Research Article
Graph of *BCI*-Algebras

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We associate a graph to any subset Y of a *BCI*-algebra X and denote it by $G(Y)$. Then we find the set of all connected components of $G(X)$ and verify the relation between X and $G(X)$, when X is commutative *BCI*-algebra or $G(X)$ is complete graph or n-star graph. Finally, we attempt to investigate the relation between some operations on graph and some operations on *BCI*-algebras.

1. Introduction

BCK- and *BCI*-algebras are two classes of abstract algebras were introduced by Imai and Iséki [1, 2], in 1966. The notion of *BCK*-algebras is originated from two different ways. One of the motivations is from classical and nonclassical propositional logic. Another motivation is based on set theory. It is known that the class of *BCK*-algebras is a proper subclass of the class of *BCI*-algebras. Many authors studied the graph theory in connection with semigroups and rings. For example, Beck [3] associated to any commutative ring R its zero divisors graph $G(R)$, whose vertices are the zero divisors of R, with two vertices a, b jointed by an edge in case $ab = 0$. In [4], Jun and Lee defined the notion of zero divisors and quasi-ideals in *BCI*-algebra and show that all zero divisors are quasi-ideal. Then, they introduced the concept of associated graph of *BCK*/*BCI*-algebra and verified some properties of this graph and proved that if X is a *BCK*-algebra, then associated graph of X is a connected graph. Moreover, if X is a *BCI*-algebra and $x \in X$ such that x is not contained in *BCK*-part of X, then there is not any edge connecting x and y, for any $y \in X$. In this paper, we associate new graph to a *BCI*-algebras X which is denoted it by $G(X)$. This definition based on branches of X. If X is a *BCK*-algebras, then this definition and last definition, which was introduced by Jun and Lee, are the same. Then, for any $a \in P$, we defined the concept of a-divisor, where P is a p-semisimple part of *BCI*-algebra X and show that it is quasi-ideal of X. Then, we explain some properties of this graph as mentioned in the abstract.

2. Preliminaries

Definition 2.1 (see [1, 2]). A *BCI*-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:

(*BCI*1) $((x * y) * (x * z)) * (z * y) = 0$;

(*BCI*2) $x * 0 = x$;

(*BCI*3) $x * y = 0$ and $y * x = 0$ imply $y = x$.

   If X is a *BCI*-algebra, then for all $x, y \in X$ the following hold:

(*BCI*4) $x * x = 0$;

(*BCI*5) $x * (x * (x * y)) = x * y$;

(*BCI*6) $(x * y) * z = (x * z) * y$;

(*BCI*7) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$, for any $z \in X$.

Moreover, the relation $\leq$ was defined by $x \leq y \Leftrightarrow x * y = 0$, for any $x, y \in X$, is a partial-order on X, which is called *BCI*-ordering of X. The set $B = \{x \in X \mid 0 * x = 0\}$ is called *BCK*-part of X. A *BCI*-algebra X is called a *BCK-algebra* if $B = X$. A *BCK*-chain is a *BCK*-algebra such that $(X, \leq)$ is a chain, where $\leq$ is the *BCI*-ordering of X. A nonzero element a of *BCK*-algebra X is called an *atom* of X if $x * a = 0$ implies $x = a$, for any nonzero element $x \in X$. Moreover, $P = \{x \in X \mid 0 * (0 * x) = x\}$ is called *p-semisimple* part of a *BCI*-algebra X. It is the set of all minimal elements of X, with respect to the *BCI*-ordering of X. The *BCI*-algebra X is called a *p-semisimple BCI*-algebra if $P = X$. For any $a \in P$, we use the notation $V_X(a)$ or simply $V(a)$ to denote the set $\{x \in X \mid a * x = 0\}$ which is called the branch of X with respect to a. It is easy to see that $X = \bigcup_{a \in P} V(a)$, and for any distinct elements $a, b \in P$, we have

(*P*1) $V(a) \cap V(b) = \emptyset$;

(*P*2) if $x \in V(a)$ and $y \in V(b)$, then $x * y \in V(a * b)$;

(*P*3) if $x \in V(a)$, then $\{y \in X \mid x \leq y\} \cup \{y \in X \mid y \leq x\} \subseteq V(a)$;

(*P*4) for all $x \in X$, $x * a = 0$ implies $x = a$.

A *BCI*-algebra X is called *commutative* if $x \leq y$ implies $x = y * (y * x)$, for any $x, y \in X$. Moreover, X is called *branchwise* commutative if $x * (x * y) = y * (y * x)$, for all $x, y \in V(a)$ and all $a \in P$. For more details, we refer to [5–11].

Lemma 2.2 (see [9]). *A BCI-algebra X is commutative if and only if it is branchwise commutative.*

Theorem 2.3 (see [10]). *Let $X = \{x_0, x_1, \ldots, x_n\}$ and $(X, *, 0)$ be a BCK-chain with x_0 as the zero element, whose BCI-ordering is supposed as follows: $x_0 < x_1 < \cdots < x_n$. Then, X is commutative if and only if the relation $*$ on X is given by $x_i * x_j = x_{i*j}$, where $i * j = \max\{0, i - j\}$, for any $i, j \in \{0, 1, \ldots, n\}$ (see [10, Theorem 2.3.3]).*

   A partial-order set $(P, \leq)$ is said to be of *finite length* if the lenghts of all chains of P are bounded. Let $x, y \in P$. Then, a chain of length n between x and y is a chain $a_0 < a_1 < \cdots < a_{n-1} < a_n$ such that $\{a_0, a_n\} = \{x, y\}$. For any $x, y \in P$, $l(x, y)$ is the greatest number in the lenghts of all chain between a and b.
   A *BCI*-algebra X is said to be finite length if it is finite length as a partial-order set.

Note 1 (see [4]). Let X be a *BCI*-algebra and $A \subseteq X$. We will use the notations $U_X(A)$ and $L_X(A)$ or simply $U(A)$ and $L(A)$ to denote the sets $\{x \in X \mid a * x = 0, \forall a \in A\}$ and $\{x \in X \mid x * a = 0, \forall a \in A\}$, respectively, that is, $U(A) = \{x \in X \mid a \leq x, \forall a \in A\}$ and $L(A) = \{x \in X \mid x \leq a, \forall a \in A\}$.

Definition 2.4 (see [4]). A nonempty subset I of *BCI*-algebra X is called a *quasi-ideal* of X if $x * y = 0$ implies $x \in I$, for any $y \in I$ and $x \in X$.

