Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 308921, 11 pages
doi:10.1155/2012/308921

Research Article

Coincidence Points for Expansive Mappings under c-Distance in Cone Metric Spaces

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Received 10 March 2012; Accepted 27 April 2012

Academic Editor: Charles Chidume

We establish some fixed (common fixed) and coincidence point results for mappings verifying some expansive type contractions in cone metric spaces with the help of the concept of a c-distance. Our results generalize, extend, and unify several well-known comparable results in the literature. Some examples are also presented.

1. Introduction and Preliminaries

Huang and Zhang [1] reintroduced the notion of cone metric spaces and established fixed point theorems for mappings on this space. After that, many fixed point theorems have been proved in normal or nonnormal cone metric spaces by some authors (see e.g., [1–26] and references contained therein).

We need to recall some basic notations, definitions, and necessary results from literature. Let $\mathbb{R}_+$ be the set of nonnegative real numbers. Let E be a real Banach space and 0_E is the zero vector of E.

Definition 1.1 (see [1]). A nonempty subset P of E is called a cone if the following conditions hold:

(i) P is closed and $P \neq \{0_E\}$,

(ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$,

(iii) $x \in P,\ -x \in P \Rightarrow x = 0_E$.

Given a cone $P \subset E$, a partial ordering $\leq_E$ with respect to P is naturally defined by $x \leq_E y$ if and only if $y - x \in P$, for $x, y \in E$. We will write $x <_E y$ to indicate that $x \leq_E y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is said solid if $\operatorname{int} P$ is non-empty.

Definition 1.2 (see [1]). Let X be a non-empty set and $d : X \times X \to P$ satisfies

(i) $d(x, y) = 0_E$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, y) \leq_E d(x, z) + d(z, y)$ for all $x, y, z \in E$.

Then, the mapping d is called a cone metric on X and the pair (X, d) is called a cone metric space.

Definition 1.3 (see [1]). Let (X, d) be a cone metric space, $\{x_n\}$ is a sequence in X and $x \in X$.

(i) If for every $c \in E$ with $0_E \ll_E c$ there is $N \in \mathbb{N}$ such that $d(x_n, x) \ll_E c$ for all $n \geq N$, then $\{x_n\}$ is said to be convergent to x. This limit is denoted by $\lim_{n\to+\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$.

(ii) If for every $c \in E$ with $0_E \ll_E c$, there is $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll_E c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X.

(iii) If every Cauchy sequence in X is convergent in X, then (X, d) is called a complete cone metric space.

Definition 1.4. Let (X, d) be a cone metric space and let $T : X \to X$ be a given mapping. We say that T is continuous on $x_0 \in X$ if for every sequence $\{x_n\}$ is X, we have

$$x_n \longrightarrow x_0 \quad \text{as } n \longrightarrow \infty \Longrightarrow Tx_n \longrightarrow Tx_0 \quad \text{as } n \longrightarrow \infty. \tag{1.1}$$

If T is continuous on each point $x_0 \in X$, then we say that T is continuous on X.

In 2011, Cho et al. [11] and Wang and Guo [27] introduced a new concept of c-distance in cone metric spaces, which is a cone version of w-distance of Kada et al. [16] and proved some fixed point theorems for some contractive-type mappings in partially ordered cone metric spaces using the c-distance. For other results, see [13, 24].

Definition 1.5 (see [11, 27]). Let (X, d) be a cone metric space. Then, a function $q : X \to E$ is called a c-distance on X if the following are satisfied:

($q1$) $q(x, y) \geq_E 0_E$ for all $x, y \in X$,

($q2$) $q(x, z) \leq_E q(x, y) + q(y, z)$ for all $x, y, z \in X$,

($q3$) for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \leq_E u$ for some $u = u_x \in P$, then $q(x, y) \leq_E u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,

($q4$) for all $c \in E$ with $0_E \ll c$, there exists $e \in E$ with $0_E \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Remark 1.6 (see [11, 27]). The c-distance q is a w-distance on X if we take (X, d) is a metric space, $E = \mathbb{R}_+$, $P = [0, \infty)$, and ($q3$) is replaced by the following condition. For any $x \in X$,

$q(x,\cdot) : X \to \mathbb{R}_+$ is lower semicontinuous. Moreover, $(q3)$ holds whenever $q(x,\cdot)$ is lower semicontinuous. Thus, if (X,d) is a metric space, $E = \mathbb{R}_+$ and $P = [0,\infty)$, then every w-distance is a c-distance. But the converse is not true in general case. Therefore, the c-distance is a generalization of the w-distance.

