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Research Article

Calculating Zeros of the q-Genocchi Polynomials Associated with p-Adic q-Integral on $\mathbb{Z}_p$

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In this paper we construct the new analogues of Genocchi the numbers and polynomials. We also observe the behavior of complex roots of the q-Genocchi polynomials $G_{n,q}(x)$, using numerical investigation. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the q-Genocchi polynomials $G_{n,q}(x)$. Finally, we give a table for the solutions of the q-Genocchi polynomials $G_{n,q}(x)$.

1. Introduction

Many mathematicians have the studied Bernoulli numbers and polynomials, the Euler numbers and polynomials, and the Genocchi numbers and the Genocchi polynomials. The Bernoulli numbers and polynomials, the Euler numbers and polynomials, and the Genocchi polynomials posses many interesting properties and arising in many areas of mathematics and physics (see [1–12]). We introduce the new analogs of the Genocchi numbers and polynomials. In the 21st century, the computing environment would make more and more rapid progress. Using computer, a realistic study for new analogs of Genocchi numbers and polynomials is very interesting. It is the aim of this paper to observe an interesting phenomenon of "scattering" of the zeros of q-Genocchi polynomials $G_{n,q}(x)$. The outline of this paper is as follows. In Section 2, we study the q-Genocchi polynomials $G_{n,q}(x)$. In Section 3, we describe the beautiful zeros of q-Genocchi polynomials $G_{n,q}(x)$ using a numerical investigation. Also we display distribution and structure of the zeros of the q-Genocchi polynomials $G_{n,q}(x)$ by using computer. By using the results of our paper, the readers can observe the regular behaviour of the roots of q-Genocchi polynomials $G_{n,q}(x)$. Finally, we carried out computer experiments that demonstrate a remarkably regular structure of the complex roots of q-Genocchi polynomials $G_{n,q}(x)$. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in C_p$. If $q \in C$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Compare [1, 2, 4, 10, 11, 13–16]. Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$X = \lim_{\overleftarrow{N}}\left(\frac{\mathbb{Z}}{dp^N\mathbb{Z}}\right), \qquad X^* = \bigcup_{\substack{0<a<dp\\ (a,p)=1}} (a+dp\mathbb{Z}_p), \quad a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a\left(\operatorname{mod} dp^N\right)\right\}, \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. For any positive integer N,

$$\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{q^a}{\left[dp^N\right]_q} \tag{1.3}$$

is known to be a distribution on X, cf. [1, 2, 4, 5, 9, 10, 13]. We say that g is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $g \in UD(\mathbb{Z}_p)$, if the difference quotients $F_g(x,y) = f(x) - f(y)/(x-y)$ have a limit $l = g'(a)$ as $(x,y) \to (a,a)$. For

$$g \in UD(\mathbb{Z}_p) = \{g \mid g : \mathbb{Z}_p \longrightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}, \tag{1.4}$$

the q-deformed bosonic p-adic integral of the function g is defined by Kim:

$$I_q(g) = \int_{\mathbb{Z}_p} g(x)\,d\mu_q(x) = \lim_{N\to\infty}\frac{1}{\left[p^N\right]_q}\sum_{0\le x<p^N} g(x)q^x. \tag{1.5}$$

Note that

$$qI_q(f_1) = I_q(f) + (q-1)f(0) + \frac{q-1}{\log q}f'(0), \tag{1.6}$$

where $f_1(x) = f(x+1)$, $f'(0) = df(0)/dx$. Now, the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{0\le x<p^N} g(x)(-q)^x. \tag{1.7}$$

If we take $g_1(x) = g(x+1)$ in (1.7), then we easily see that

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0). \tag{1.8}$$

From (1.8), we obtain

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l), \tag{1.9}$$

where $g_n(x) = g(x+n)$. First, we introduce the Genocchi numbers and the Genocchi polynomials. The Genocchi numbers G_n are defined by the generating function:

$$F(t) = \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.10}$$

Compare [4, 9–11, 17], where we use the technique method notation by replacing G^n by $G_n (n \ge 0)$ symbolically. We consider the Genocchi polynomials $G_n(x)$ as follows:

$$F(x,t) = \frac{2t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \tag{1.11}$$

Note that $G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}$. In the special case $x = 0$, we define $G_n(0) = G_n$.

2. An Analogue of the Genocchi Numbers and Polynomials

The versions of q-Genocchi numbers and polynomials, which were derived from different considerations and different formulas, were defined by Kim [13, 14]. Kim [14] treated analogue of the Genocchi numbers, which is called q-analogue of the Genocchi numbers. Kim defined the q-extension of the Genocchi numbers and polynomials as follows:

$$\begin{aligned} F_q(t) &= \sum_{n=0}^{\infty} c_{n,q} \frac{t^n}{n!} = e^{t/(1-q)} \sum_{n=0}^{\infty} \frac{(2n+1)}{[2n+1]_q} [n]_q \left(\frac{1}{q-1}\right)^{n-1} \frac{t^n}{n!}, \\ F_q(x,t) &= \sum_{n=0}^{\infty} c_{n,q}(x) \frac{t^n}{n!} = e^{t/(1-q)} \sum_{n=0}^{\infty} \frac{(2n+1)}{[2n+1]_q} [n]_q \left(\frac{1}{q-1}\right)^{n-1} q^{nx} \frac{t^n}{n!}. \end{aligned} \tag{2.1}$$

In [14], Kim introduced the q-analogue of the Genocchi polynomials as follows:

$$G_q(x,t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}. \tag{2.2}$$

We now consider another construction q-Genocchi numbers and polynomials. In (1.8), if we take $g(x) = e^{xt}$, then one has

$$(\log q + t) \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]_q (\log q + t)}{qe^t + 1}. \tag{2.3}$$

Let us define the q-Genocchi numbers and polynomials as follows:

$$(\log q + t) \int_{\mathbb{Z}_p} e^{yt} d\mu_{-q}(y) = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}, \tag{2.4}$$

$$(\log q + t) \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}. \tag{2.5}$$

Note that $G_{n,q}(0) = G_{n,q}$, $\lim_{q\to 1} G_{n,q} = G_n$, where G_n are the nth Genocchi numbers. By (2.4) and (2.5), we obtain the following Witt's formula.

