex Combinations

In this section we show that the class $T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions. For $f(z) = z - \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|\overline{z}^k$ and $F(z) = z - \sum_{k=2}^{\infty}|A_k|z^k + \sum_{k=1}^{\infty}|B_k|\overline{z}^k$, we define the convolution of two harmonic functions f and F as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{k=2}^{\infty}|a_k|\,|A_k|z^k + \sum_{k=1}^{\infty}|b_k|\,|B_k|\overline{z}^k. \tag{4.1}$$

Using the definition, we show that the class $T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$ is closed under convolution.

Theorem 4.1. *For $0 \leq \delta < \gamma < 1$, let $f \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$ and $F \in T\mathcal{SC}_{\mathcal{H}}(b,\delta,\lambda)$. Then, $f * F \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda) \subset T\mathcal{SC}_{\mathcal{H}}(b,\delta,\lambda)$.*

Proof. Let $f(z) = z - \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|\overline{z}^k$ and $F(z) = z - \sum_{k=2}^{\infty}|A_k|z^k + \sum_{k=1}^{\infty}|B_k|\overline{z}^k$ be in $T\mathcal{SC}_{\mathcal{H}}(b,\delta,\lambda)$. Then, $f * F \in T\mathcal{SC}_{\mathcal{H}}(b,\delta,\lambda)$. We note that $|A_k| \leq 1$ and $|B_k| \leq 1$. In view of Theorem 2.2 and the inequality $0 \leq \delta \leq \gamma < 1$, we have

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)[(k-1)+|b|(1-\delta)]}{(1-\delta)|b|}|a_k|\,|A_k| + \sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)[(k+1)-|b|(1-\delta)]}{(1-\delta)|b|}|b_k|\,|B_k| \\ &\quad\leq \sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)[(k-1)+|b|(1-\delta)]}{(1-\delta)|b|}|a_k| + \sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)[(k+1)-|b|(1-\delta)]}{(1-\delta)|b|}|b_k| \\ &\quad\leq \sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_k| \\ &\quad\leq 1, \end{aligned} \tag{4.2}$$

by Theorem 2.2, $f \in T\mathcal{S}C_{\mathscr{H}}(b,\gamma,\lambda)$. By the same token, we then conclude that $f * F \in T\mathcal{S}C_{\mathscr{H}}(b,\gamma,\lambda) \subset T\mathcal{S}C_{\mathscr{H}}(b,\delta,\lambda)$. □

Next, we show that the class $T\mathcal{S}C_{\mathscr{H}}(b,\gamma,\lambda)$ is closed under convex combination of its members.

Theorem 4.2. *The class $T\mathcal{S}C_{\mathscr{H}}(b,\gamma,\lambda)$ is closed under convex combinations.*

Proof. Suppose that $f_i(z) \in T\mathcal{S}C_{\mathscr{H}}(b,\gamma,\lambda)$, where f_i is given by

$$f_i(z) = z - \sum_{k=2}^{\infty}|a_{i,k}|z^k + \sum_{k=1}^{\infty}|b_{i,k}|\overline{z}^k, \qquad i = 1,2,3,\ldots. \tag{4.3}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \le t_i \le 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i|a_{i,k}|\right)z^k + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i|b_{i,k}|\right)\overline{z}^k. \tag{4.4}$$

Since,

$$\sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}|a_{i,k}| + \sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_{i,k}| \le 1, \tag{4.5}$$

from the above equation we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}\sum_{i=1}^{\infty} t_i|a_{i,k}| \\
&\quad + \sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}\sum_{i=1}^{\infty} t_i|b_{i,k}| \\
&= \sum_{i=1}^{\infty} t_i\left\{\sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)\left[(k-1)+|b|(1-\gamma)\right]}{(1-\gamma)|b|}|a_{i,k}|\right. \\
&\qquad \left.+\sum_{k=1}^{\infty}\frac{k(k\lambda+\lambda-1)\left[(k+1)-|b|(1-\gamma)\right]}{(1-\gamma)|b|}|b_{i,k}|\right\} \\
&\le \sum_{i=1}^{\infty} t_i = 1.
\end{aligned} \tag{4.6}$$

This is the required condition by (2.14) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in T\mathcal{S}C_{\mathscr{H}}(b,\gamma,\lambda)$. □

5. Class-Preserving Integral Operator

In this section, we consider the closure property of the class $T\mathcal{S}C_{\mathcal{H}}(b,\gamma,\lambda)$ under the Bernardi integral operator $\mathcal{L}_c[f(z)]$, which is defined by

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c}\int_0^z \xi^{c-1} f(\xi)d\xi \quad (c > -1). \tag{5.1}$$