Example 1.7 (see [11, 27]). Let (X,d) be a cone metric space and let P be a normal cone. Define a mapping $q : X \times X \to E$ by $q(x,y) = d(x,y)$ for all $x,y \in X$. Then, q is a c-distance.

Example 1.8 (see [11, 27]). Let (X,d) be a cone metric space and let P be a normal cone. Define a mapping $q : X \times X \to E$ by $q(x,y) = d(u,y)$ for all $x,y \in X$, where u is a fixed point in X. Then, q is a c-distance.

Example 1.9. Let (X,d) be a cone metric space and let P be a normal cone. Define a mapping $q : X \times X \to E$ by $q(x,y) = (d(x,u) + d(u,y))/2$ for all $x,y \in X$, where u is a fixed point in X. Then q is a c-distance.

Remark 1.10 (see [11, 27]). (1) $q(x,y) = q(y,x)$ *does not necessarily hold for all* $x,y \in X$.
(2) $q(x,y) = 0_E$ is not necessarily equivalent to $x = y$ for all $x,y \in X$.

Lemma 1.11 (see [11, 27]). *Let (X,d) be a cone metric space and let q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be a sequences in X and $x,y,z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to 0_E. The following hold.*

(1) *If $q(x_n,y) \leq_E u_n$ and $q(x_n,z) \leq_E u_n$, then $y = z$.*
(2) *If $q(x_n,y_n) \leq_E u_n$ and $q(x_n,z) \leq_E u_n$, then $\{y_n\}$ converges to z.*
(3) *If $q(x_n,x_m) \leq_E u_n$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*
(4) *If $q(y,x_n) \leq_E u_n$, then $\{x_n\}$ is a Cauchy sequence in X.*

Let $f,g : X \to X$ be two selfmaps on a nonempty set X. Recall that a point $x \in X$ is called a coincidence point of the pair (f,g) if $fx = gx$. The point $y = fx = gx$ is called a point of coincidence. If $x = fx = gx$, then x is said a common fixed point of f and g.

The purpose of this paper is to give some common fixed and coincidence point theorems for mappings verifying some expansive type contractions on cone metric spaces via a c-distance. Also, some examples are presented.

2. Main results

First, we present the following useful lemma, which is a variant of (2.2) in Lemma 1.11.

Lemma 2.1. *Let (X,d) be a cone metric space and let q be a c-distance on X. Let $\{x_n\}$ be a sequence in X. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are tow sequences in P converging to 0_E. If $q(x_n,y) \leq_E \alpha_n$ and $q(x_n,z) \leq_E \beta_n$, then $y = z$.*

Proof. Let $c \gg 0_E$ be arbitrary. Since $\alpha_n \to 0_E$, so there exists $N_1 \in \mathbb{N}$ such that $\alpha_n \ll c/2$ for all $n \geq N_1$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $\beta_n \ll c/2$ for all $n \geq N_1$. Thus, for all $N \geq \max\{N_1,N_2\}$, we have

$$q(x_n,y) \ll \frac{c}{2}, \quad q(x_n,z) \ll \frac{c}{2}. \tag{2.1}$$

Take $e = c/2$, so by $(q4)$, we get that $d(y,z) \ll c$ for each $c \gg 0_E$, hence $y = z$. □

Now, we present coincidence point results in the frame work of cone metric spaces in terms of a c-distance. Note that (2.2) is called an expansive type contraction.

Theorem 2.2. *Let (X, d) be a cone metric space and let q be a c-distance on X. Let $f, T : X \to X$ be two functions such that there are three nonnegative real numbers a, b, and c with $a + b + c > 1$ such that*

$$q(Tx, Ty) \geq_E aq(fx, fy) + bq(Tx, fx) + cq(Ty, fy). \tag{2.2}$$

Assume the following hypotheses:

(1) $b < 1$ *and* $a \neq 0$,

(2) $f(X) \subseteq T(X)$,

(3) $(T(X), d)$ *is a complete subset of* (X, d).