Theorem 2.1. *For $q \in \mathbb{C}_p$ with $|1-q|_p \le p^{-1/(p-1)}$, we have*

$$\begin{gathered} n \int_{\mathbb{Z}_p} x^{n-1} d\mu_{-q}(x) + \log q \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = G_{n,q}, \\ n \int_{\mathbb{Z}_p} (x+y)^{n-1} d\mu_{-q}(y) + \log q \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) = G_{n,q}(x). \end{gathered} \tag{2.6}$$

By the above theorem, easily see that

$$G_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} G_{k,q}. \tag{2.7}$$

Let q be a complex number with $|q| < 1$. By the meaning of (1.10) and (1.11), let us define the q-Genocchi numbers $G_{n,q}$ and polynomials $G_{n,q}(x)$ as follows:

$$\begin{gathered} F_q(t) = \frac{[2]_q (\log q + t)}{qe^t + 1} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}, \\ F_q(x,t) = \frac{[2]_q (\log q + t)}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}. \end{gathered} \tag{2.8}$$

For q-Euler numbers, Kim constructed q-Euler numbers which can be uniquely determined by

$$q(qE_q+1)^n + E_{n,q} = \begin{cases} [2]_q, & \text{if } n=0, \\ 0, & \text{if } n>0, \end{cases} \tag{2.9}$$

with the usual convention of symbolically replacing E_q^n by $E_{n,q}$, where $E_{n,q}$ denotes the q-Euler numbers. For q-Genocchi numbers, we have the following theorem.

Theorem 2.2. *q-Genocchi numbers $G_{n,q}$ are defined inductively by*

$$G_{0,q} = \frac{[2]_q \log q}{1+q}, \qquad q(G_q+1)^n + G_{n,q} = \begin{cases} [2]_q, & \text{if } n=1, \\ 0, & \text{if } n>1, \end{cases} \tag{2.10}$$

with the usual convention about replacing $(G_q)^n$ by $G_{n,q}$ in the binomial expansion.

Proof. From (2.4), we obtain

$$\frac{[2]_q(\log q + t)}{qe^t+1} = \sum_{n=0}^{\infty} G_{n,q}\frac{t^n}{n!} = \sum_{n=0}^{\infty} (G_q)^n \frac{t^n}{n!} = e^{G_q t}, \tag{2.11}$$

which yields

$$[2]_q(\log q + t) = (qe^t+1)e^{G_q t} = qe^{(G_q+1)t} + e^{G_q t}. \tag{2.12}$$

Using the Taylor expansion of exponential function, we have

$$\begin{aligned} [2]_q \log q + [2]_q t &= \sum_{n=0}^{\infty} \left\{ q(G_q+1)^n + (G_q)^n \right\} \frac{t^n}{n!} \\ &= q(G_q+1)^0 + (G_q)^0 + q(G_q+1)^1 + (G_q)^1 \\ &\quad + \sum_{n=2}^{\infty} \left\{ q(G_q+1)^n + (G_q)^n \right\} \frac{t^n}{n!}. \end{aligned} \tag{2.13}$$

The result follows by comparing the coefficients.

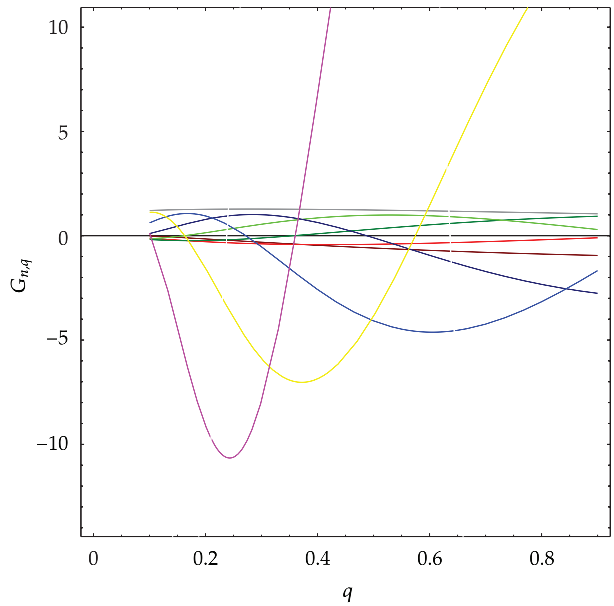

Figure 1: Curves of $G_{n,q}$.

Here is the list of the first q-Genocchi numbers $G_{n,q}$:

$$\begin{aligned} G_{0,q} &= \log q, \\ G_{1,q} &= -\frac{-1 - q + q\log q}{(1+q)}, \\ G_{2,q} &= \frac{q(-2 - 2q - \log q + q\log q)}{(1+q)^2}, \\ G_{3,q} &= -\frac{q(3 - 3q^2 + \log q - 4q\log q + q^2\log q)}{(1+q)^3}, \\ &\vdots \end{aligned} \tag{2.14}$$

We display the shapes of the q-Genocchi numbers $G_{n,q}$. For $n = 1, \ldots, 10$, we can draw a curve of $G_{n,q}$, $1/10 \le q \le 9/10$, respectively. This shows the ten curves combined into one. We display the shape of $G_{n,q}$: (Figure 1).