Theorem 5.1. *Let $f(z) \in T\mathcal{S}C_{\mathcal{H}}(b,\gamma,\lambda)$; then $\mathcal{L}_c[f(z)] \in T\mathcal{S}C_{\mathcal{H}}(b,\gamma,\lambda)$.*

Proof. From the representation of $\mathcal{L}_c[f(z)]$, it follows that

$$\begin{aligned}
\mathcal{L}_c[f(z)] &= \frac{c+1}{z^c}\int_0^z \xi^{c-1}h(\xi)d\xi + \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1}g(\xi)d\xi} \\
&= \frac{c+1}{z^c}\int_0^z \xi^{c-1}\left(\xi - \sum_{k=2}^{\infty}|a_k|\xi^k\right)d\xi + \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1}\left(\sum_{k=1}^{\infty}|b_k|\xi^k\right)d\xi} \\
&= z - \sum_{k=2}^{\infty}A_k z^k + \sum_{k=1}^{\infty}B_k z^k,
\end{aligned} \tag{5.2}$$

where $A_k = ((c+1)/(c+k))|a_k|$ and $B_k = ((c+1)/(c+k))|b_k|$. Hence,

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}\left(\frac{c+1}{c+k}|a_k|\right) \\
&\quad + \sum_{k=1}^{\infty}\frac{k|k\lambda+\lambda-1|[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}\left(\frac{c+1}{c+k}|b_k|\right) \\
&\quad \le \sum_{k=2}^{\infty}\frac{k(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| \\
&\quad + \sum_{k=1}^{\infty}\frac{k|k\lambda+\lambda-1|[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \le 1,
\end{aligned} \tag{5.3}$$

and since $f \in T\mathcal{S}C_{\mathcal{H}}(b,\gamma,\lambda)$, by Theorem 2.2, $\mathcal{L}_c[f(z)] \in T\mathcal{S}C_{\mathcal{H}}(b,\gamma,\lambda)$. □

Acknowledgments

The authors would like to thank the referees for their insightful suggestions and comments. The authors would like to thank Professor S. Yalçin for his suggestions during the preparation of the paper.

References

[1] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academiae Scientiarum Fennicae. Series A I*, vol. 9, pp. 3–25, 1984.

[2] B. A. Frasin, "Comprehensive family of harmonic univalent functions," *SUT Journal of Mathematics*, vol. 42, no. 1, pp. 145–155, 2006.

[3] J. M. Jahangiri, "Coefficient bounds and univalence criteria for harmonic functions with negative coefficients," *Annales Universitatis Mariae Curie-Skłodowska. Sectio A*, vol. 52, no. 2, pp. 57–66, 1998.

[4] J. M. Jahangiri, "Harmonic functions starlike in the unit disk," *Journal of Mathematical Analysis and Applications*, vol. 235, no. 2, pp. 470–477, 1999.

[5] J. M. Jahangiri, G. Murugusundaramoorthy, and K. Vijaya, "Starlikeness of harmonic functions defined by Ruscheweyh derivatives," *Indian Academy of Mathematics*, vol. 26, no. 1, pp. 191–200, 2004.

[6] J. M. Jahangiri, Y. C. Kim, and H. M. Srivastava, "Construction of a certain class of harmonic close-to-convex functions associated with the Alexander integral transform," *Integral Transforms and Special Functions*, vol. 14, no. 3, pp. 237–242, 2003.

[7] H. Silverman, "Harmonic univalent functions with negative coefficients," *Journal of Mathematical Analysis and Applications*, vol. 220, no. 1, pp. 283–289, 1998.

[8] H. Silverman and E. M. Silvia, "Subclasses of harmonic univalent functions," *New Zealand Journal of Mathematics*, vol. 28, no. 2, pp. 275–284, 1999.

[9] S. Yalçin and M. Öztürk, "On a subclass of certain convex harmonic functions," *Journal of the Korean Mathematical Society*, vol. 43, no. 4, pp. 803–813, 2006.

[10] Y. Avcı and E. Złotkiewicz, "On harmonic univalent mappings," *Annales Universitatis Mariae Curie-Skłodowska. Sectio A*, vol. 44, pp. 1–7, 1990.

[11] S. Yalçin and M. Öztürk, "Harmonic functions starlike of the complex order," *Matematichki Vesnik*, vol. 58, no. 1-2, pp. 7–11, 2006.

[12] M. A. Nasr and M. K. Aouf, "Starlike function of complex order," *The Journal of Natural Sciences and Mathematics*, vol. 25, no. 1, pp. 1–12, 1985.