Proposition 2.5 (see [12]). *Let $(X, *_1, 0)$ and $(Y, *_2, 0)$ be two BCK-algebras such that $X \cap Y = \{0\}$ and $*$ be the binary operation on $X \cup Y$ as follows: for any $a, b \in X \cup Y$;*

$$a * b = \begin{cases} a *_1 b & \text{if } a, b \in X, \\ a *_2 b & \text{if } a, b \in Y, \\ a & \text{otherwise}. \end{cases} \tag{2.1}$$

Then, $(X \cup Y, *, 0)$ is a BCK-algebra. We denote $(X \cup Y, *, 0)$ by $X \oplus Y$.

Definition 2.6 (see [4]). For any $x \in X$, the set $\{y \in X \mid L(\{x, y\}) = \{0\}\}$ is called the set of all zero divisors of x.

Theorem 2.7 (see [4]). *For any element x of BCK-algebra X, the set of all zero divisors of x is a quasi-ideal of X containing the zero element $\{0\}$.*

Let G be a graph, $E(G)$ be the set of all edges of G and $V(G)$ be the set of all vertexes of G. For any $S \subseteq E(G)$, the graph with vertex set $V(G)$ and edge set $E(G) - S$ is denoted by $G - S$. The edge which connect two vertices x, y is denoted by xy. Note that, xy and yx are the same. For any $T \subseteq V(G)$, the graph with vertex set $V(G) - T$ and edge set $E(G) - T'$ is denoted by $G - T$, where $T' = \{xy \in E(G) \mid x \in T, y \in G\}$. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph $G = (V, E)$ is *connected*, if any vertices x, y of G linked by a path in G, otherwise the graph is *disconnected*. A *tree* is a connected graph with no cycles. The degree of a vertex v in a graph G, denoted by $\deg(v)$, is the number of edges of G incident with v and $\deg(G) = \sum_{x \in V} \deg(x) = 2|E|$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of G, respectively. If G and H are two graphs such that $V(G) \cap V(H) = \emptyset$, then the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ is called the *disjoint union* of G and H and we denote it by $G \cup H$. Any graph $G = (V, E)$ may be expressed uniquely as a disjoint union of connected graphs. These graphs are called the *connected components* or simply *components* of G. The number of components of G is denoted by $c(G)$. A graph G is called *empty graph* if $E(G) = \emptyset$. Moreover, a graph K is called *complete graph* if $xy \in E(G)$, for any distinct elements $x, y \in V(G)$. For more details we refer to [13, 14].

Definition 2.8. Let G and H be two graphs such that $V(G) \cap V(H) = \emptyset$. We denote the graph $G + H$, for the graph, whose vertex set and edge set are $V(G) \cup V(H)$ and $E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$, respectively.

Definition 2.9 (see [4]). By the associated graph of *BCK/BCI*-algebra X, denoted $\Gamma(X)$, we means the graph whose vertices are just the elements of X, and for distinct $x, y \in \Gamma(X)$, there is an edge connecting x and y if and only if $L(\{x, y\}) = \{0\}$.

Definition 2.10 (see [13, 15]). Let G be a finite graph with $|V(G)| = n$. Then, the adjacency matrix of G is an $n \times n$ matrix $A_G = [a_{i,j}]$ such that $a_{i,j}$ is the number of edges joining v_i and v_j. Moreover, we denote the characteristic polynomial of the matrix A_G, by $\chi(G, \lambda)$. That is $\chi(G, \lambda) = \det(\lambda I - A_G)$, where I is a $n \times n$ identity matrix.

Proposition 2.11 (see [15, Proposition 2.3]). *Let G be a graph and $\chi(G, \lambda) = \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_n$. be the characteristic polynomial of the adjacent matrix G. Then, $c_1 = 0$, $-c_2$ is the number of edges of G and $-c_3$ is twice the number of triangles in G.*

3. Graph Based on *BCI*-Algebras

From now on, in this paper, $(X, *, 0)$ or simply X is a *BCI*-algebra, B is a *BCK*-part and P is a p-semisimple part of X, unless otherwise state. For all $x, y \in X$, we use $x \lneqq y$ to denote $x \leq y$ and $x \neq y$.

Definition 3.1. Note that the set of all 0-divisors of x and the set of all zero divisors of x are the same. Let $x \in X$. Then, there exists $a \in P$ such that $x \in V(a)$. We will use the notation Z_x to denote the set of all $y \in X$ such that $L(\{x, y\}) = \{a\}$, that is, $Z_x = \{y \in X \mid L(\{x, y\}) = \{a\}\}$, which is called the set of a-divisor of X.

Note that the set of all 0-divisors of x and the set of all zero divisors of x are the same.

Lemma 3.2. *Let $a, b \in P$ and $x, y \in X$. Then*

(i) *one has*

$$L(\{x, a\}) = \begin{cases} a & \text{if } x \in V(a), \\ \emptyset & \text{otherwise}, \end{cases} \tag{3.1}$$

(ii) *if $a \neq b$, $x \in V(a)$ and $y \in V(b)$, then $L(\{x, y\}) = \emptyset$.*

Proof. (i) Clearly, if $x \in V(a)$, then $L(\{x, a\}) = \{a\}$. Let $x \notin V(a)$. Then, $y \in L(\{x, a\})$ implies $y \leq x$ and $y \leq a$. Since $a \in P$, we have $a = y$ and so $a \leq x$, which is impossible. Therefore, $L(\{x, a\}) = \emptyset$.

(ii) Since $a \neq b$, we have $L(\{x, y\}) \subseteq V(a) \cap V(b) = \emptyset$.   □

Lemma 3.3. *For any $x, y \in X$, if $x * y = 0$, then $L(\{x\}) \subseteq L(\{y\})$ and $Z_y \subseteq Z_x$.*

Proof. Let $x * y = 0$, for some $x, y \in X$. Then, $x \leq y$. Clearly, $L\{x\} \subseteq L\{y\}$. If $x \in V(a)$, where $a \in P$, then $a \leq x \leq y$ and so by (P3), $y \in V(a)$, too. Now, let $u \in Z_y$. Then, $L(\{u, y\}) = \{a\}$ and so $a \in L(\{u, x\}) \subseteq L(\{u, y\}) = \{a\}$. Hence, $L(\{x, u\}) = a$ and so $u \in Z_x$. Therefore, $Z_y \subseteq Z_x$.   □

Jun and Lee in [4] proved that if x is an element of *BCK*-algebra X, then the set of all zero divisors of x is a quasi-ideal of X. In Theorem 3.4, we will show that if x is an element of *BCI*-algebra X, then Z_x is a quasi-ideal of X.

