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Research Article

Generalized Derivations in Semiprime Gamma Rings

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Let M be a 2-torsion-free semiprime Γ-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M, \alpha, \beta \in \Gamma$, and let $D : M \to M$ be an additive mapping such that $D(x\alpha x) = D(x)\alpha x + x\alpha d(x)$ for all $x \in M, \alpha \in \Gamma$ and for some derivation d of M. We prove that D is a generalized derivation.

1. Introduction

Hvala [1] first introduced the generalized derivations in rings and obtained some remarkable results in classical rings. Daif and Tammam El-Sayiad [2] studied the generalized derivations in semiprime rings. The authors consider an additive mapping $G : R \to R$ of a ring R with the property $G(x^2) = G(x)x + xD(x)$ for some derivation D of R. They prove that G is a Jordan generalized derivation.

Aydin [3] studied generalized derivations of prime rings. The author proved that if I is an ideal of a noncommutative prime ring R, a is a fixed element of R and F is an generalized derivation on R associated with a derivation d then the condition $F([x, a]) = 0$ or $[F(x), a] = 0$ for all $x \in I$ implies $d(x) = \lambda[x, a]$.

Çeven and Öztürk [4] have dealt with Jordan generalized derivations in Γ-rings and they proved that every Jordan generalized derivation on some Γ-rings is a generalized derivation.

Generalized derivations of semiprime rings have been treated by Ali and Chaudhry [5]. The authors proved that if F is a commuting generalized derivation of a semiprime ring

R associated with a derivation d then $d(x)[y,z] = 0$ for all $x,y,z \in R$ and d is central. They characterized a decomposition of R relative to the generalized derivations.

Atteya [6] proved that if U is nonzero ideal of a semiprime ring R and R admits a generalized derivation D such that $D(xy) - xy \in Z(R)$ then R contains a nonzero central ideal.

Rehman [7, 8] studied the commutativity of a ring R by means of generalized derivations acting as homomorphisms and antihomomorphisms.

In this paper, we prove the following results

Let M be a 2-torsion-free semiprime Γ-ring satisfying the following assumption:

$$a\alpha b\beta c = a\beta b\alpha c \quad \forall a,b,c \in M,\ \alpha,\beta \in \Gamma, \tag{$*$}$$

and $D : M \to M$ be an additive mapping. If there exists a derivation d of M such that $D(x\alpha x) = D(x)\alpha x + x\alpha d(x)$ for all $x \in M$, $\alpha \in \Gamma$, then D is a Jordan generalized derivation.

2. Preliminaries

Let M and Γ be additive abelian groups. M is called a Γ-ring if there exists a mapping $M \times M \times M \to M$ such that for all $a,b,c \in M$, $\alpha,\beta \in \Gamma$ the following conditions are satisfied:

(i) $a\beta b \in M$,

(ii) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$, $(a\alpha b)\beta c = a\alpha(b\beta c)$.

For any $a,b \in M$ and for $\alpha,\beta \in \Gamma$ the expressions $a\alpha b - b\alpha a$ is denoted by $[a,b]_\alpha$ and $\alpha a\beta - \beta a\alpha$ are denoted by $[\alpha,\beta]_a$. Then one has the following identities:

$$\begin{aligned} [a\beta b, c]_\alpha &= a\beta[b,c]_\alpha + [a,c]_\alpha \beta b + a[\beta,\alpha]_c b, \\ [a, b\beta c]_\alpha &= b\beta[a,c]_\alpha + [a,b]_\alpha \beta c + b[\beta,\alpha]_a c, \end{aligned} \tag{2.1}$$

for all $a,b,c \in M$ and for all $\alpha,\beta \in \Gamma$. Using the assumption $(*)$ the above identities reduce to

$$\begin{aligned} [a\beta b, c]_\alpha &= a\beta[b,c]_\alpha + [a,c]_\alpha \beta b, \\ [a, b\beta c]_\alpha &= b\beta[a,c]_\alpha + [a,b]_\alpha \beta c, \end{aligned} \tag{2.2}$$

for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$.

Further M stands for a prime Γ-ring with center $Z(M)$. The ring M is n-torsion-free if $nx = 0$, $x \in M$ implies $x = 0$, where n is a positive integer, M is prime if $a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$, and it is semiprime if $a\Gamma M\Gamma a = 0$ implies $a = 0$. An additive mapping $T : M \to M$ is called a left (right) centralizer if $T(x\alpha y) = T(x)\alpha y (T(x\alpha y) = x\alpha T(y))$ for $x,y \in M$, $\alpha \in \Gamma$ and it is called a Jordan left (right) centralizer if

$$T(x\alpha x) = T(x)\alpha x (T(x\alpha x) = x\alpha T(x)) \quad \forall x \in M,\ \alpha \in \Gamma. \tag{2.3}$$

A mapping $\theta : M \times M \to M$ is called biadditive if it is additive in both arguments. An additive mapping $D : M \to M$ is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ for all

$x, y \in M$, $\alpha \in \Gamma$ and it is called a Jordan derivation if $D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$ for all $x \in M$, $\alpha \in \Gamma$. A derivation D is inner if there exists $a \in M$, such that $D(x) = a\alpha x - x\alpha a$ holds for all $x \in M$, $\alpha \in \Gamma$. Every derivation is a Jordan derivation. The converse is in general not true. An additive mapping $D : M \to M$ is said to be a generalized derivation if there exists a derivation $d : M \to M$ such that $D(x\alpha y) = D(x)\alpha y + x\alpha d(y)$ for all $x, y \in M$, $\alpha \in \Gamma$. The maps of the form $x \to a\alpha x + x\alpha b$ where a, b are fixed elements in M and for all $\alpha \in \Gamma$ called the generalized inner derivation. An additive mapping $D : M \to M$ is said to be a Jordan generalized derivation if there exists a derivation $d : M \to M$ such that $D(x\alpha x) = D(x)\alpha x + x\alpha d(x)$ for all $x \in M$, $\alpha \in \Gamma$. Hence the concept of a generalized derivation covers both the concepts of a derivation and a left centralizers and the concept of a Jordan generalized derivation covers both the concepts of a Jordan derivation and a left Jordan centralizers. An example of a generalized derivation and a Jordan generalized derivation is given in [4].