Because

$$\frac{\partial}{\partial x} F_q(t,x) = tF_q(t,x) = \sum_{n=0}^{\infty} \frac{d}{dx} G_{n,q}(x) \frac{t^n}{n!}, \tag{2.15}$$

it follows the important relation

$$\frac{d}{dx}G_{n,q}(x) = nG_{n-1,q}(x). \tag{2.16}$$

Here is the list of the first the q-Genocchi polynomials $G_{n,q}(x)$:

$$\begin{aligned} G_{0,q}(x) &= \log q, \\ G_{1,q}(x) &= \frac{(1+q-q\log q + x\log q + qx\log q)}{(1+q)}, \\ &\vdots \end{aligned} \tag{2.17}$$

Since

$$\begin{aligned} \sum_{l=0}^{\infty} G_{l,q}(x+y)\frac{t^l}{l!} &= \frac{[2]_q \log q + [2]_q t}{qe^t + 1} e^{(x+y)t} \\ &= \sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty} y^m \frac{t^m}{m!} \\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} G_{n,q}(x)\frac{t^n}{n!} y^{l-n}\frac{t^{l-n}}{(l-n)!}\right) \\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n} G_{n,q}(x) y^{l-n}\right)\frac{t^l}{l!}, \end{aligned} \tag{2.18}$$

we have the following theorem. □

Theorem 2.3. *q-Genocchi polynomials $G_{n,q}(x)$ satisfy the following relation:*

$$G_{l,q}(x+y) = \sum_{n=0}^{l}\binom{l}{n} G_{n,q}(x) y^{l-n}. \tag{2.19}$$

It is easy to see that

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!} &= \frac{[2]_q \log q + [2]_q t}{qe^t+1}e^{xt} \\ &= \frac{[2]_q}{m[2]_{q^m}}\sum_{a=0}^{m-1}(-1)^a q^a \frac{[2]_{q^m}\log q^m + [2]_{q^m} mt}{q^m e^{mt}+1}e^{(a/m+x/m)(mt)} \\ &= \frac{[2]_q}{m[2]_{q^m}}\sum_{a=0}^{m-1}(-1)^a q^a \sum_{n=0}^{\infty} G_{n,q^m}\left(\frac{a+x}{m}\right)\frac{(mt)^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(m^{n-1}\frac{[2]_q}{[2]_{q^m}}\sum_{a=0}^{m-1}(-1)^a q^a G_{n,q^m}\left(\frac{a+x}{m}\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.20}$$

Hence we have the following theorem.

Theorem 2.4. *For any positive integer m(=odd), one obtains*

$$G_{n,q}(x) = m^{n-1}\frac{[2]_q}{[2]_{q^m}}\sum_{i=0}^{m-1}(-1)^i q^i G_{n,q^m}\left(\frac{i+x}{m}\right), \quad \text{for } n \geq 0. \tag{2.21}$$

3. Distribution and Structure of the Zeros

In this section, we investigate the zeros of the q-Genocchi polynomials $G_{n,q}(x)$ by using computer. We display the shapes of the q-Genocchi polynomials $G_{n,q}(x)$. For $n = 1, \dots, 10$, we can draw a curve of $G_{n,q}(x), -2 \leq x \leq 2$, respectively. This shows the ten curves combined into one. We display the shape of $G_{n,q}(x)$ (Figures 2, 3, 4, and 5).

We plot the zeros of $G_{n,q}(x), x \in \mathbb{C}$ for $n = 10, 20, 25, 30, q = 1/3$ (Figures 6, 7, 8, and 9).

Next, we plot the zeros of $G_{n,q}(x), x \in \mathbb{C}$ for $n = 30$, $q = 1/2, 1/3, 1/4, 1/5$. (Figures 10, 11, 12, and 13).

In Figures 6, 7, 8, 9, 10, 11, 12, and 13, $G_{n,q}(x), x \in \mathbb{C}$, has $\text{Im}(x) = 0$ reflection symmetry. This translates to the following open problem: prove or disprove: $G_{n,q}(x), x \in \mathbb{C}$, has $\text{Im}(x) = 0$ reflection symmetry. Our numerical results for numbers of real and complex zeros of $G_{n,q}(x), q = 1/2, 1/3$, are displayed in Table 1.

Figure 15 shows the distribution of real zeros of $G_{n,q}(x)$ for $1 \leq n \leq 20$.

In Figure 15(a), we choose $q = 1/10$. In Figure 15(b), we choose $q = 3/10$. In Figure 15(c), we choose $q = 5/10$. In Figure 15(d), we choose $q = 6/10$.

We calculated an approximate solution satisfying $G_{n,q}(x), q = 1/2, 1/3, x \in \mathbb{R}$. The results are given in Tables 2 and 3.

The plot above shows $G_{n,q}(x)$ for real $1/10 \leq q \leq 9/10$ and $-3 \leq x \leq 3$, with the zero contour indicated in black (Figure 16). In Figure 16(a), we choose $n = 2$. In Figure 16(b), we choose $n = 3$. In Figure 16(c), we choose $n = 4$. In Figure 16(d), we choose $n = 5$.