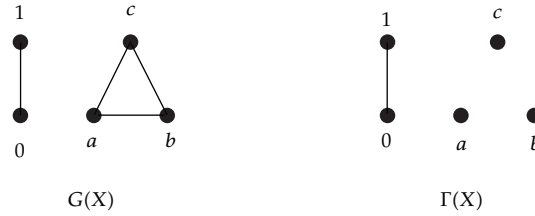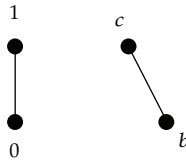**Theorem 3.4.** *Z_x is a quasi-ideal of X, for any $x \in X$.*

Figure 1

Figure 2

Proof. Let $x \in X$ such that $x \in V(a)$, for some $a \in P$ and $u, v \in X$ such that $u * v = 0$ and $v \in Z_x$. Then, $u \leq v$ and $L(\{v, x\}) = \{a\}$. Hence, $v \in V(a)$ and so by (P3), $u \in V(a)$. Since $a \in L(\{u, x\}) \subseteq L(\{v, x\}) = \{a\}$, then $L(\{u, x\}) = \{a\}$. Therefore, $u \in Z_x$ and so Z_x is a quasi-ideal of X. $\square$

Definition 3.5. Let $Y \subseteq X$, and $G(Y)$ be a simple graph, whose vertices are just the elements of Y and for distinct $x, y \in Y$, there is an edge connecting x and y, denoted by xy if and only if $L(\{x, y\}) = \{a\}$, for some $a \in P$. If $Y = X$, then $G(X)$ is called a *BCI-graph of* X.

Clearly, if X is a *BCK*-algebra, then $G(X) = \Gamma(X)$. But it is not true, in general.

Example 3.6. (i) Let $X = \{0, 1, a, b, c\}$. Define the binary operation "$*$" on X by the following table:

$*$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	1
c	c	a	1	1	0

Then, $(X, *, 0)$ is a *BCI*-algebra, $P = \{0, a\}$, $V(0) = \{0, 1\}$, and $V(a) = \{a, b, c\}$. Moreover, $L(\{1, 0\}) = \{0\}$ and $L(\{a, b\}) = L(\{a, c\}) = L(\{b, c\}) = \{a\}$ and so $E(G(X)) = \{10, ac, bc, ab\}$. Therefore, the graphs $G(X)$ and $\Gamma(X)$ which are given by **Figure 1** are different.

(ii) Let $(X, *, 0)$ be the *BCI*-algebra in Example 3.6(i) and $Y = \{0, 1, b, c\}$. Then, $G(Y)$ is given by **Figure 2**.

Example 3.7 (see [4]). Let $X = \{0, a, b, c\}$. Define the binary operation "$*$" on X by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Then, $(X, *, 0)$ is a BCK-algebra and $P = \{0\}$ and so $G(X) = \Gamma(X)$. Moreover, $L(\{0, a\}) = L(\{0, b\}) = L(\{0, c\}) = L(\{a, c\}) = L(\{b, c\}) = L(\{a, b\}) = \{0\}$. Therefore, the graph $G(X)$ is given by Figure 3.

Proposition 3.8. (i) $G(V(a))$ *is a connected graph, for any* $a \in P$,

(ii) $G(X) = \bigcup_{a \in P} G(V(a))$,

(iii) $G(X)$ *is a graph with* $|P|$ *components.*

Proof. (i) Let $a \in P$ and $x, y \in V(a)$. Then, by Lemma 3.2(i) $xa, ya \in E(G(X))$ and so there is a path from x to y in $G(X)$.

(ii) Since $X = \bigcup_{a \in P} V(a)$, then $V(G(X)) = V(\bigcup_{a \in P} G(V(a)))$. Clearly, $E(\bigcup_{a \in P} G(V(a))) \subseteq E(G(X))$. Now, let $xy \in E(G(X))$. Then, $L(\{x, y\}) = \{a\}$, for some $a \in P$ and so $xy \in G(V(a))$. Hence, $E(\bigcup_{a \in P} G(V(a))) = E(G(X))$. Therefore, $G(X) = \bigcup_{a \in P} G(V(a))$.

(iii) We want to show that there is not any path between elements of $V(a)$ and $V(b)$, for all distinct elements $a, b \in P$. Let a, b be distinct elements of P, $x \in V(a)$ and $y \in V(b)$. If there is a path $x, x_1, x_2, \dots, x_n, y$, which link x to y, then $a = L(\{x, x_1\}) \neq \emptyset$ and so by Lemma 3.2(ii), $x_1 \in V(a)$. By a similar way, we have $x_2, \dots, x_n, y \in V(a)$, which is impossible. Hence, there is not any path between x and y. Therefore, by (i), $G(X)$ is a graph with $|P|$ components. $\square$

Theorem 3.9. X *is a p-semisimple BCI-algebra if and only if* $G(X)$ *is an empty graph.*

Proof. Clearly, if X is a p-semisimple BCI-algebra, then by Lemma 3.2(ii), $G(X)$ is an empty graph. Conversely, let $G(X)$ be an empty graph and $x \in X$. Since, $0 * (0 * x) \in P$ and $x \in V(0 * (0 * x))$, then $L(\{0 * (0 * x), x\}) = \{0 * (0 * x)\}$. If $0 * (0 * x) \lneqq x$, then there exists an edge between x and $0 * (0 * x)$ in $G(X)$, which is impossible. Hence, $x = 0 * (0 * x)$ and so X is a p-semisimple BCI-algebra. $\square$

Definition 3.10. Let $a \in P$. The element $x \in V(a)$ is called an a-atom if $x \neq a$ and $y * x = 0$ implies $y = a$ or $y = x$, for all $y \in X$.

Note that if X is a BCK-algebra then the concept of 0-atom and atom are the same.

Lemma 3.11. *Let X be a BCI-algebra of finite order. Then, for any $x \in X - P$, there is an a-atom $b \in X$, such that $b \leq x$, for some $a \in P$.*

Proof. Let $x \in X - P$. Then, there is $a \in P$ such that $x \in V(a)$. Let $S = \{L(x, u) \mid u * x = 0, u \in V(a) - \{a\}\}$. Clearly, $0 \in S$. Since X is of finite length, then S has the greatest element. Let $L(x, u)$ be the greatest element of S. Then, we show that v is an a-atom of X. Let $w * v = 0$, for some $w \in X$. Then, $w \leq v$ and so by (P3), $w \in V(a)$. If $w \lneqq v$, then $1 + L(x, u) \leq L(x, w)$, which is impossible. Therefore, $w = v$ or $w = a$ and so v is an atom of X. $\square$

Theorem 3.12. *Let X be a finite leght BCI-algebra and $a \in P$. Then*

(i) $G(V(a))$ *is a tree if and only if* $V(a) = \{a\}$ *or* $V(a)$ *has only one a-atom.*

(ii) $G(X)$ *is a tree if and only if* $X = \{0\}$ *or X is a BCK-algebra with only one atom.*
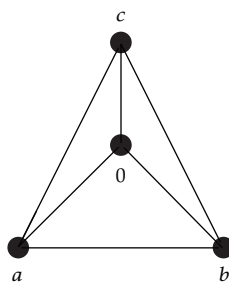
Figure 3