3. Main Results

We start from the following subsidiary results.

Lemma 3.1. *Let M be a semiprime Γ-ring. If $a, b \in M$ are such that $a\alpha x\beta b = 0$ for all $x \in M$, $\alpha, \beta \in \Gamma$, then $a\alpha b = b\alpha a = 0$.*

Proof. Let $x \in M$. Then

$$\begin{aligned} a\alpha b\beta x\gamma a\alpha b &= a\alpha\left(b\beta x\gamma a\right)\alpha b = 0, \\ b\alpha a\beta x\gamma b\alpha a &= b\alpha\left(a\beta x\gamma b\right)\alpha a = 0. \end{aligned} \tag{3.1}$$

By semiprimeness of M with respect to $\beta, \gamma \in \Gamma$, it follows that $a\alpha b = b\alpha a = 0$. □

Lemma 3.2. *Let M be a semiprime Γ-ring and $\theta, \varphi : M \times M \to M$ biadditive mappings. If $\theta(x, y)\alpha w\beta\varphi(x, y) = 0$ for all $x, y, w \in M$, then $\theta(x, y)\alpha w\beta\varphi(u, v) = 0$ for all $x, y, u, v, w \in M$, $\alpha, \beta \in \Gamma$.*

Proof. First we replace x with $x + u$ in the relation $\theta(x, y)\alpha w\beta\varphi(x, y) = 0$, and use the biadditivity of the θ and φ. Then we have

$$\theta(x+u, y)\alpha w\beta\varphi(x+u, y) = 0 \Longrightarrow \theta(x, y)\alpha w\beta\varphi(u, y) = -\theta(u, y)\alpha w\beta\varphi(x, y). \tag{3.2}$$

Then

$$\begin{aligned} &(\theta(x, y)\alpha w\beta\varphi(u, y))\gamma z\delta(\theta(x, y)\alpha w\beta\varphi(u, y)) \\ &\quad = -(\theta(u, y)\alpha w\beta\varphi(x, y))\gamma z\delta(\theta(u, y)\alpha w\beta\varphi(u, y)) = 0. \end{aligned} \tag{3.3}$$

Hence $\theta(x, y)\alpha w\beta\varphi(u, y) = 0$ by semiprimeness of M with respect to $\gamma, \delta \in \Gamma$.

Now we replace y by $y + v$ and obtain the assertion of the lemma with the similar observation as above. □

Lemma 3.3. *Let M be a semiprime Γ-ring satisfying the assumption $(*)$ and a be a fixed element of M. If $a\beta[x,y]_\alpha = 0$ for all $x,y \in M$, $\alpha,\beta \in \Gamma$, then $a \in Z(M)$.*

Proof. First we calculate the following expressions using the assumption $(*)$,

$$\begin{aligned}[z,a]_\alpha \beta x \delta [z,a]_\alpha &= z\alpha a\beta x\delta[z,a]_\alpha - a\alpha z\beta x\delta[z,a]_\alpha \\ &= z\alpha a\beta[z,x\alpha a]_\delta - z\alpha a\beta[z,x]_\delta \alpha a - a\alpha[z,z\delta x\alpha a]_\beta + a\alpha[z,z\delta x]_\beta \alpha a.\end{aligned} \tag{3.4}$$

Since $a\beta[x,y]_\alpha = 0$ for all $x,y \in M$, $\alpha,\beta \in \Gamma$, we get $[z,a]_\alpha\beta x\delta[z,a]_\alpha = 0$. By the semiprimeness of M we get $[z,a]_\alpha = 0$ for all $\alpha \in \Gamma$. Hence $a \in Z(M)$. □

Lemma 3.4. *Let M be a Γ-ring satisfying the condition $(*)$ and $D : M \to M$ be a Jordan generalized derivation with the associated derivation d. Let $x,y,z \in M$ and $\alpha,\beta \in \Gamma$. Then*

(i) $D(x\alpha y + y\alpha x) = D(x)\alpha y + D(y)\alpha x + x\alpha d(y) + y\alpha d(x)$.

In particular, if M is 2-torsion-free, then

(ii) $D(x\alpha y\beta x) = D(x)\alpha y\beta x + x\beta d(y\beta x)$,

(iii) $D(x\alpha y\beta z + z\beta y\alpha x) = D(x)\alpha y\beta z + D(z)\beta y\alpha x + x\alpha d(y\beta z) + z\beta d(y\alpha x)$.

Proof. (i) We have $D(x\alpha x) = D(x)\alpha x + x\alpha d(x)$, for all $x \in M$, $\alpha \in \Gamma$. Then replacing x by $x+y$, and following the series of implications below we get the result:

$$\begin{aligned}&D((x+y)\alpha(x+y)) \\ &\quad = D(x+y)\alpha(x+y) + (x+y)\alpha d(x+y) \\ &\Longrightarrow D(x\alpha x + x\alpha y + y\alpha x + y\alpha y) \\ &\quad = D(x)\alpha x + D(y)\alpha x + D(x)\alpha y + D(y)\alpha x + x\alpha d(y) + x\alpha d(x) + y\alpha d(y) + y\alpha d(x) \\ &\Longrightarrow D(x\alpha x + y\alpha y) + D(x\alpha y + y\alpha x) \\ &\quad = (D(x)\alpha x + x\alpha d(x) + D(y)\alpha y + y\alpha d(y)) + D(x)\alpha y \\ &\qquad + D(y)\alpha x + x\alpha d(y) + y\alpha d(x) \\ &\Longrightarrow D(x\alpha y + y\alpha x) \\ &\quad = D(x)\alpha y + D(y)\alpha x + x\alpha d(y) + y\alpha d(x), \quad \forall x,y \in M,\ \alpha \in \Gamma.\end{aligned} \tag{3.5}$$