Then T and f have a coincidence point, say $u \in X$. One has $q(Tu, Tu) = 0_E$.

Also, if $a > 1$, then the point of coincidence $x = Tu = fu$ is unique.

Proof. Let $x_0 \in X$. Since $f(X) \subseteq T(X)$, we can choose $x_1 \in X$ such that $Tx_1 = fx_0$. Again since $f(X) \subseteq T(X)$, we can choose $x_2 \in X$ such that $Tx_2 = fx_1$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $Tx_n = fx_{n-1}$ for all $n \geq 1$.

By (2.2), we have

$$\begin{aligned} q(fx_{n-1}, fx_n) &= q(Tx_n, Tx_{n+1}) \\ &\geq_E aq(fx_n, fx_{n+1}) + bq(Tx_n, fx_n) + cq(Tx_{n+1}, fx_{n+1}) \\ &= aq(fx_n, fx_{n+1}) + bq(fx_{n-1}, fx_n) + cq(fx_n, fx_{n+1}). \end{aligned} \tag{2.3}$$

Therefore,

$$(1 - b)q(fx_{n-1}, fx_n) \leq_E (a + c)q(fx_n, fx_{n+1}). \tag{2.4}$$

Set $\lambda = (1 - b)/(a + c)$. By hypotheses, we have $\lambda \in (0, 1)$. Also,

$$q(fx_n, fx_{n+1}) \leq_E \lambda q(fx_{n-1}, fx_n) \quad \forall n \geq 1. \tag{2.5}$$

By induction, we get that

$$q(fx_n, fx_{n+1}) \leq_E \lambda^n q(fx_0, fx_1) \quad \forall n \geq 0. \tag{2.6}$$

Let $m > n$. By $(q2)$, we have

$$q(fx_n, fx_m) \leq_E \sum_{i=n}^{m-1} q(fx_i, fx_{i+1}) \leq_E \frac{\lambda^n}{1 - \lambda} q(fx_0, fx_1). \tag{2.7}$$

Since $\lambda < 1$, so by Lemma 1.11(3), the sequence $\{Tx_n = fx_{n-1}\}$ is Cauchy in $(T(X), d)$, which is complete, hence there exists $u \in X$ such that $Tx_n \to Tu$. Thus, $fx_n \to Tu$ as $n \to \infty$, that is,

$$\lim_{n\to+\infty} d(Tx_n, Tu) = \lim_{n\to+\infty} d(fx_n, Tu) = 0. \tag{2.8}$$

We claim that $Tu = fu$. Recall that $q(fx_n, fx_m) \leq_E (\lambda^n/(1-\lambda))q(fx_0, fx_1)$ and $fx_n \to Tu$, so by $(q3)$ and as $m \to \infty$, we get that

$$q(fx_n, Tu) \leq_E \frac{\lambda^n}{1-\lambda} q(fx_0, fx_1). \tag{2.9}$$

From (2.2), we have

$$\begin{aligned} q(fx_{n-1}, Tu) &= q(Tx_n, Tu) \\ &\geq_E aq(fx_n, fu) + bq(Tx_n, fx_n) + cq(Tu, fu) \\ &\geq_E aq(fx_n, fu). \end{aligned} \tag{2.10}$$

Since $a \neq 0$, so it follows that

$$q(fx_n, fu) \leq_E \frac{1}{a} q(fx_{n-1}, Tu). \tag{2.11}$$

By (2.9), we obtain

$$q(fx_n, fu) \leq_E \frac{1}{a} q(fx_{n-1}, Tu) \leq_E \frac{1}{a}\frac{\lambda^{n-1}}{1-\lambda} q(fx_0, fx_1). \tag{2.12}$$

Set $\alpha_n =: (\lambda^n/(1-\lambda))q(fx_0, fx_1)$ and $\beta_n =: (1/a)(\lambda^{n-1}/(1-\lambda))q(fx_0, fx_1)$. Since $\lambda < 1$, so $\alpha_n, \beta_n \to 0_E$ as $n \to \infty$. Thus, by (2.9), (2.12), and Lemma 2.1, get that $Tu = fu$.