Proof. (i) Let $G(V(a))$ be a tree. If $V(a) = \{a\}$, we do not have any thing to prove. Let $V(a) \neq \{a\}$ and A be the set of all a-atoms of $V(a)$. Then, by Lemma 3.11, we have $1 \leq |A|$. Let $x, y \in A$, $u \leq x$ and $u \leq y$, for some $u \in X$. Then, by (P3), $u \in V(a)$. Since x, y are a-atoms of $V(a)$, then $u = a$ or $x = u = y$. Hence, $x = y$ or $xy \in E(G(X))$. If $xy \in E(G(X))$, then by Lemma 3.2(i), we have $xy, ax, ay \in E(G(X))$. Hence, $G(X)$ has a cycle, which is impossible. Therefore, $x = y$ and so $|A| = 1$. Conversely, let $V(a)$ has only one a-atom. By Proposition 3.8, $G(V(a))$ is a connected graph. If $V(a) = \{a\}$, then clearly, $G(V(a))$ is a tree. Let $V(a) \neq \{a\}$, $x, y \in V(a) - \{a\}$ and u be an a-atom of X. Then, by Lemma 3.11, $u \in L(\{x, y\})$ and so $L(\{x, y\}) \neq \{a\}$. Hence, $E(G(X)) = \{xa \mid x \in V(a) - \{a\}\}$ and so $G(V(a))$ does not have any cycle. Therefore, $G(V(a))$ is a tree.

(ii) Let $G(X)$ be a tree. Then, $G(X)$ is a connected graph and so by Proposition 3.8, $|P| = 1$. Hence, $P = \{0\}$ and so, X is a BCK-algebra. Since X is a BCK-algebra, we have $X = V(0)$ and so by (i), $X = \{0\}$ or X is a BCK-algebra with only one atom. The converse is straight consequent of (i). $\qquad \square$

Example 3.13. Let $X = \mathbb{R}^+ = \{x \in \mathbb{R} \mid 0 \leq x\}$ and $x * y = \max\{x - y, 0\}$, for any $x, y \in X$. Then, $(X, *, 0)$ is a BCK-algebra. Clearly, X does not have any atoms. Since X is a BCK-algebra, then by Proposition 3.8, $G(X)$ is a connected graph. Moreover, $L(\{x, y\}) \neq \{0\}$, for any $x, y \in X - \{0\}$. Hence, $E(G(X)) = \{x0 \mid x \in X - \{0\}\}$. Therefore, $G(X)$ is a tree.

Example 3.14. Let "$*$" be the binary operation in Example 3.13, $A = \{a_i \mid i \in \mathbb{R}^+\}$ and $B = \{b_i \mid i \in \mathbb{R}^+\}$, where $a_0 = 0 = b_0$ and $A \cap B = \{0\}$. Let a binary operation $*'$ on X is defined as follows: for all $i, j \in \mathbb{R}^+$,

$$a_i *' a_j = a_{i*j};$$
$$b_i *' b_j = b_{i*j};$$
$$a_i *' b_j = a_i;$$
$$b_i *' a_j = b_i. \tag{3.2}$$

By Example 3.13, $(A, *_1, 0)$ and $(B, *_2, 0)$ are two BCK-algebras, where $a_i *_1 a_j = a_{i*j}$ and $b_i *_2 b_j = b_{i*j}$, for any $i, j \in \mathbb{R}^+$. Hence, by Proposition 2.5, $(X, *', 0)$ is a BCK-algebra and so by Proposition 3.8, $G(X)$ is a connected graph. Let $a_i \in A - \{a_0\}$ and $b_i \in B - \{a_0\}$. Since X is a BCK-algebra, then $0 \in L(\{a_i, b_i\})$. Let $u \in L(\{a_i, b_i\})$, for some $u \in X$. If $u = b_j$, for some $j \in \mathbb{R}^+$, then $u = u *' a_i = 0$. If $u = a_j$, for some $j \in \mathbb{R}^+$, then $u = u *' b_i = 0$. Hence,
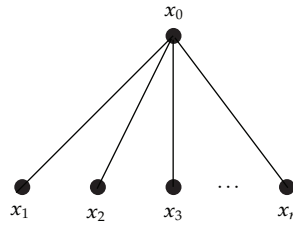
Figure 4: n-star.

$L(\{a_i, b_i\}) = \{0\}$ and so $a_i b_i \in E(G(X))$. Since $0a_i, 0b_i, a_i b_i \in E(G(X))$, then $G(X)$ has a cycle and so it is not a tree. But, X does not have any atoms.

Proposition 3.15. *Let X be a BCI-algebra such that the set $\{|V(a)| : a \in P\}$ is bounded. Then, $\Delta(G(X)) = \max\{|V(a)| - 1 : a \in P\}$.*

Proof. Since the set $\{|V(a)| : a \in P\}$ is bounded, then there is a $u \in P$, such that $|V(u)| - 1 = \max\{|V(a)| - 1 : a \in P\}$. We show that $\deg(x) \leq \deg(u)$, for any $x \in X$. Let $x \in X$. Then, there is $a \in P$ such that $x \in V(a)$. Since $V(a) \cap V(b) = \emptyset$, for any $b \in P - \{a\}$, then by Proposition 3.8(ii) and Lemma 3.2(i), $\deg(x) \leq |V(a)| - 1 \leq |V(u)| - 1 = \deg(u)$. Therefore, $\Delta(G(X)) = \deg(u) = |V(u)| - 1 = \max\{|V(a)| - 1 : a \in P\}$. $\qquad\square$

Corollary 3.16. (i) *If X is a finite BCI-algebra, then $\Delta(G(X)) = \max\{|V(a)| - 1 : a \in P\}$.*
    (ii) *If X is a finite BCK-algebra, then $\Delta(G(X)) = |X| - 1$.*

Proof. (i) If X is a finite *BCI*-algebra, then $\{|V(a)| : a \in P\}$ is bounded. Therefore, by Proposition 3.15, $\Delta(G(X)) = \max\{|V(a)| - 1 : a \in P\}$.
    (ii) If X is a *BCK*-algebra, then $P = \{0\}$ and so $\{|V(a)| - 1 : a \in P\} = \{V(0) - 1\} = \{|X| - 1\}$. Hence, by Proposition 3.15, $\Delta(G(X)) = |X| - 1$. $\qquad\square$

Definition 3.17. Let G be a graph with $n + 1$ vertices. Then G is called an n-star graph if it has Figure 4.