We will consider the more general open problem. In general, how many roots does $G_{n,q}(x)$ have? Prove or disprove: $G_{n,q}(x)$ has n distinct solutions. Find the numbers of complex zeros $C_{G_{n,q}(x)}$ of $G_{n,q}(x), \text{Im}(x) \neq 0$. Prove or give a counterexample: *Conjecture*: since

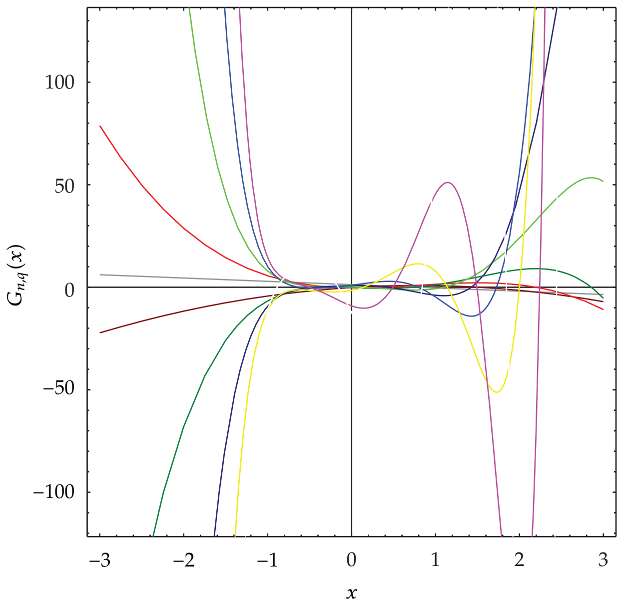

Figure 2: Curves of $G_{n,1/5}(x)$.

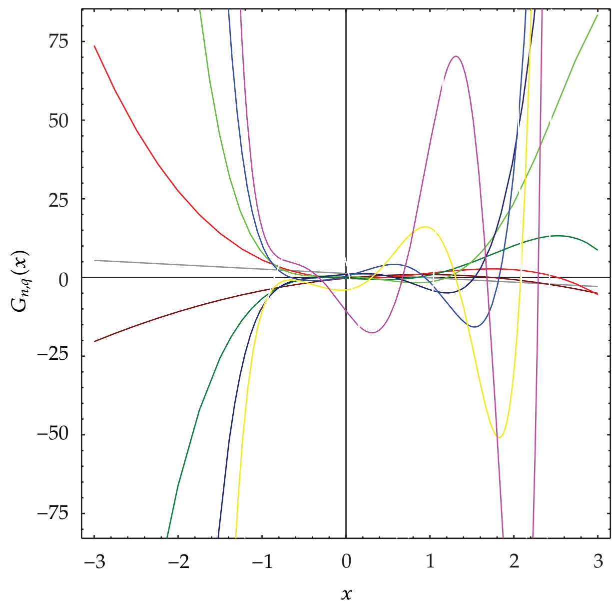

Figure 3: Curves of $G_{n,1/4}(x)$.

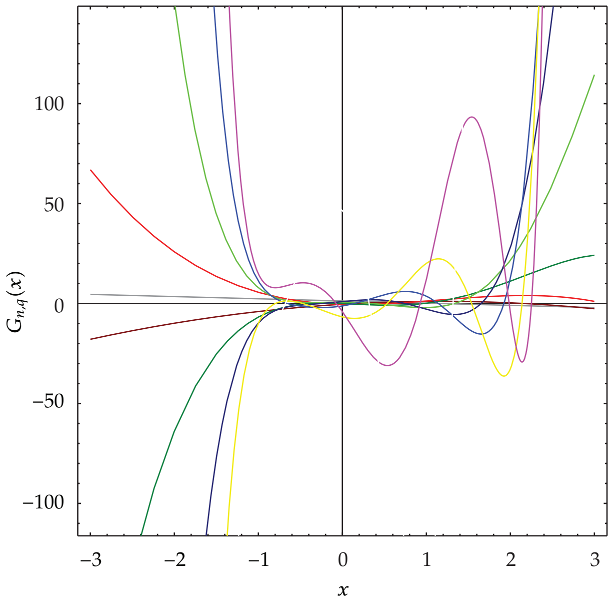

Figure 4: Curves of $G_{n,1/3}(x)$.

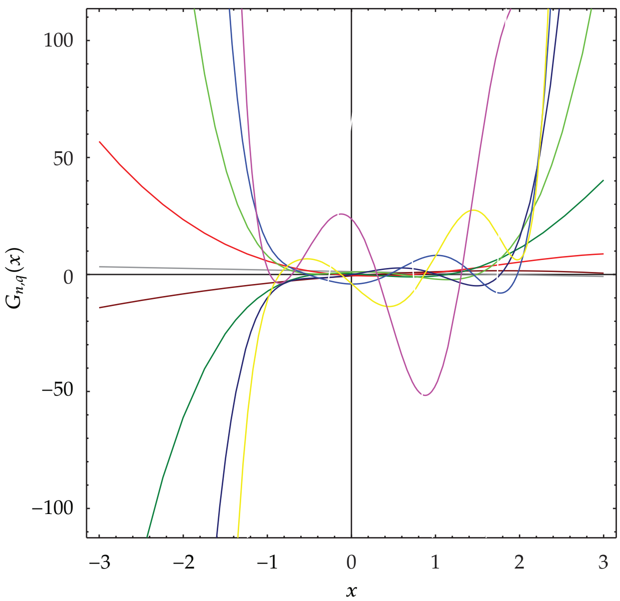

Figure 5: Curves of $G_{n,1/2}(x)$.

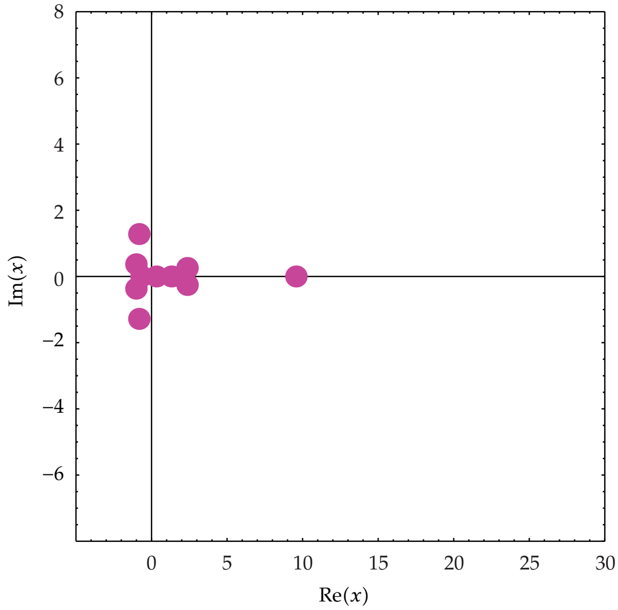

Figure 6: Zero of $G_{10,1/3}(x)$.