Using (2.2), we get

$$q(Tu, Tu) \geq_E (a+b+c)q(Tu, Tu). \tag{2.13}$$

Since $a+b+c > 1$, so $q(Tu, Tu) = 0_E$. Now, we prove that if $a > 1$, then the point of coincidence $x = Tu = fu$ is unique.

Let x and y be two points of coincidence of T and f, that is, there exist $u, v \in X$ such that $x = Tu = fu$ and $y = Tv = fv$. By the above, we have $q(Tu, Tu) = q(Tv, Tv) = 0_E$. By (2.2), the following holds:

$$\begin{aligned} q(Tu, Tv) &\geq_E aq(fu, fv) + bq(Tu, fu) + cq(Tv, fv) \\ &= aq(Tu, Tv) + bq(Tu, Tu) + cq(Tv, Tv) = aq(Tu, Tv). \end{aligned} \tag{2.14}$$

If $a > 1$, we conclude that $q(Tu, Tv) = 0$.

Let $c \gg 0_E$ be arbitrary. Take $e = c \gg 0_E$. Since $q(Tu, Tu) = q(Tu, Tv) = 0_E \ll e$, then by the condition $(q4)$, we get that $d(Tu, Tv) \ll c$. Thus, $x = Tu = Tv = y$. This completes the proof. □

Now, we state the following corollaries.

Corollary 2.3. *Let (X, d) be a cone metric space and let q be a c-distance on X. Let $f, T : X \to X$ be two functions. Assume there exists $a > 1$ such that*

$$q(Tx, Ty) \geq_E aq(fx, fy). \tag{2.15}$$

Assume the following hypotheses:

(1) $f(X) \subseteq T(X)$,

(2) $(T(X), d)$ *is a complete subset of* X.

Then, T and f have a coincidence point, say $u \in X$. One has $q(Tu, fu) = 0_E$ and $x = Tu = fu$ is the unique point of coincidence of T and f.

Proof. It follows by taking $b = c = 0$ in Theorem 2.2. □

Corollary 2.4. *Let (X, d) be a cone metric space and let q be a c-distance on X. Let $f, T : X \to X$ be two functions. Assume there exist two nonnegative real numbers a and b, with $a + b > 1$ such that*

$$q(Tx, Ty) \geq_E aq(fx, fy) + bq(Tx, fx). \tag{2.16}$$

Assume the following hypotheses:

(1) $b < 1$,

(2) $f(X) \subseteq T(X)$,

(3) $(T(X), d)$ *is a complete subset of* X.

Then, T and f have a coincidence point, say $u \in X$. One has $q(Tu, Tu) = 0_E$ and $x = Tu = fu$ is the unique point of coincidence of T and f.

Proof. It follows by taking $c = 0$ in Theorem 2.2. □

Corollary 2.5. *Let (X, d) be a complete cone metric space and let q be a c-distance on X. Let $T : X \to X$ be a surjective function. Assume there are three nonnegative real numbers a, b and c with $a + b + c > 1$ such that*

$$q(Tx, Ty) \geq_E aq(x, y) + bq(Tx, x) + cq(Ty, y). \tag{2.17}$$

If $b < 1$ and $a \neq 0$, then T has a fixed point, say $u \in X$. One has $q(Tu, Tu) = 0_E$. Also if $a > 1$, then the fixed point u is unique.

Proof. It follows by taking $f = \mathrm{Id}_X$, the identity on X, in Theorem 2.2. Note that when T is surjective, so $T(X) = X$, that is, the hypothesis (2) in Theorem 2.2 holds. □

The next result is similar to Theorem 2.2, except that the contractive condition (2.2) is replaced by

$$q(Ty,Tx) \geq_E aq(fy,fx) + bq(Tx,fx) + cq(Ty,fy). \tag{2.18}$$

Note that this contractive condition is studied since it is different to (2.2) because of Remark 1.10.(2). Its proof is essentially the same as for Theorem 2.2 and so is omitted.