(ii) Replace y by $x\beta y + y\beta x$ in the above relation (3.5), then we get,

$$\begin{aligned}&D(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) \\ &\quad = D(x)\alpha(x\beta y + y\beta x) + D(x\beta y + y\beta x)\alpha x \\ &\qquad + x\alpha d(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha d(x), \quad \forall x,y,z \in M,\ \alpha,\beta \in \Gamma. \\ &\Longrightarrow D(x\alpha x\beta y + y\beta x\alpha x) + D(x\alpha y\beta x) + D(x\beta y\alpha x) \\ &\quad = D(x)\alpha x\beta y + D(x)\alpha y\beta x + D(x)\beta y\alpha x + D(y)\beta x\alpha x + x\beta d(y)\alpha x + y\beta d(x)\alpha x \\ &\qquad + x\alpha d(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha d(x), \quad \forall x,y,z \in M,\ \alpha,\beta \in \Gamma.\end{aligned} \tag{3.6}$$

Using the assumption (∗), we conclude that

$$D(x\alpha x\beta y + y\beta x\alpha x) + 2D(x\alpha y\beta x) = D(x)\alpha x\beta y + D(x)\alpha y\beta x + D(x)\beta y\alpha x + D(y)\beta x\alpha x + x\beta d(y)\alpha x + y\beta d(x)\alpha x + x\alpha d(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha d(x), \quad \forall x, y, z \in M,\ \alpha, \beta \in \Gamma. \tag{3.7}$$

Again, replacing x by $x\beta x$ in (3.5)

$$\begin{aligned} D(x\beta x\alpha y + y\alpha x\beta x) &= D(x\beta x)\alpha y + D(y)\alpha x\beta x + x\beta x\alpha d(y) \\ &\quad + y\alpha d(x\beta x), \quad \forall x, y \in M,\ \alpha, \beta \in \Gamma, \\ \Longrightarrow D(x\beta x\alpha y + y\alpha x\beta x) &= D(x)\beta x\alpha y + x\beta d(x)\alpha y + D(y)\alpha x\beta x + x\beta x\alpha d(y) \\ &\quad + y\alpha d(x)\beta x + y\alpha x\beta d(x), \quad \forall x, y \in M,\ \alpha, \beta \in \Gamma. \end{aligned} \tag{3.8}$$

Adding both sides $2D(x\alpha y\beta x)$, we get,

$$\begin{aligned} &\Longrightarrow D(x\beta x\alpha y + y\alpha x\beta x) + 2D(x\alpha y\beta x) \\ &\quad = D(x)\beta x\alpha y + x\beta d(x)\alpha y + D(y)\alpha x\beta x + x\beta x\alpha d(y) \\ &\qquad + y\alpha d(x)\beta x + y\alpha x\beta d(x) + 2D(x\alpha y\beta x), \quad \forall x, y \in M,\ \alpha, \beta \in \Gamma. \end{aligned} \tag{3.9}$$

Comparing (3.7) and (3.9) we obtain,

$$2D(x\alpha y\beta x) = 2\{D(x)\alpha y\beta x + x\alpha y\beta d(x) + x\alpha d(y)\beta x\}. \tag{3.10}$$

Since M is 2-torsion-free, it gives

$$D(x\alpha y\beta x) = D(x)\alpha y\beta x + x\alpha d(y\beta x). \tag{3.11}$$

(iii) Replace x for $x + z$ in (3.12), we get,

$$\begin{aligned} &D((x+z)\alpha y\beta(x+z)) \\ &\quad = D(x+z)\alpha y\beta(x+z) + (x+z)\alpha d(y\beta(x+z)) \\ &\Longrightarrow D(x\alpha y\beta x + x\alpha y\beta z + z\alpha y\beta x + z\alpha y\beta z) \\ &\quad = D(x)\alpha y\beta x + D(z)\alpha y\beta x + D(x)\alpha y\beta z + D(z)\alpha y\beta z + x\alpha d(y\beta x) \\ &\qquad + x\alpha d(y\beta z) + z\alpha d(y\beta x + y\beta z) \\ &\Longrightarrow D(x\alpha y\beta x + z\alpha y\beta z) + D(x\alpha y\beta z + z\alpha y\beta x) \\ &\quad = (D(x)\alpha y\beta x + D(z)\alpha y\beta z + x\alpha d(y\beta x) + z\alpha d(y\beta z)) \\ &\qquad + (D(z)\alpha y\beta x + D(x)\alpha y\beta z + x\alpha d(y\beta z) + z\alpha d(y\beta x)) \end{aligned}$$

$$
\begin{aligned}
&\Longrightarrow D(x\alpha y\beta z + z\alpha y\beta x) \\
&\quad = D(x)\alpha y\beta z + D(z)\alpha y\beta x + x\alpha d(y\beta z) + z\alpha d(y\beta x) \\
&\Longrightarrow D(x\alpha y\beta z + z\beta y\alpha x) \\
&\quad = D(x)\alpha y\beta z + D(z)\alpha y\beta x + x\alpha d(y\beta z) + z\alpha d(y\beta x), \quad \forall x, y \in M,\ \alpha, \beta \in \Gamma.
\end{aligned}
\tag{3.12}
$$

□

Definition 3.5. Let M be a Γ-ring and $D : M \to M$ be a Jordan generalized derivation with the associated derivation d. Define $G_\alpha(x,y) = D(x\alpha y) - D(x)\alpha y - x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Lemma 3.6. *The function $G_\alpha(x,y)$ has the following properties:*

(i) $G_\alpha(x,y) + G_\alpha(y,x) = 0$.

(ii) $G_\alpha(x,y+z) = G_\alpha(x,y) + G_\alpha(x,z)$.

(iii) $G_\alpha(x+y,z) = G_\alpha(x,z) + G_\alpha(y,z)$.

(iv) $G_{\alpha+\beta}(x,y) = G_\alpha(x,y) + G_\beta(x,y)$.

Proof. The results easily follow from Lemma 3.4(i). □

Remark 3.7. D is a generalized derivation if and only if $G_\alpha(x,y) = 0$ for all $x, y \in M$, $\alpha \in \Gamma$.

Theorem 3.8. *Let M be a 2-torsion-free semiprime Γ-ring satisfying the condition $(*)$. Let $x, y, z \in M$ and $\beta, \delta \in \Gamma$. Then*

(i) $G_\alpha(x,y)\beta z\delta[x,y]_\alpha = 0$ *for all* $z \in M$,

(ii) $G_\alpha(x,y)\beta[x,y]_\alpha = 0$.