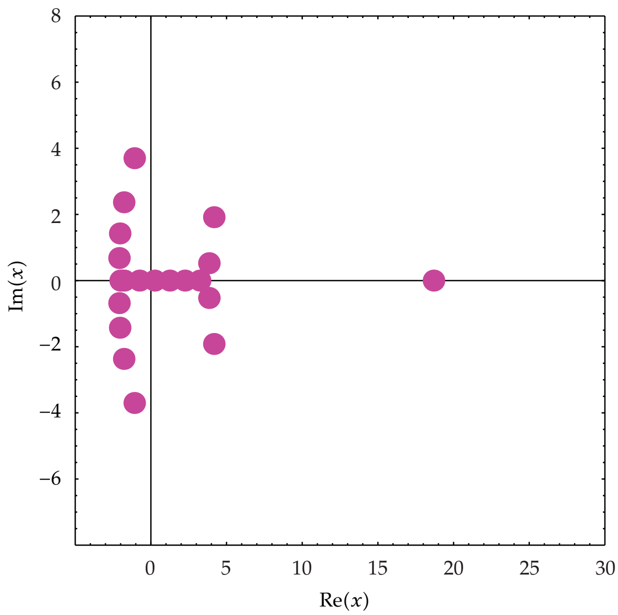

Figure 7: Zero of $G_{20,1/3}(x)$.

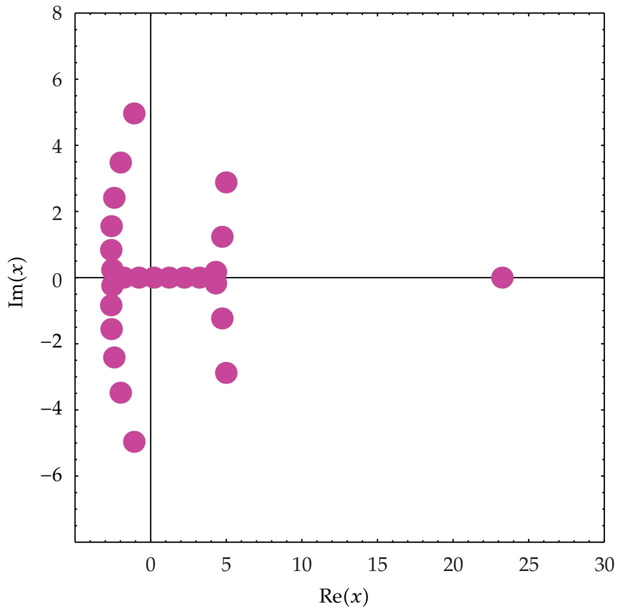

Figure 8: Zero of $G_{25,1/3}(x)$.

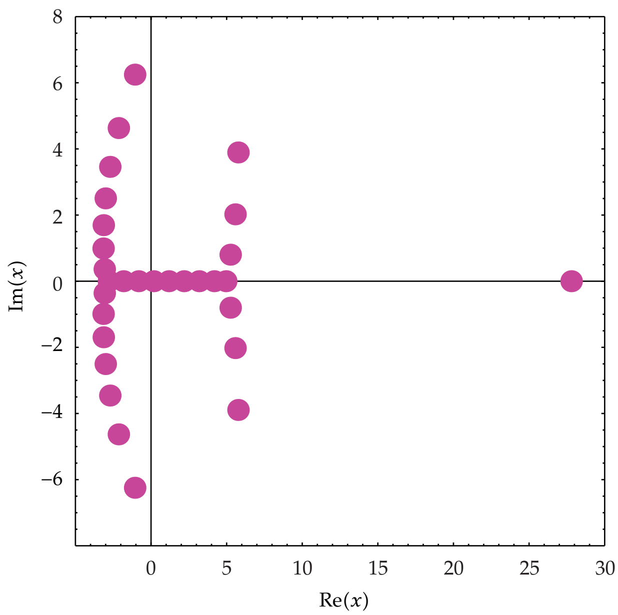

Figure 9: Zero of $G_{30,1/3}(x)$.

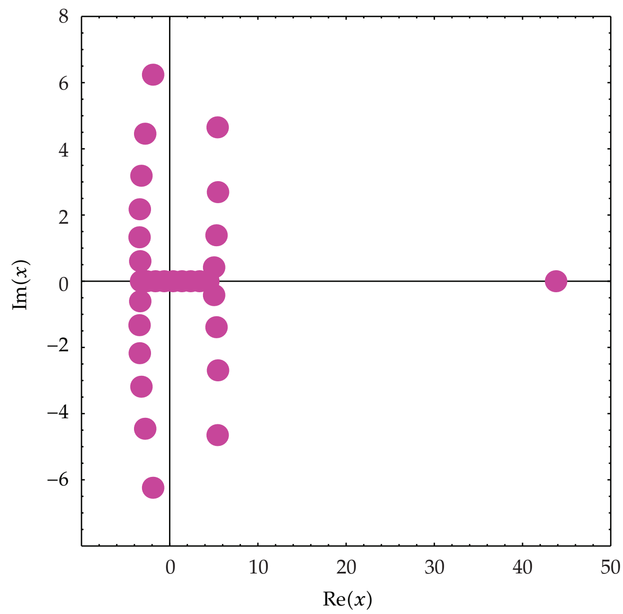

Figure 10: Zero of $G_{30,1/2}(x)$.