Theorem 2.6. *Let (X,d) be a cone metric space and let q be a c-distance on X. Let $f,g : X \to X$ be two functions such that there are three nonnegative real numbers a, b, and c with $a+b+c>1$ such that*

$$q(Ty,Tx) \geq_E aq(fy,fx) + bq(Tx,fx) + cq(Ty,fy). \tag{2.19}$$

Assume the following hypotheses:

(1) $c<1$ *and* $a \neq 0$,

(2) $f(X) \subseteq T(X)$,

(3) $(T(X),d)$ *is a complete subset of* X.

Then, T and f have a coincidence point say $u \in X$. One has $q(Tu,Tu)=0_E$. If $a>1$, $x=Tu=fu$ is the unique point of coincidence of T and f.

Remark 2.7. Let (X,d) be a cone metric space and let P be a normal cone. Take in Theorem 2.2 or Theorem 2.6 the c-distance $q : X \times X \to E$ defined by $q(x,y)=d(x,y)$ for all $x,y \in X$. Then, the inequalities (2.2) and (2.19) correspond to the contractive condition given in Theorem 2.1 of Shatanawi and Awawdah [28]. Thus, our results (Theorems 2.2 and 2.6) extend and generalize the results in [28].

Remark 2.8. Some similar results as above corollaries could be derived from Theorem 2.6.

Now, we present the following example.

Example 2.9. Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\|_E = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E,\ x(t) \geq 0,\ t \in [0,1]\}$. Let $X=[0,\infty)$ and let $d : X \times X \to E$ be defined by $d(x,y)(t) = |x-y|2^t$. Then, (X,d) is a cone metric space. Let, further, $q : X \times X \to E$ be defined by $q(x,y)(t) = 2^t y$. It is easy to check that q is a c-distance. Consider the mappings $f,T : X \to X$ defined by

$$Tx = \frac{3}{4}x, \qquad fx = \frac{9}{20}x. \tag{2.20}$$

Take $a=4/3$ and $b=c=0$ (we have $a+b+c>1$, $b<1$ and $a>1$). For all $x,y \in X$, we have

$$q(Tx,Ty) = 2^t\frac{3}{4}y \geq 2^t\frac{3}{5}y = aq(fx,fy). \tag{2.21}$$

All hypotheses of Corollary 2.3 are satisfied, and $u=0$ is a coincidence point of f and T. Also, $q(Tu,Tu)=q(0,0)=0$ and $x=Tu=T0=0$ is the unique point of coincidence of T and f.

Next, we present a common fixed point theorem for two maps involving some expansive type contractions given by the conditions (2.22).

Theorem 2.10. *Let (X, d) be a complete cone metric space and let q be a c-distance on X. Let $T, S : X \to X$ be two mappings. Suppose that T and S satisfy the following inequalities:*

$$
\begin{gathered}
q(T(Sx), Sx) + kq(T(Sx), x) \geq_E aq(Sx, x), \\
q(S(Tx), Tx) + kq(S(Tx), x) \geq_E bq(Tx, x)
\end{gathered}
\tag{2.22}
$$

for all $x \in X$ and some nonnegative real numbers a, b, and k with $a > 1 + 2k$ and $b > 1 + 2k$. If T and S are continuous and surjective, then T and S have a common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since T is surjective, there exists $x_1 \in X$ such that $x_0 = Tx_1$. Also, since S is surjective, there exists $x_2 \in X$ such that $x_2 = Sx_1$. Continuing this process, we construct a sequence (x_n) in X such that $x_{2n} = Tx_{2n+1}$ and $x_{2n+1} = Sx_{2n+2}$ for all $n \in \mathbb{N}$. Now, for $n \in \mathbb{N}$, we have

$$
q(T(Sx_{2n+2}), Sx_{2n+2}) + kq(T(Sx_{2n+2}), x_{2n+2}) \geq_E aq(Sx_{2n+2}, x_{2n+2}). \tag{2.23}
$$

Thus, we have

$$
q(x_{2n}, x_{2n+1}) + kq(x_{2n}, x_{2n+2}) \geq_E aq(x_{2n+1}, x_{2n+2}). \tag{2.24}
$$

By $(q2)$, we have $q(x_{2n}, x_{2n+1}) + q(x_{2n+1}, x_{2n+2}) \geq_E p(x_{2n}, x_{2n+2})$. Hence, we get that

$$
q(x_{2n}, x_{2n+1}) + kq(x_{2n}, x_{2n+1}) + kq(x_{2n+1}, x_{2n+2}) \geq_E aq(x_{2n+1}, x_{2n+2}). \tag{2.25}
$$