Proof. (i) From Lemma 3.4(iii) we get

$$
D(x\alpha y\beta z + z\beta y\alpha x) = D(x)\alpha y\beta z + D(z)\beta y\alpha x + x\alpha d(y\beta z) + z\beta d(y\alpha x). \tag{3.13}
$$

We set

$$
A = x\alpha y\beta z\delta y\alpha x + y\alpha x\beta z\delta x\alpha y. \tag{3.14}
$$

Since

$$
\begin{aligned}
D(A) &= D(x\alpha y\beta z\delta y\alpha x + y\alpha x\beta z\delta x\alpha y) \\
&= D(x\alpha(y\beta z\delta y\alpha x) + y\alpha(x\beta z\delta x\alpha y)) \\
&= D(x)\alpha y\beta z\delta y\alpha x + x\alpha d(y\beta z\delta y\alpha x) + D(y)\alpha x\beta z\delta x\alpha y + y\alpha d(x\beta z\delta x\alpha y)
\end{aligned}
$$

$$= D(x)\alpha y\beta z\delta y\alpha x + x\alpha d(y)\beta z\delta y\alpha x + x\alpha y\beta d(z\delta y\alpha x) + D(y)\alpha x\beta z\delta x\alpha y$$
$$+ y\alpha d(x)\beta z\delta x\alpha y + y\alpha x\beta d(z\delta x\alpha y), \quad \forall x, y, z \in M,\ \alpha, \beta, \delta \in \Gamma. \tag{3.15}$$

Again

$$\begin{aligned} D(A) &= D(x\alpha y\beta z\delta y\alpha x + y\alpha x\beta z\delta x\alpha y) \\ &= D((x\alpha y)\beta z\delta(y\alpha x) + (y\alpha x)\beta z\delta(x\alpha y)) \\ &= D(x\alpha y)\beta z\delta y\alpha x + D(y\alpha x)\beta z\delta x\alpha y + x\alpha y\beta d(z\delta y\alpha x) \\ &\quad + y\alpha x\beta d(z\delta x\alpha y), \quad \forall x, y, z \in M,\ \alpha, \beta, \delta \in \Gamma. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16) we find,

$$\begin{gathered} (D(x\alpha y) - D(x)\alpha y - x\alpha d(y))\beta z\delta y\alpha x + (D(y\alpha x) - D(y)\alpha x - y\alpha d(x))\beta z\delta x\alpha y = 0. \\ \Longrightarrow G_\alpha(x, y)\beta z\delta y\alpha x + G_\alpha(y, x)\beta z\delta x\alpha y = 0, \\ \Longrightarrow G_\alpha(x, y)\beta z\delta y\alpha x - G_\alpha(x, y)\beta z\delta x\alpha y = 0, \\ \Longrightarrow G_\alpha(x, y)\beta z\delta[y, x]_\alpha = 0, \\ \Longrightarrow G_\alpha(x, y)\beta z\delta[x, y]_\alpha = 0, \quad \text{for } x, y \in M,\ \alpha, \beta, \delta \in \Gamma. \end{gathered} \tag{3.17}$$

(ii) According to Lemma 3.4(ii) we have,

$$D(x\alpha y\beta z) = D(x)\alpha y\beta z + x\alpha d(y\beta z). \tag{3.18}$$

Replace z by $x\alpha y$ in (3.18) we find

$$\begin{gathered} D(x\alpha y\beta x\alpha y) = D(x)\alpha y\beta x\alpha y + x\alpha d(y\beta x\alpha y) \\ \Longrightarrow D((x\alpha y)\beta(x\alpha y)) = D(x)\alpha y\beta x\alpha y + x\alpha d(y\beta x\alpha y) \\ \Longrightarrow D(x\alpha y)\beta x\alpha y + x\alpha y\beta d(x\alpha y) - D(x)\alpha y\beta x\alpha y - x\alpha d(y\beta x\alpha y) = 0 \\ \Longrightarrow (D(x\alpha y) - D(x)\alpha y - x\alpha d(y))\beta x\alpha y) + x\alpha y\beta d(x\alpha y) - x\alpha y\beta d(x\alpha y) = 0 \\ \Longrightarrow G_\alpha(x, y)\beta x\alpha y = 0. \end{gathered} \tag{3.19}$$

Similarly $G_\alpha(x, y)\beta y\alpha x = 0$. Therefore,

$$G_\alpha(x, y)\beta x\alpha y - G_\alpha(x, y)\beta y\alpha x = G_\alpha(x, y)\beta[x, y]_\alpha = 0. \tag{3.20}$$

□

Lemma 3.9. *$G_\alpha(x, y) \in Z(M)$, for all $x, y \in M$, $\alpha \in \Gamma$.*

Proof. From Theorem 3.8(ii), we have $G_\alpha(x, y)\beta[x, y]_\alpha = 0$. Therefore due to Lemma 3.4(ii) $G_\alpha(x, y) \in Z(M)$. □

Theorem 3.10. *Let M be a 2-torsion-free semiprime Γ-ring satisfying the assumption $(*)$ and $D : M \to M$ be a Jordan generalized derivation with associated derivation d on M. Then D is a generalized derivation.*

Proof. In particular, $r\gamma G_\alpha(x,y), G_\alpha(x,y)\gamma r \in Z(M)$ for all $r \in M$, $\alpha, \gamma \in \Gamma$. This gives

$$\begin{aligned} x\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta y &= G_\alpha(x,y)\delta G_\alpha(x,y)\beta y\alpha x \\ &= y\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x \\ &= y\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta x. \end{aligned} \tag{3.21}$$