Theorem 3.18. *Let X be a BCK-algebra with $m + 1$ elements and only one atom. Then $G(X)$ is an m-star graph. Moreover, if G is an m-star graph, then there is a BCK-algebra X such that $G(X) = G$.*

Proof. Since, X is a *BCK*-algebra, then $G(X)$ is a connected graph. Let a be an atom of X. Then by Lemma 3.11, $a \leq x$, for all $x \in X - \{0\}$ and so $E(G(X)) = \{0x \mid x \in X - \{0\}\}$. Therefore, $G(X)$ is a m-star graph. Now, let G be a m-star graph, for some $m \in \mathbb{N}$ and $X = V(G) = \{x_0, x_1, \ldots, x_m\}$ be a chain such that $x_i < x_{i+1}$, for any $i \in \{0, 1, \ldots, m - 1\}$. Define the binary operation $*'$ on X by $x_i *' x_j = x_{i*j}$, where $i * j = \max\{0, i - j\}$, for any $i, j \in \{0, 1, \ldots, m\}$. By Theorem 2.3, X is a commutative *BCK*-chain and so $L(\{x, y\}) \neq \{0\}$, for any $x, y \in X - \{0\}$. Hence, by Lemma 3.2(i), $E(G(X)) = \{x_0 x_1, x_0 x_2, \ldots, x_0 x_m\}$ and so $G(X)$ is an m-star graph. $\qquad\square$

Theorem 3.19. *X is a commutative BCI-algebra and $x * y = x * a$, for any $a \in P$ and $x, y \in V(a)$, where $x \neq y$ if and only if $G(X)$ is a graph with complete components.*

Proof. Let $G(X)$ be a graph with complete components, $a \in P$ and $x, y \in V(a)$ such that $x \neq y$. Since $x * y \in V(0)$ by $(BCI6)$, we get $x * (x * y) \in L(\{x, y\})$. Now, since $x, y \in V(a)$ and $G(X)$ is a graph with complete components, then by Proposition 3.8, $xy \in E(G(X))$ and so $L(\{x, y\}) = \{a\}$. Hence, $x * (x * y) = a$ and so by $(BCI6)$ and $(BCI4)$, we get $x * a \leq x * y$. On the other hand, since $a \leq y$, then by $(BCI7)$, $x * y \leq x * a$. Therefore, $x * y = x * a$. Now, we show that X is a commutative BCI-algebra. Let $a \in P$ and $x, y \in V(a)$. Clearly, if $x = y$, then $x * (x * y) = y * (y * x)$. If $x \neq y$, then $x * (x * y) = x * (x * a) \leq a$. Since $a \in P$, we get $x * (x * a) = a$. By the similar way, we get $y * (y * x) = a$ and so $x * (x * y) = a = y * (y * x)$. Hence, X is branchwise commutative. Therefore, by Lemma 2.2, X is commutative BCI-algebra. Conversely, let X be a commutative BCI-algebra such that $x * y = x * a$, for any $a \in P$ and distinct elements $x, y \in V(a)$. Let $a \in P$ and $x, y \in V(a)$. If $u \in L(\{x, y\})$, then by $(P3)$, $u * a = u * x = 0$ and so $u \leq a$. Since $u \in P$, we have $a = u$. Hence, $L(\{x, y\}) = \{a\}$ and so $xy \in E(G(X))$. Therefore, by Proposition 3.8, all components of $G(X)$ are complete graph. $\qquad\square$

Corollary 3.20. *The graph $G(X)$ is a complete graph if and only if X is a BCK-algebra and*

$$x * y = \begin{cases} 0 & \text{if } y = x, \\ x & \text{otherwise.} \end{cases} \tag{3.3}$$

Definition 3.21 (see [10]). Let $(X_1, *_1, 0)$ be a BCK-algebra and $(X_2, *_2, 0)$ be a BCI-algebra such that $X_1 \cap X_2 = \{0\}$. Define the binary operation "$*$" on $X_1 \cup X_2$ by

$$x * y = \begin{cases} x *_1 y & \text{if } x, y \in X_1; \\ x *_2 y & \text{if } x, y \in X_2; \\ 0 *_2 y & \text{if } x \in X_1, y \in X_2 - \{0\}; \\ x & \text{if } x \in X_2, \ y \in X_1. \end{cases} \tag{3.4}$$

The $(X_1 \cup X_2, *, 0)$ is a BCI-algebra, whose BCK-part B contains X_1 and p-semisimple part P is contained in X_2. This algebra is called the *Li's union* of X_1 and X_2 and is denoted by $X_1 \cup_L X_2$.

Lemma 3.22. *Let $(X_1, *_1, 0)$ be a BCK-algebra, $(X_2, *_2, 0)$ be a BCI-algebra such that $X_1 \cap X_2 = \{0\}$ and Y be the Li's union of X_1 and X_2. Then*

$$L_Y(\{x, y\}) = \begin{cases} L_{X_1}(\{x, y\}) & \text{if } x, y \in X_1; \\ L_{X_2}(\{x, y\}) & \text{if } x, y \in X_2 - V_{X_2}(0); \\ L_{X_2}(\{x, y\}) \cup X_1 & \text{if } x, y \in V_{X_2}(0) - \{0\}; \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.5}$$

Proof. If $x, y \in X_1$, then

$$u \in L_{X_1}(\{x, y\}) \iff u *_1 x = u * x = 0 = u * y = u *_1 y \iff u \in L_Y(\{x, y\}). \tag{3.6}$$

Hence, $L_{X_1}(\{x, y\}) = L_Y(\{x, y\})$. Let $x, y \in V_{X_2}(a)$, for some $a \in P - \{0\}$. Then by Definition 3.21, $a \in Q - \{0\}$, where Q is a p-semisimple part of Y. Let $u \in L_Y(\{x, y\})$. Then

by $(P3)$, $u \in V_Y(a)$ and so $u \in X_2 - \{0\}$. Hence, $u*_2x = u * x = 0 = u * y = u*_2y$ and so $u \in L_{X_2}(\{x,y\})$. Now, let $u \in L_{X_2}(\{x,y\})$. Then $u * x = u*_2x = 0 = u*_2y = u * y$ and so $u \in L_Y(\{x,y\})$. Hence, $L_{X_2}(\{x,y\}) = L_Y(\{x,y\})$, for any $a \in P - \{0\}$ and $x,y \in V(a)$. If $x,y \in V_{X_2}(0) - \{0\}$, then $u * x = 0*_2x = 0$ and $u * y = 0*_2y = 0$, for any $u \in X_1$. Clearly, by definition of "$*$", $L_{X_2}(\{x,y\}) \subseteq L_Y(\{x,y\})$ and so $L_{X_2}(\{x,y\}) \cup X_1 \subseteq L_Y(\{x,y\})$. Let $u \in L_Y(\{x,y\})$. Then $u * x = 0$. Since $x \in V_{X_2}(0) - \{0\}$ by definition of "$*$", $u \in X_1$ or $u \in L_{X_2}(\{x,y\})$ and so $L_Y(\{x,y\}) \subseteq L_{X_2}(\{x,y\}) \cup X_1$. The final part is straight consequent of Lemma 3.2. □