Therefore,

$$
q(x_{2n+1}, x_{2n+2}) \leq_E \frac{1+k}{a-k} q(x_{2n}, x_{2n+1}). \tag{2.26}
$$

On other hand, we have

$$
q(S(Tx_{2n+1}), Tx_{2n+1}) + kq(S(Tx_{2n+1}), x_{2n+1}) \geq_E bq(Tx_{2n+1}, x_{2n+1}). \tag{2.27}
$$

Thus,

$$
q(x_{2n-1}, x_{2n}) + kq(x_{2n-1}, x_{2n+1}) \geq_E bq(x_{2n}, x_{2n+1}). \tag{2.28}
$$

Again, using $(q2)$, we have

$$
q(x_{2n-1}, x_{2n}) + kq(x_{2n-1}, x_{2n}) + kq(x_{2n}, x_{2n+1}) \geq_E bq(x_{2n}, x_{2n+1}). \tag{2.29}
$$

Hence,

$$q(x_{2n}, x_{2n+1}) \preceq_E \frac{1+k}{b-k} q(x_{2n-1}, x_{2n}). \tag{2.30}$$

Let

$$\lambda = \max\left\{\frac{1+k}{a-k}, \frac{1+k}{b-k}\right\}. \tag{2.31}$$

Then, by combining (2.26) and (2.30), we have

$$q(x_n, x_{n+1}) \preceq_E \lambda q(x_{n-1}, x_n) \quad \forall n \geq 1. \tag{2.32}$$

Repeating (2.32) n-times, we get

$$q(x_n, x_{n+1}) \preceq_E \lambda^n q(x_0, x_1). \tag{2.33}$$

Thus, for $m > n$, we have

$$\begin{aligned} q(x_n, x_m) &\preceq_E q(x_n, x_{n+1}) \cdots + q(x_{m-1}, x_m) \\ &\preceq_E \left(\lambda^n + \cdots + \lambda^{m-1}\right) q(x_0, x_1) \\ &\preceq_E \frac{\lambda^n}{1-\lambda} q(x_0, x_1). \end{aligned} \tag{2.34}$$

By assumption, we get that $0 \leq \lambda < 1$. By Lemma 1.11(3), $\{x_n\}$ is a Cauchy sequence in the complete cone metric space (X, d). Then, there exists $v \in X$ such that $x_n \to v$ as $n \to +\infty$. Since $x_{2n+1} \to v$ and $x_{2n} \to v$ as $n \to +\infty$, so clearly, the fact that T and S are continuous and uniqueness of limit yields that $v = Tv = Sv$, that is, v is a common fixed point of T and S. □

Corollary 2.11. *Let (X, d) be a complete cone metric space and let q be a c-distance on X. Let $T : X \to X$ be a continuous surjective mapping satisfying*

$$q(T(Tx), Tx) + kq(T(Tx), x) \succeq_E aq(Tx, x) \tag{2.35}$$

for all $x \in X$ and some nonnegative real numbers a and k with $a > 1 + 2k$. Then, T has a fixed point.

Proof. It follows from Theorem 2.10 by taking $S = T$ and $b = a$. □

Remark 2.12. Corollary 4.1 of [29], Theorem 4 of [30], and Corollary 2.8 of [28] are particular cases of Corollary 2.11.

We give the following examples illustrating our result obtained by Theorem 2.10.

Example 2.13. Let $E = \mathbb{R}$ with $\|x\|_E = |x|$, $P = [0, \infty)$, and $X = [0, \infty)$. Let $d, q : X \times X \to E$ be defined by $q(x, y) = d(x, y) = |x - y|$. Then, obviously, (X, d) is a cone metric space and q is a c-distance. Consider two mappings $S, T : X \to X$ defined by $Tx = 2x$ and $Sx = 3x$. For every $k > 0$, take $a = (2 + 5k)/2$ and $b = 4 + 5k$. We have $a, b > 1 + 2k$ and the conditions (2.22) hold. So T and S have a (unique) common fixed point, which is $v = 0$.

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