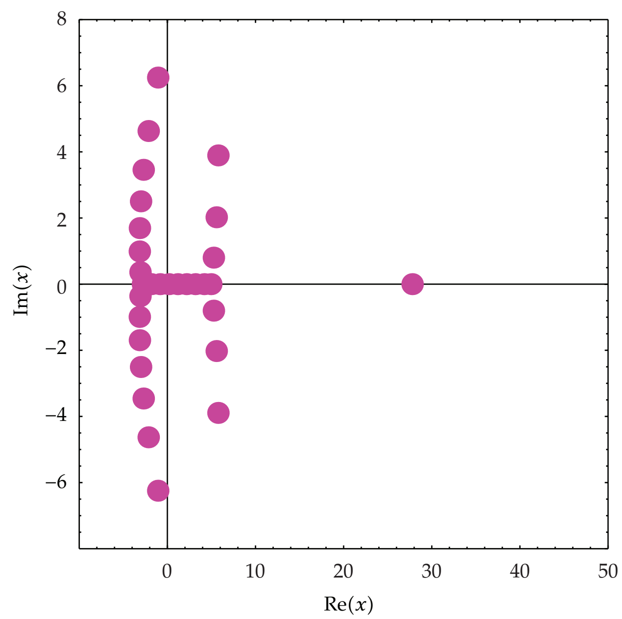

Figure 11: Zero of $G_{30,1/3}(x)$.

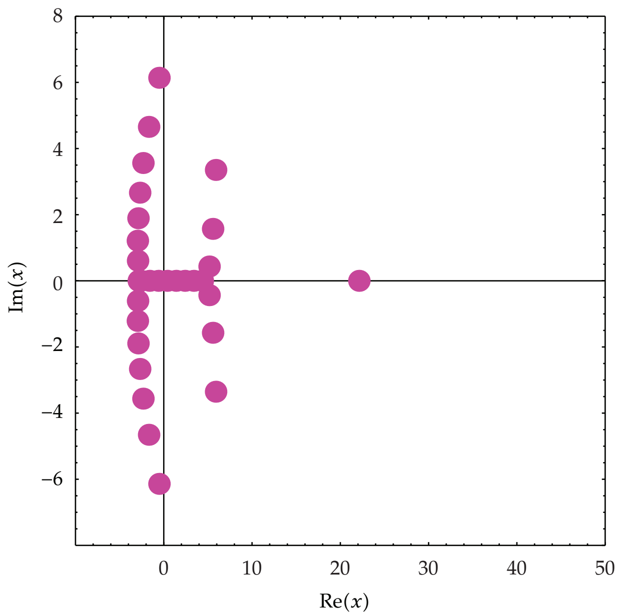

Figure 12: Zero of $G_{30,1/4}(x)$.

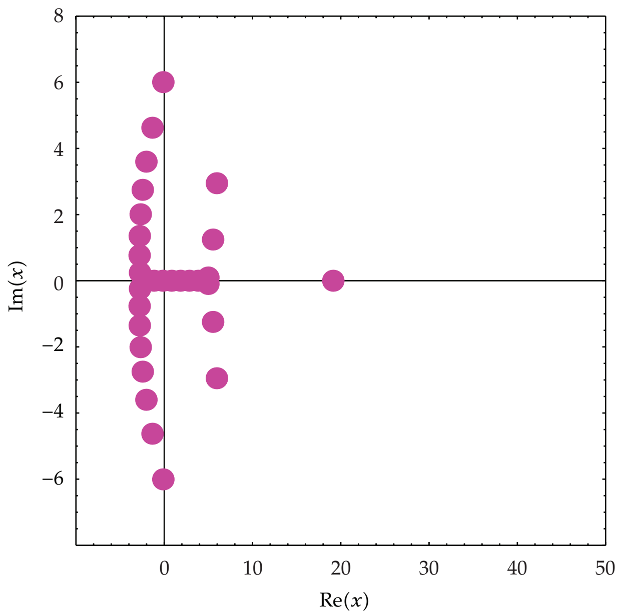

Figure 13: Zero of $G_{30,1/5}(x)$.

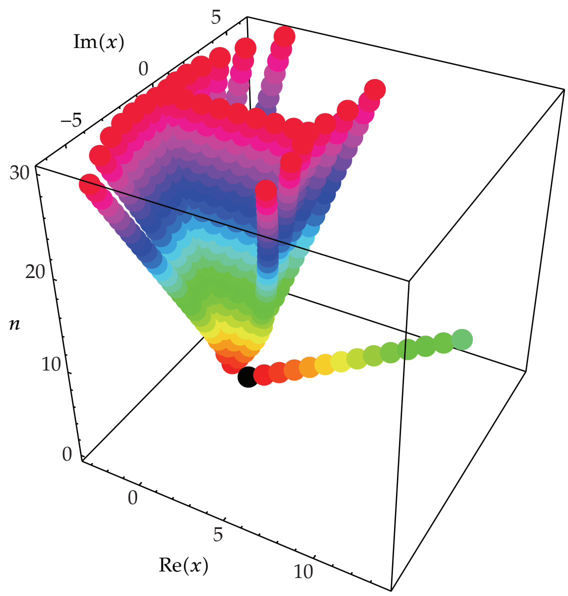

Figure 14: Stacks of zeros $G_{n,1/3}(x)$ for $1 \leq n \leq 30$.