Then $4D(x\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta y) = 4D(y\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x)$. Now we will compute each side of this equality by using (3.15) and the above relation,

$$\begin{aligned} &4D(x\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta y) \\ &\quad= 2D(x\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta y + G_\alpha(x,y)\delta G_\alpha(x,y)\beta y\alpha x) \\ &\quad= 2D(x)\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta y + 2x\alpha d(G_\alpha(x,y)\delta G_\alpha(x,y)\beta y) \\ &\qquad+ 2D(G_\alpha(x,y)\delta G_\alpha(x,y)\beta y)\alpha x + 2G_\alpha(x,y)\delta G_\alpha(x,y)\beta y\alpha d(x) \\ &\quad= 2D(x)\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta y + D(G_\alpha(x,y)\delta G_\alpha(x,y)\beta y + y\beta G_\alpha(x,y)\delta G_\alpha(x,y))\alpha x \\ &\qquad+ 2x\alpha d(G_\alpha(x,y)\delta G_\alpha(x,y)\beta y) + 2G_\alpha(x,y)\delta G_\alpha(x,y)\beta y\alpha d(x) \\ &\quad= 2D(x)\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta y + D(G_\alpha(x,y))\delta G_\alpha(x,y)\beta y\alpha x \\ &\qquad+ G_\alpha(x,y)\delta d(G_\alpha(x,y))\beta y\alpha x + D(y)\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x \\ &\qquad+ G_\alpha(x,y)\delta G_\alpha(x,y)\beta d(y)\alpha x + y\beta d(G_\alpha(x,y)\delta G_\alpha(x,y))\alpha x \\ &\qquad+ 2x\alpha d(G_\alpha(x,y)\delta G_\alpha(x,y)\beta y) + 2G_\alpha(x,y)\delta G_\alpha(x,y)\beta y\alpha d(x). \end{aligned} \tag{3.22}$$

So we get

$$\begin{aligned} &4D(x\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha y) \\ &\quad= 2D(x)\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha y + D(G_\alpha(x,y))\delta G_\alpha(x,y)\beta y\alpha x \\ &\qquad+ G_\alpha(x,y)\delta d(G_\alpha(x,y))\beta yx + D(y)\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x \\ &\qquad+ G_\alpha(x,y)\delta G_\alpha(x,y)\beta d(y)\alpha x + y\beta d(G_\alpha(x,y)\delta G_\alpha(x,y))\alpha x \\ &\qquad+ 2x\alpha d(G_\alpha(x,y)\delta G_\alpha(x,y)\beta y) + 2G_\alpha(x,y)\delta G_\alpha(x,y)\beta y\alpha d(x), \quad x,y \in M,\ \alpha,\beta,\delta \in \Gamma. \end{aligned} \tag{3.23}$$

Moreover,

$$\begin{aligned} &4D(y\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x) \\ &\quad= 2D(y\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x + G_\alpha(x,y)\delta G_\alpha(x,y)\beta x\alpha y) \\ &\quad= 2D(y)\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x + 2y\beta d(G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x) \\ &\qquad+ 2D(G_\alpha(x,y)\delta G_\alpha(x,y)\beta x)\alpha y + 2G_\alpha(x,y)\delta G_\alpha(x,y)\beta x\alpha d(y) \end{aligned}$$

$$
\begin{aligned}
&= 2D(y)\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x + D(G_\alpha(x,y)\delta G_\alpha(x,y)\beta x + x\beta G_\alpha(x,y)\delta G_\alpha(x,y))\alpha y \\
&\quad + 2y\beta d(G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x) + 2G_\alpha(x,y)\delta G_\alpha(x,y)\beta x\alpha d(y) \\
&= 2D(y)\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x + D(G_\alpha(x,y))\delta G_\alpha(x,y)\beta x\alpha y \\
&\quad + G_\alpha(x,y)\delta d(G_\alpha(x,y))\beta x\alpha y + D(x)\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha y \\
&\quad + G_\alpha(x,y)\delta G_\alpha(x,y)\alpha d(x)\beta y + x\alpha d(G_\alpha(x,y)\delta G_\alpha(x,y))\beta y \\
&\quad + 2y\beta d(G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x) + 2G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x\beta d(y).
\end{aligned}
\tag{3.24}
$$

So we get

$$
\begin{aligned}
&4D(y\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x) \\
&\quad = 2D(y)\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x + D(G_\alpha(x,y))\delta G_\alpha(x,y)\alpha x\beta y \\
&\qquad + G_\alpha(x,y)\delta d(G_\alpha(x,y))\alpha x\beta y + D(x)\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha y \\
&\qquad + G_\alpha(x,y)\delta G_\alpha(x,y)\alpha d(x)\beta y + x\alpha d(G_\alpha(x,y)\delta G_\alpha(x,y))\beta y \\
&\qquad + 2y\beta d(G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x) + 2G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x\beta d(y), \\
&\qquad\qquad x,y \in M,\ \alpha,\beta,\delta \in \Gamma.
\end{aligned}
\tag{3.25}
$$

Comparing the results of (3.22) and (3.24) and using the above relations

$$
\begin{aligned}
G_\alpha(x,y)\beta y\alpha x &= G_\alpha(x,y)\beta y\alpha x \\
&= x\alpha G_\alpha(x,y)\beta y \\
&= x\alpha G_\alpha(x,y)\beta y \\
&= G_\alpha(x,y)\beta x\alpha y, \\
G_\alpha(x,y)\delta d(G_\alpha(x,y))\beta y\alpha x &= d(G_\alpha(x,y))\delta G_\alpha(x,y)\beta y\alpha x \\
&= d(G_\alpha(x,y))\delta G_\alpha(x,y)\alpha x\beta y \\
&= G_\alpha(x,y)\delta d(G_\alpha(x,y))\alpha x\beta y, \\
x\beta G_\alpha(x,y)\delta d(G_\alpha(x,y))\alpha y &= d(G_\alpha(x,y))\alpha x\delta G_\alpha(x,y)\beta y \\
&= d(G_\alpha(x,y))\delta G_\alpha(x,y)\beta x\alpha y \\
&= d(G_\alpha(x,y))\delta G_\alpha(x,y)\alpha y\beta x \\
&= G_\alpha(x,y)\beta y\delta d(G_\alpha(x,y))\alpha x \\
&= y\beta G_\alpha(x,y)\delta d(G_\alpha(x,y))\alpha x,
\end{aligned}
\tag{3.26}
$$

we obtain

$$
\begin{aligned}
&D(x)\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta y + x\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta d(y) \\
&\quad = D(y)\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta x + y\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta d(x),
\end{aligned}
\tag{3.27}
$$

which gives

$$\varphi_\alpha(x,y)\beta G_\alpha(x,y)\delta G_\alpha(x,y) = \varphi_\alpha(y,x)\alpha G_\alpha(x,y)\delta G_\alpha(x,y), \tag{3.28}$$

where $\varphi_\alpha(x,y)$ stands for $D(x)\alpha y + x\alpha d(y)$.