Proposition 3.23. *Let $(X_1, *_1, 0)$ be a BCK-algebra, $(X_2, *_2, 0)$ be a BCI-algebra such that $X_1 \cap X_2 = \{0\}$, $X_1 \neq \{0\}$ and $S = \{xy \in E(G(X_2)) \mid x, y \in V_{X_2}(0) - \{0\}\}$. Then $G(X_1 \cup_L X_2) = (G(X_1) \cup G(X_2)) - S$.*

Proof. Clearly, $V(G(X_1 \cup_L X_2)) = X_1 \cup X_2 = V((G(X_1) \cup G(X_2)) - S)$. Let $Y = X_1 \cup_L X_2$, Q be a p-semisimple part of Y and $xy \in E(G(Y))$. Then $x \neq y$ and $L_Y\{x,y\} = \{a\}$, for some $a \in Q$. If $a \neq 0$, then by Definition 3.21, $x, y \in X_2$ and so by Lemma 3.22, $L_{X_2}(\{x,y\}) = \{a\}$. Hence, $xy \in E(G(X_2) - S) \subseteq E((G(X_1) \cup G(X_2)) - S)$. Let $a = 0$. Then $L_Y(\{x,y\}) = \{0\}$ and so $x, y \in V_Y(0) = V_{X_1}(0) \cup V_{X_2}(0)$. If $x, y \in V_{X_1}(0)$, then by Lemma 3.22, $xy \in E(G(X_1)) \subseteq E((G(X_1) \cup G(X_2)) - S)$. If $x = 0$ or $y = 0$, then clearly, $xy \in E(G(X_1) \cup G(X_2) - S)$. If $x, y \in V_{X_2}(0) - \{0\}$, then by Lemma 3.22, $L_{X_2}(\{x,y\}) \cup X_1 \subseteq L_Y(\{x,y\}) = \{0\}$ and so $X = 0$, which is impossible. If $x \in V_{X_1}(0) - \{0\}$ and $y \in V_{X_2}(0) - \{0\}$, then $x * y = 0 * y = 0$ and so $x \in L_Y(\{x,y\}) = \{0\}$, which is impossible. By the similar way, we get that $x \in V_{X_2}(0) - \{0\}$ and $y \in V_{X_1}(0) - \{0\}$ is impossible. Hence, $E(G(Y)) \subseteq E(G(X_1) \cup G(X_2) - S)$. Now, let $xy \in E((G(X_1) \cup G(X_2)) - S)$. Then $x \neq y$ and $xy \in E(G(X_1))$ or $xy \in E(G(X_2) - S)$. By Lemma 3.22, $E(G(X_1)) \subseteq E(G(Y))$. Let $xy \in E(G(X_2) - S)$. Then $L_{X_2}(\{x,y\}) = \{a\}$, for some $a \in P_2$ and $a \notin S$, where P_2 is a p-semisimple part of X_2. If $a \neq 0$, then $a \in Q - \{0\}$ and by Lemma 3.22, $L_Y(\{x,y\}) = \{a\}$. Hence, by Lemma 3.2, $xy \in E(G(Y))$. If $a = 0$, then $x, y \in V_{X_2}(0)$ and so $xy \in E(G(X_2) - S)$ implies $x = 0$ or $y = 0$. Hence, $xy \in E(G(Y))$. Therefore, $E((G(X_1) \cup G(X_2)) - S) \subseteq E(G(Y))$ and so $E((G(X_1) \cup G(X_2)) - S) = E(G(Y))$. □

Corollary 3.24. *Let $(X_1, *_1, 0)$ be a BCK-algebra, $(X_2, *_2, 0)$ be a BCI-algebra such that $X_1 \cap X_2 = \emptyset$ and the graph $G(V_{X_2}(0))$ be a tree. Then $G(X_1 \cup_L X_2) = G(X_1) \cup G(X_2)$.*

Proof. If $X_1 = \{0\}$, then $X_2 = X_1 \cup_L X_2$ and so $G(X_1 \cup_L X_2) = G(X_2) = G(X_1) \cup G(X_2)$. Now, let $G(V_{X_2}(0))$ be a tree. Then for any distinct elements $x, y \in V_{X_2}(0) - \{0\}$, we have $xy \notin E(G(V_{X_2}(0)))$. Hence, by Proposition 3.8(ii), $xy \notin E(G(X_2))$ and so $S = \emptyset$, where $S = \{xy \in E(G(X_2)) \mid x, y \in V_{X_2}(0) - \{0\}\}$. Therefore, by Proposition 3.23, $G(X_1 \cup_L X_2) = G(X_1) \cup G(X_2)$. □

Example 3.25. (i) Let $X_1 = \{0, a, b, c\}$ and $X_2 = \{0, 1, 2, 3\}$. Define the binary operations $*_1$ and $*_2$ on X_1 and X_2, respectively, by the following tables:

$*_1$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	c	0

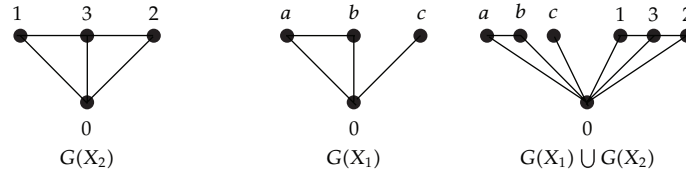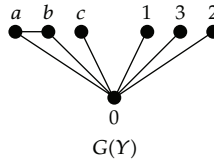| $*_2$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

Figure 5

Figure 6

Then $(X_1, *_1, 0)$ and $(X_2, *_2, 0)$ are *BCK*-algebras (see [10]). Also, $G(X_1)$, $G(X_2)$ and $G(X_1) \cup G(X_2)$ are given by Figure 5.

Let $Y = X_1 \cup_L X_2$. Then $L_Y(\{x, 0\}) = \{0\}$, for any $x \in Y$. Hence, by Lemma 3.22, $x \in L_Y(\{x, y\})$ and $L_Y(\{0, z\}) = \{0\} = L_Y(\{a, b\})$, for any $x \in \{1, 2, 3\}$, $z \in X_1$ and $y \in \{a, b, c\}$. Moreover, $X_1 \subseteq L_Y(\{a, b\}) \cap L_Y(\{a, c\}) \cap L_Y(\{b, c\})$. Hence, $E(G(Y)) = \{0x \mid x \in Y - \{0\}\} \cup \{ab\}$ and so $G(Y)$ is given by Figure 6.

If S is the set was defined in the Proposition 3.23, then $S = \{13, 23\}$. Therefore, $G(Y) = G(X_1) \cup (G(X_2) - S)$.