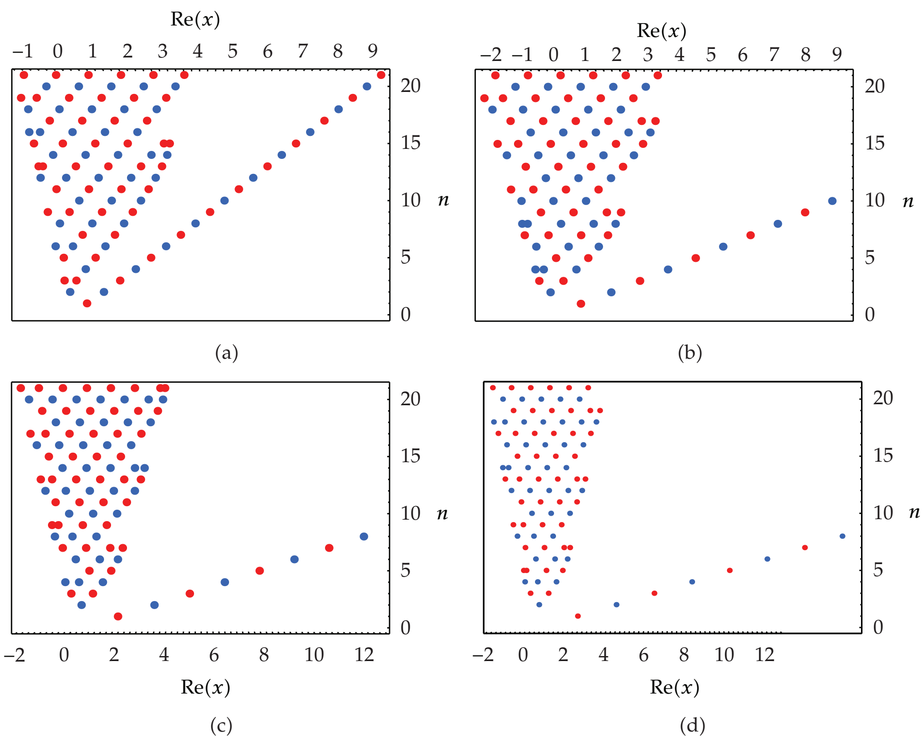

Figure 15: Plot of real zeros of $G_{n,q}(x)$ for $1 \leq n \leq 20$.

Table 1: Numbers of real and complex zeros of $G_{n,q}(x)$.

Degree n	$q = 1/2$		$q = 1/3$	
	Real zeros	Complex zeros	Real zeros	Complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	4	0	4	0
5	3	2	3	2
6	4	2	4	2
7	5	2	5	2
8	4	4	6	2
9	5	4	5	4
10	4	6	4	6

Table 2: Approximate solutions of $G_{n,1/2}(x) = 0, x \in \mathbb{R}$.

Degree n	x
1	1.7760
2	0.2583, 3.294
3	−0.1698, 0.7313, 4.767
4	−0.4188, 0.1527, 1.145, 6.225
5	0.5848, 1.492, 7.677
6	0.01656, 1.017, 1.772, 9.126
7	−0.5269, 0.4468, 1.452, 1.974, 10.573
8	−0.8536, −0.1221, 0.8779, 12.019
9	−0.969, −0.707, 0.3088, 1.309, 13.46
10	−0.2604, 0.7396, 1.738, 14.91

Table 3: Approximate solutions of $G_{n,1/3}(x) = 0, x \in \mathbb{R}$.

Degree n	x
1	1.1602
2	0.1523, 2.168
3	−0.2107, 0.5657, 3.126
4	−0.3621, −0.02976, 0.9703, 4.062
5	0.3561, 1.333, 4.989
6	−0.2472, 0.7504, 1.652, 5.910
7	−0.6547, 0.1435, 1.144, 1.923, 6.828
8	−0.798, −0.4682, 0.5362, 1.540, 2.139, 7.744
9	−0.07080, 0.9292, 1.98, 2.25, 8.659
10	−0.673, 0.3221, 1.322, 9.573

n is the degree of the polynomial $G_{n,q}(x)$, the number of real zeros $R_{G_{n,q}(x)}$ lying on the real plane $\mathrm{Im}(x) = 0$ is then $R_{G_{n,q}(x)} = n - C_{G_{n,q}(x)}$, where $C_{G_{n,q}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{G_{n,q}(x)}$ and $C_{G_{n,q}(x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane and the equation of a trajectory curve running through the complex zeros on any one of the arcs. For $n = 1, \ldots, 10$, we can draw a plot of the $G_{n,q}(x)$, respectively. This shows the ten curves combined into one. These figures give mathematicians