On the other hand, we have

$$4D(x\alpha y\beta G_\alpha(x,y)\delta G_\alpha(x,y)) = 4D(x\alpha G_\alpha(x,y)\delta y\beta G_\alpha(x,y)). \tag{3.29}$$

Now we use (3.15) and the properties of $G_\alpha(x,y)$, to derive

$$\begin{aligned}
&4D(x\alpha y\beta G_\alpha(x,y)\delta G_\alpha(x,y))\\
&\quad= 2D(x\alpha y\beta G_\alpha(x,y)\delta G_\alpha(x,y) + G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x\beta y)\\
&\quad= 2D(x\alpha y)\beta G_\alpha(x,y)\delta G_\alpha(x,y) + 2x\alpha y\beta d(G_\alpha(x,y)\delta G_\alpha(x,y))\\
&\qquad+ 2D(G_\alpha(x,y)\delta G_\alpha(x,y))\alpha x\beta y + 2G_\alpha(x,y)\delta G_\alpha(x,y)\beta d(x\alpha y),
\end{aligned} \tag{3.30}$$

which gives

$$\begin{aligned}
4D(x\alpha y\beta G_\alpha(x,y)\delta G_\alpha(x,y)) &= 2D(x\alpha y)\beta G_\alpha(x,y)\delta G_\alpha(x,y) + 2x\alpha y\beta d(G_\alpha(x,y)\delta G_\alpha(x,y))\\
&\quad+ 2D(G_\alpha(x,y)\delta G_\alpha(x,y))\alpha x\beta y\\
&\quad+ 2G_\alpha(x,y)\delta G_\alpha(x,y)\beta d(x\alpha y), \quad x,y\in M,\ \alpha,\beta,\delta\in\Gamma.
\end{aligned} \tag{3.31}$$

Moreover,

$$\begin{aligned}
&4D(x\alpha G_\alpha(x,y)\delta y\beta G_\alpha(x,y))\\
&\quad= 2D(x\alpha G_\alpha(x,y)\delta y\beta G_\alpha(x,y) + y\beta G_\alpha(x,y)\delta x\alpha G_\alpha(x,y))\\
&\quad= 2D(G_\alpha(x,y)\alpha x)\delta G_\alpha(x,y)\beta y + 2G_\alpha(x,y)\alpha x\delta d(G_\alpha(x,y)\beta y)\\
&\qquad+ 2D(G_\alpha(x,y)\beta y)\delta G_\alpha(x,y)\alpha x + 2G_\alpha(x,y)\beta y\delta d(G_\alpha(x,y)\alpha x)\\
&\quad= D(x\alpha G_\alpha(x,y) + G_\alpha(x,y)\alpha x)\delta G_\alpha(x,y)\beta y + 2G_\alpha(x,y)\alpha x\delta d(G_\alpha(x,y)\beta y)\\
&\qquad+ D(y\beta G_\alpha(x,y) + G_\alpha(x,y)\beta y)\delta G_\alpha(x,y)\alpha x + 2G_\alpha(x,y)\beta y\delta d(G_\alpha(x,y)\alpha x)\\
&\quad= D(x)\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta y + D(G_\alpha(x,y))\delta G_\alpha(x,y)\alpha x\beta y\\
&\qquad+ x\alpha d(G_\alpha(x,y))\delta G_\alpha(x,y)\beta y + G_\alpha(x,y)\alpha d(x)\delta G_\alpha(x,y)\beta y\\
&\qquad+ 2G_\alpha(x,y)\alpha x\delta d(G_\alpha(x,y)\beta y) + D(y)\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x\\
&\qquad+ D(G_\alpha(x,y))\delta G_\alpha(x,y)\beta y\alpha x + y\beta d(G_\alpha(x,y))\delta G_\alpha(x,y)\alpha x\\
&\qquad+ G_\alpha(x,y)\beta d(y)\delta G_\alpha(x,y)\alpha x + 2G_\alpha(x,y)\beta y\delta d(G_\alpha(x,y)\alpha x).
\end{aligned} \tag{3.32}$$

So we obtain

$$\begin{aligned}
&4D(x\alpha G_\alpha(x,y)\delta y\beta G_\alpha(x,y)) \\
&\quad = D(x)\alpha G_\alpha(x,y)\delta G_\alpha(x,y)\beta y + D(G_\alpha(x,y))\delta G_\alpha(x,y)\beta x\alpha y \\
&\qquad + x\alpha d(G_\alpha(x,y))\delta G_\alpha(x,y)\beta y + G_\alpha(x,y)\alpha d(x)\delta G_\alpha(x,y)\beta y \\
&\qquad + 2G_\alpha(x,y)\alpha x\delta d(G_\alpha(x,y)\beta y) + D(y)\beta G_\alpha(x,y)\delta G_\alpha(x,y)\alpha x \\
&\qquad + D(G_\alpha(x,y))\delta G_\alpha(x,y)\beta y\alpha x + y\beta d(G_\alpha(x,y))\delta G_\alpha(x,y)\alpha x \\
&\qquad + G_\alpha(x,y)\beta d(y)\delta G_\alpha(x,y)\alpha x + 2G_\alpha(x,y)\beta y\delta d(G_\alpha(x,y)\alpha x), \\
&\qquad\qquad x,y\in M,\ \alpha,\beta,\delta\in\Gamma.
\end{aligned} \tag{3.33}$$

Comparing (3.31) and (3.33), we derive

$$\begin{aligned}
&2D(x\alpha y)\beta G_\alpha(x,y)\delta G_\alpha(x,y) \\
&\quad = \varphi_\alpha(x,y)\beta G_\alpha(x,y)\delta G_\alpha(x,y) + \varphi_\alpha(y,x)\beta G_\alpha(x,y)\delta G_\alpha(x,y), \quad x,y\in M,\ \alpha,\beta,\delta\in\Gamma.
\end{aligned} \tag{3.34}$$

Finally using (3.31) we get $D(x\alpha y)\beta G_\alpha(x,y)\delta G_\alpha(x,y) = \varphi_\alpha(x,y)\beta G_\alpha(x,y)\delta G_\alpha(x,y)$. But $G_\alpha(x,y) = D(x\alpha y) - \varphi_\alpha(x,y)$. By the semiprimeness of M, we have $\varphi_\alpha(x,y)\beta G_\alpha(x,y) = 0$. Again by the primeness of M, we get $G_\alpha(x,y)$. The proof is complete. □

It is clear that if we let the derivation d to be the zero derivation in the above theorem, we get the following result.