(ii) Let X be the *BCI*-algebra was defined in Example 3.6(i) and X_1 be the *BCK*-algebra in (i). Then the *BCK*-part of X_2 is a tree. Hence, by Corollary 3.24, $G(X_1 \cup_L X) = G(X_1) \cup G(X)$.

Lemma 3.26. *Let* $(X, *_1, 0)$ *and* $(Y, *_2, 0)$ *be two BCK-algebras and* $x, y \in X \oplus Y$. *Then*

$$
L_{X \oplus Y}\{x, y\} = \begin{cases} L_X\{x, y\} & \text{if } x, y \in X; \\ L_Y\{x, y\} & \text{if } x, y \in Y; \\ \{0\} & \text{otherwise.} \end{cases} \tag{3.7}
$$

Proof. Let $x, y \in X \cup Y$.

(1) If $x, y \in X$, then for any $u \in L_X\{x, y\}$, we have $u * x = 0 = u * y$. If $u \in Y$, then by Proposition 2.5, $u = u * x = 0$ and so $u \in X$. Moreover, if $u \in X$, then $u *_1 x = u * x = 0$ and $u *_1 y = u * y = 0$ and so $u \in L_X\{x, y\}$. Hence, $L_{X \oplus Y}\{x, y\} \subseteq L_X\{x, y\}$. Now, let $u \in L_X\{x, y\}$. Then $u, x, y \in X$ and $u *_1 x = 0 = u *_1 y$ and so $u * x = u *_1 x = 0 = u *_1 y = u * y$. Hence, $L_X\{x, y\} \subseteq L_{X \oplus Y}\{x, y\}$. Therefore, $L_{X \oplus Y}\{x, y\} = L_X\{x, y\}$, for all $x, y \in X$. By the similar way, we can prove that $L_{X \oplus Y}\{x, y\} = L_Y\{x, y\}$, for all $x, y \in Y$.

(2) If $x \in X$ and $y \in Y$. Since $X \oplus Y$ is a *BCK*-algebra, we have $0 \in L_{X \oplus Y}\{x, y\}$. Let $u \in L_{X \oplus Y}\{x, y\}$. Then $u * x = 0 = u * y$. Since $x \in X$ and $y \in Y$, then by definition of
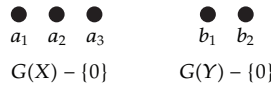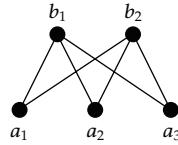
Figure 7

Figure 8: $(G(X) - \{0\}) + (G(Y) - \{0\})$.

"$*$", we have $u * x = u$ or $u * y = u$ and so $u = 0$. Therefore, $L_{X \oplus Y}\{x, y\} = \{0\}$. By the similar way, if $x \in Y$ and $y \in X$, then we can prove that $L_{X \oplus Y}\{x, y\} = \{0\}$.

$\square$

Theorem 3.27. *Let* $(X, *_1, 0)$ *and* $(Y, *_2, 0)$ *be two BCK-algebras. Then* $G(X \oplus Y) - \{0\} = (G(X) - \{0\}) + (G(Y) - \{0\})$.

Proof. Clearly, $V(G(X \oplus Y) - \{0\}) = V((G(X) - \{0\}) + (G(Y) - \{0\}))$. Let $xy \in E(G(X \oplus Y) - \{0\})$. Then $x \neq 0$, $y \neq 0$ and $L_{X \oplus Y}\{x, y\} = \{0\}$. If $x, y \in X$ or $x, y \in Y$, then by Lemma 3.26, $L_X\{x, y\} = \{0\}$ or $L_Y\{x, y\} = \{0\}$ and so $xy \in E(G(X) - \{0\})$ or $xy \in E(G(Y) - \{0\})$. Hence, by Definition 2.8, $xy \in E((G(X) - \{0\}) + (G(Y) - \{0\}))$. If $x \in X$ and $y \in Y$ or $x \in Y$ and $y \in X$, then $x \in V((G(X) - \{0\}))$ and $y \in V(G(Y) - \{0\})$ and so $xy \in E((G(X) - \{0\}) + (G(Y) - \{0\}))$. Now, let $xy \in E((G(X) - \{0\}) + (G(Y) - \{0\}))$. Then $xy \in E(G(X) - \{0\}) \cup E(G(Y) - \{0\})$ or $x \in X - \{0\}$ and $y \in Y - \{0\}$. If $xy \in E(G(X) - \{0\}) \cup E(G(Y) - \{0\})$, then $x \neq 0$, $y \neq 0$ and $L_X\{x, y\} = \{0\}$ or $L_Y\{x, y\} = \{0\}$. Hence, by Lemma 3.26, $xy \in E(G(X \oplus Y) - \{0\})$. If $x \in X - \{0\}$ and $y \in Y - \{0\}$, then by Lemma 3.26, $L_{X \oplus Y} = \{0\}$ and so $xy \in E(G(X \oplus Y) - \{0\})$. Therefore, $E(G(X \oplus Y) - \{0\}) = E((G(X) - \{0\}) + (G(Y) - \{0\}))$. $\square$

Corollary 3.28. *Let* $(X, *_1, 0)$ *and* $(Y, *_2, 0)$ *be two finite BCK-algebras. Then* $|E(X \oplus Y)| = (|X| + |Y| - 1) + (|X| - 1) \cdot (|Y| - 1)$.

Proof. Straightforward. $\square$

Example 3.29. Let $X = \{a_0, a_1, a_2, a_3\}$, $Y = \{b_0, b_1, b_2\}$, $a_0 = b_0$ and "$*$" be the operation was defined in Example 3.13. Then $(X, *_1, a_0)$ and $(Y, *_2, b_0)$ are *BCK*-chain, where $a_i *_1 a_j = a_{i*j}$ and $b_i *_1 b_j = b_{i*j}$. Since they are *BCK*-chain, then by Lemma 3.2, $G(X)$ is 3-star graph and $G(Y)$ is 2-star graph and so $G(X) - \{0\}$ and $G(Y) - \{0\}$ have Figure 7.

Hence, by Definition 2.8, $(G(X) - \{0\}) + (G(Y) - \{0\})$ is given by Figure 8.

On the other hand, by Lemma 3.26, $G(X \oplus Y)$ is given by Figure 9 and so $G(X \oplus Y) - \{0\}$ is the graph on Figure 8.