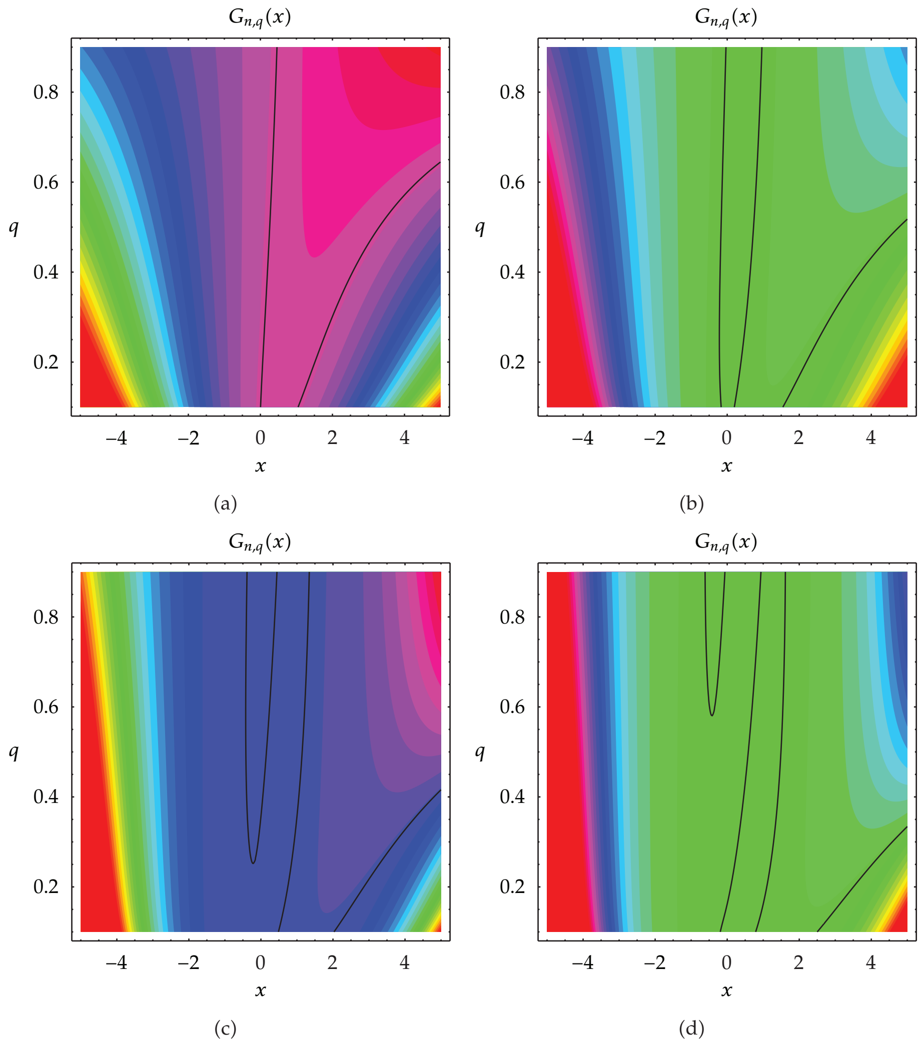

Figure 16: Zero contour of $G_{n,q}(x)$.

an unbounded capacity to create visual mathematical investigations of the behavior of the $G_{n,q}(x)$ and roots of the $G_{n,q}(x)$ (Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 and 16). Moreover, it is possible to create new mathematical ideas and analyze them in ways that generally are not possible by hand. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the q-Genocchi polynomials $G_{n,q}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [15–19].

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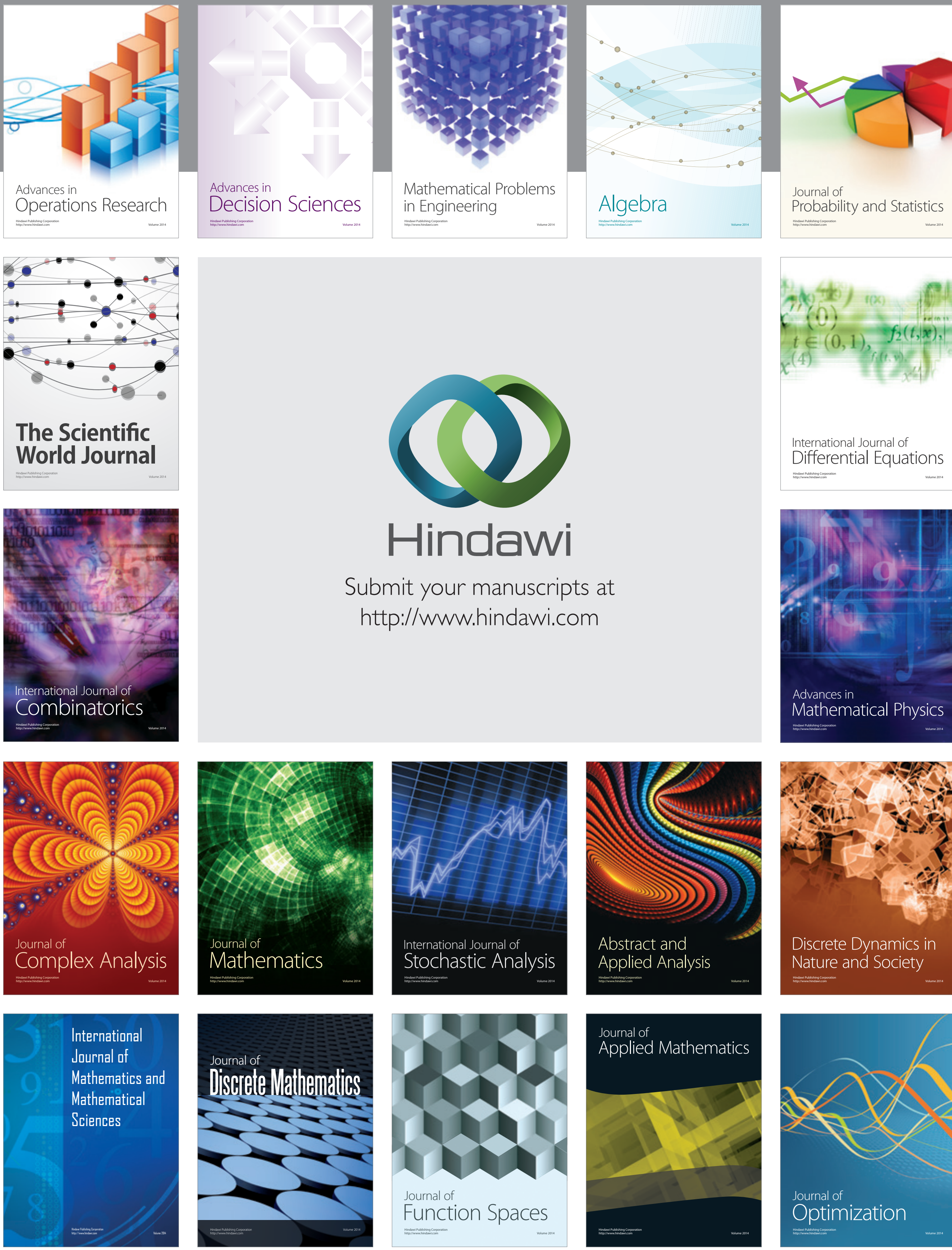
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