Theorem 3.11. *Let M be a 2-torsion-free semiprime Γ-ring and $D : M \to M$ be an additive mapping which satisfies $D(x\alpha x) = D(x)\alpha x$ for all $x\in M$, $\alpha\in\Gamma$. Then D is a left centralizer*

Proof. We have

$$D(x\alpha x) = D(x)\alpha x. \tag{3.35}$$

If we replace x by $x+y$, we get

$$D(x\alpha y + y\alpha x) = D(x)\alpha y + D(y)\alpha x. \tag{3.36}$$

By replacing y with $x\alpha y + y\alpha x$ and using (3.5), we arrive at

$$D(x\alpha(x\alpha y + y\alpha x) + (x\alpha y + y\alpha x)\alpha x) = D(x)\alpha x\alpha y + D(x)\alpha y\alpha x + D(x)\alpha y\alpha x + D(y)\alpha x\alpha x. \tag{3.37}$$

But on the other hand,

$$D(x\alpha x\alpha y + y\alpha x\alpha x) + 2D(x\alpha y\alpha x) = D(x)\alpha x\alpha y + D(y)\alpha x\alpha x + 2D(x\alpha y\alpha x). \tag{3.38}$$

Comparing (3.37) and (3.38) we obtain

$$D(x\alpha y\alpha x) = D(x)\alpha y\alpha x. \tag{3.39}$$

If we linearize (3.39) in x, we get

$$D(x\alpha y\alpha z + z\alpha y\alpha x) = D(x)\alpha y\alpha z + D(z)\alpha y\alpha x. \tag{3.40}$$

Now we shall compute $D(x\alpha y\alpha z\alpha y\alpha x + y\alpha x\alpha z\alpha x\alpha y)$ in two different ways. If we use (3.39) we have

$$D(x\alpha y\alpha z\alpha y\alpha x + y\alpha x\alpha z\alpha x\alpha y) = D(x)\alpha y\alpha z\alpha y\alpha x + D(y)\alpha x\alpha z\alpha x\alpha y. \tag{3.41}$$

But if we use (3.40) we have

$$D(x\alpha y\alpha z\alpha y\alpha x + y\alpha x\alpha z\alpha x\alpha y) = D(x\alpha y)\alpha z\alpha y\alpha x + D(y\alpha x)\alpha z\alpha x\alpha y. \tag{3.42}$$

Comparing (3.41) and (3.42) and introducing a bi-additive mapping $G_\alpha(x,y) = D(x\alpha y) - D(x)\alpha y$ we arrive at

$$G_\alpha(x,y)\alpha z\alpha y\alpha x + G_\alpha(y,x)\alpha z\alpha x\alpha y = 0. \tag{3.43}$$

Equality (3.36) can be rewritten in this notation as $G_\alpha(x,y) = -G_\alpha(y,x)$. Using this fact and (3.43) we obtain

$$G_\alpha(x,y)\alpha z\alpha[x,y]_\alpha = 0. \tag{3.44}$$

Using first Lemma 3.2 and then Lemma 3.1 we have

$$G_\alpha(x,y)\alpha z\alpha[u,v]_\alpha = 0. \tag{3.45}$$

Now fix some $x, y \in M$ and using Lemma 3.3 we get $G_\alpha(x,y) \in Z$.

In particular, $r\beta G_\alpha(x,y)$, $G_\alpha(x,y)\beta r \in Z$ for all $r \in M$. This gives

$$\begin{aligned} x\beta G_\alpha(x,y)\delta G_\alpha(x,y)\beta y &= G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y\Gamma x \\ &= y\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x \\ &= y\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x. \end{aligned} \tag{3.46}$$

Therefore $4D(x\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y) = 4D(y\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x)$. Both sides of this equality will be computed in few steps using (3.36),

$$\begin{aligned}
&2D(x\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y + G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y\Gamma x)\\
&\quad = 2D(y\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x + G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x\Gamma y),\\
&2D(x)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y + 2D(G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y)\Gamma x\\
&\quad = 2D(y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x + 2D(G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x)\Gamma y,\\
&2D(x)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y + D(G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y + y\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y))\Gamma x\\
&\quad = 2D(y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x + D(G_\alpha(x,y))\Gamma G_\alpha(x,y)\Gamma x + x\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y,\\
&2D(x)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y + D(G_\alpha(x,y))\Gamma G_\alpha(x,y)\Gamma y\Gamma x + D(y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x\\
&\quad = 2D(y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x + D(G_\alpha(x,y))\Gamma G_\alpha(x,y)\Gamma x\Gamma y\\
&\qquad + D(x)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y,\\
&D(x)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y + D(G_\alpha(x,y))\Gamma G_\alpha(x,y)\Gamma y\Gamma x\\
&\quad = D(y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x + D(G_\alpha(x,y))\Gamma G_\alpha(x,y)\Gamma x\Gamma y.
\end{aligned} \tag{3.47}$$

Since $G_\alpha(x,y)\Gamma y\Gamma x = G_\alpha(x,y)\Gamma y\Gamma x = x\Gamma G_\alpha(x,y)\Gamma y = x\Gamma G_\alpha(x,y)\Gamma y = G_\alpha(x,y)\Gamma x\Gamma y$, we obtain

$$D(x)\Gamma G_\alpha(x,y)\Gamma y = D(y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x. \tag{3.48}$$

On the other hand, we also have

$$\begin{aligned}
&4D(x\Gamma y\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)) = 4D(x\Gamma G_\alpha(x,y)\Gamma y\Gamma G_\alpha(x,y)),\\
&2D(x\Gamma y\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y) + G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x\Gamma y)\\
&\quad = 2D(x\Gamma G_\alpha(x,y)\Gamma y\Gamma G_\alpha(x,y) + y\Gamma G_\alpha(x,y)\Gamma x\Gamma G_\alpha(x,y)),\\
&2D(x\Gamma y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y) + 2D(G_\alpha(x,y))\Gamma G_\alpha(x,y)\Gamma x\Gamma y\\
&\quad = 2D(G_\alpha(x,y)\Gamma x)\Gamma G_\alpha(x,y)\Gamma y + 2D(G_\alpha(x,y)\Gamma y)\Gamma G_\alpha(x,y)\Gamma x,\\
&2D(x\Gamma y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y) + 2D(G_\alpha(x,y))\Gamma G_\alpha(x,y)\Gamma x\Gamma y\\
&\quad = D(x\Gamma G_\alpha(x,y) + G_\alpha(x,y)\Gamma x)\Gamma G_\alpha(x,y)\Gamma y + D(y\Gamma G_\alpha(x,y) + G_\alpha(x,y)\Gamma y)\Gamma G_\alpha(x,y)\Gamma x,\\
&2D(x\Gamma y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y) + 2D(G_\alpha(x,y))\Gamma G_\alpha(x,y)\Gamma x\Gamma y\\
&\quad = D(x)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma y + D(G_\alpha(x,y))\Gamma G_\alpha(x,y)\Gamma x\Gamma y\\
&\qquad + D(y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma x + D(G_\alpha(x,y))\Gamma G_\alpha(x,y)\Gamma x\Gamma y,\\
&2D(x\Gamma y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\\
&\quad = D(x)\Gamma y\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y) + D(y)\Gamma x\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y).
\end{aligned} \tag{3.49}$$

Finally using (3.48) we arrive at $D(x\Gamma y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y) = D(x)\Gamma y\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)$, but this means that $G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y) = 0$. Hence,

$$G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma M\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y) = G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma M = 0,$$
$$G_\alpha(x,y)\Gamma M\Gamma G_\alpha(x,y) = G_\alpha(x,y)\Gamma G_\alpha(x,y)\Gamma M = 0, \tag{3.50}$$

which implies $G_\alpha(x,y) = 0$. The proof is complete. □

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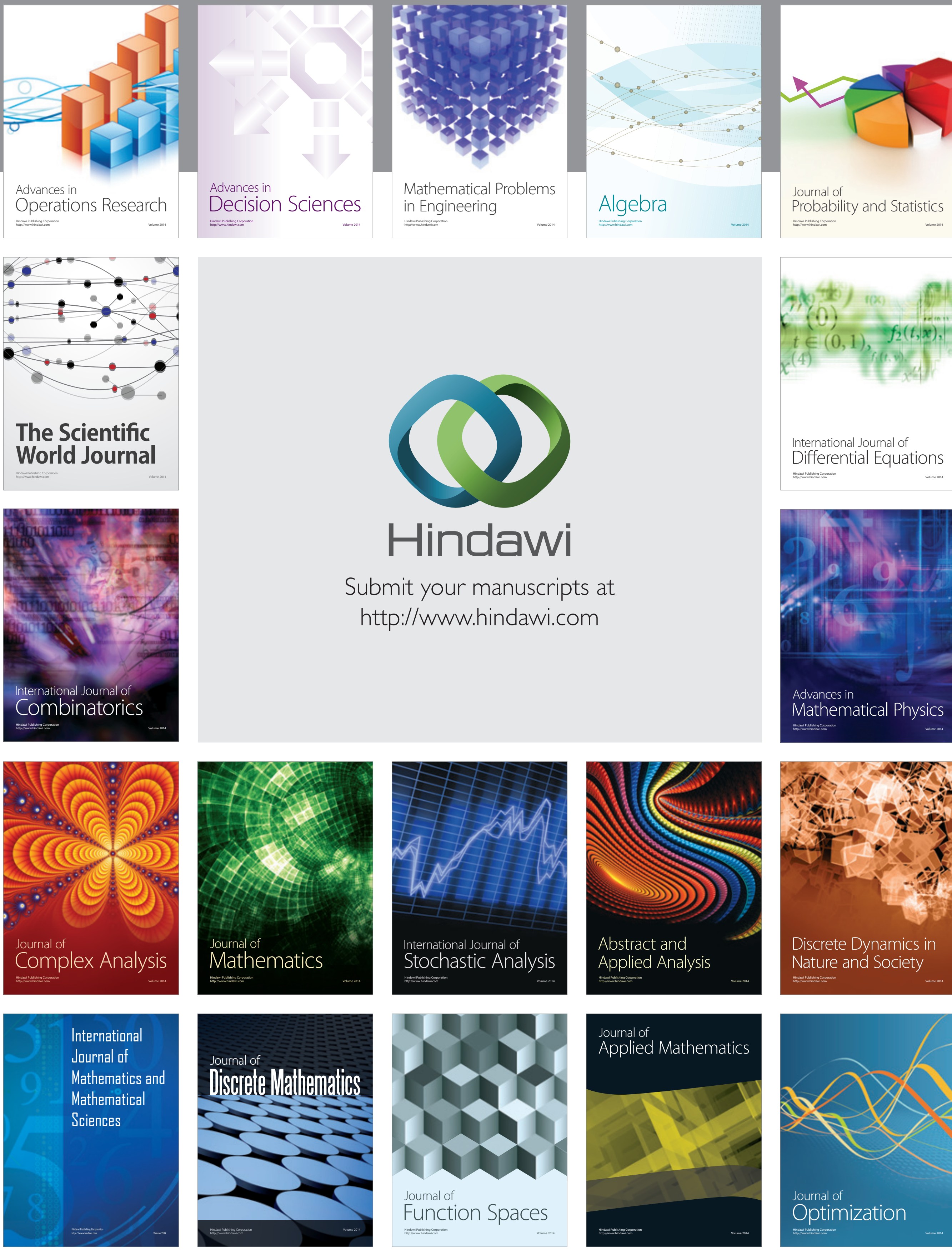
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