Let I be an ideal of X. Define a binary relation θ on X as follows: $(x, y) \in \theta$ if and only if $x * y, y * x \in I$, for all $x, y \in X$. Then, θ is a congruence relation and it is called the
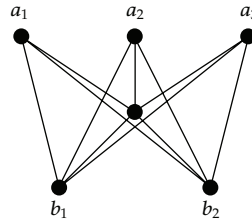
Figure 9: $G(X \oplus Y) - \{0\}$.

equivalence relation induced by I. If $X/I = \{[x] \mid x \in X\}$, then $(X/I, *, [0])$ is a BCI-algebra, where $[x] * [y] = [x * y]$, for all $x, y \in X$ (see [10]). Moreover, let $G = (V, E)$ be a graph and Π be a partition of X. The graph whose vertexes are the elements of Π and for distinct elements $u, v \in \Pi$, there is an edge connecting u and v if and only if $xy \in E$, for some $x \in u$ and $y \in v$, is denoted by G/Π. Now, we want to verify the relation between the $G(X/I)$ and $G(X)/\Pi$, where I is an ideal of X, θ is a congruence relation induced by I and Π is a partition induced by θ.

Theorem 3.30. *Let I be an ideal of X and Π be the partition of X induced by I.*

  (i) *If X is a BCK-algebra, then $G(X)/\Pi$ is a subgraph of $G(X/I)$.*

  (ii) *If X/I is a commutative BCI-algebra, then $G(X)/\Pi$ is a subgraph of $G(X/I)$.*

Proof. (i) Clearly, $V(G(X/I)) = \{[x] \mid x \in X\} = V(G(X)/\Pi)$. Let $[x][y] \in E(G(X)/\Pi)$. Then there are $u \in [x]$ and $v \in [y]$, such that $uv \in E(G(X))$. Hence, $L_X\{u, v\} = \{0\}$. Let $[z] \in L_{X/I}\{[x], [y]\}$. Since $[u] = [x]$ and $[v] = [y]$, then $[z * u] = [z] * [u] = [0] = [z] * [v] = [z * v]$ and so $z * u \in I$ and $z * v \in I$. Hence, $z * u = a$ and $z * v = b$, for some $a, b \in I$. Since X is a BCK-algebra, then by $(BCI6)$, $(z * a) * b \in L_X\{u, v\} = \{0\}$. Since I is an ideal of X and $a, b \in I$, then $z \in I$ and so $z * 0 \in I$ and $0 * z = 0 \in I$. Hence, $[z] = [0]$ and so $L_{X/I}\{[x], [y]\} = \{[0]\}$. Therefore, $[x][y] \in E(G(X/I))$ and so $G(X)/\Pi$ is a subgraph of $G(X/I)$.

    (ii) Clearly, $V(G(X/I)) = \{[x] \mid x \in X\} = V(G(X)/\Pi)$. Let $[x][y] \in E(G(X)/\Pi)$. Then there are $u \in [x]$ and $v \in [y]$, such that $uv \in E(G(X))$ and so $L_X\{u, v\} = \{a\}$, for some $a \in P$. Since $u, v \in V(a)$, then by $(P2)$, $v * u, u * v \in V(0)$ and so by $(BCI6)$, $v * (v * u), u * (u * v) \in L_X\{u, v\} = \{a\}$. Hence, $v * (v * u) = a = u * (u * v)$ and so $[v] * ([v] * [u]) = [a] = [u] * ([u] * [v])$. Let $[z] \in L_{X/I}\{[x], [y]\}$. Then by $(P3)$, $[z] \in V_{X/I}([a])$. Since X/I is a commutative BCI-algebra, then by Lemma 2.2, $[v] * ([v] * [z]) = [z] * ([z] * [v]) = [z]$ and so by $(BCI7)$, we have $[z] = [v] * ([v] * [z]) \leq [v] * ([v] * [u]) = [a]$. Clearly, $a \in Q$, where Q is a p-semisimple part of X/I. Hence, $[a] = [z]$ and so $L_{X/I}\{[x], [y]\} = \{[a]\}$. Therefore, $[x][y] \in E(G(X/I))$ and so $G(X)/\Pi$ is a subgraph of $G(X/I)$. $\qquad\square$

Example 3.31. Let $Y = \{0, 1, 2, a, b\}$ and $X = \{0, 1, 2, 3, 4\}$. Define the binary operations "$*_1$" and "$*_2$" on Y and X, respectively by the following tables:

$*_1$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	0	a	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	a	1	0

$*_2$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	1
3	3	3	3	0	3
4	4	4	4	4	0

(i) It is easy to see that $(Y, *_1, 0)$ is a *BCI*-algebra and $J = \{0, 1\}$ is an ideal of Y (see [10, Example 2.5.5]). Moreover, $[0] = \{0, 1, a, b\}$ and $[2] = \{2\}$. Hence, $Y/J = \{[0], [2]\}$ is a commutative *BCI*-algebra and so $E(G(Y/I)) = \{[0][2]\}$. Moreover, Since $L_Y\{2, 0\} = \{0\}$, then $[0][2] \in E(G(X)/\Phi$, where Φ is the partition of Y induced by J. Therefore, $G(X/J) = G(X)/\Phi$.

(ii) We can prove that $(X, *_2, 0)$ is a *BCK*-algebra and $I = \{0, 1, 2\}$ is an ideal of X (see [10]). Moreover, $X/I = \{[0], [3], [4]\}$, where $[0] = \{0, 1, 2\}$, $[3] = \{3\}$ and $[4] = \{4\}$. It is easy to check that $E(G(X/I)) = \{[0][3], [0][4], [3][4]\}$ and $E(G(X)/\Pi) = \{[0][3], [0][4]\}$.

4. Characteristic Polynomials of Graphs of *BCI*-Algebras

In this section, we verify the characteristic polynomials of graph of *BCI*-algebras. Then we fined the relation between Characteristic polynomial of $G(X)$ and Characteristic polynomial of graphs of branches of X, for any *BCI*-algebra X.

Theorem 4.1. *Let X be a finite BCI-algebra. Then $\chi(G(X), \lambda) = \prod_{a \in P} \chi(G(V(a)), \lambda)$.*

Proof. Let $m \in \mathbb{N}$, $P = \{a_1, \ldots, a_m\}$, $V(a_t) = \{x_{1,t}, \ldots, x_{r,t}\}$ and $A_t = [b_{i,j}]$ be the adjacency matrix of $G(V(a_i))$, for all $t \in \{1, 2, \ldots, m\}$. Then $X = \{x_{1,1}, x_{2,1}, \ldots, x_{r_1,1}, x_{1,2}, x_{2,2}, \ldots, x_{r_2,2}, \ldots, x_{1,m}, x_{2,m}, \ldots, x_{r_m,m}\}$. Since $V(a_i) \cap V(a_j) = \emptyset$, for all distinct $i, j \in \{1, 2, \ldots, m\}$, then by Proposition 3.8(ii), the adjacent matrix of $G(X)$ is of the form

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \tag{4.1}$$

where A_j is a $m_j \times m_j$ matrix, for all $j \in \{1, 2, \ldots, m\}$. By the properties of determinant, we have $\chi(G(X), \lambda) = \det(\lambda I - A) = \det(\lambda I_1 - A_1) \times \det(\lambda I_2 - A_2) \times \cdots \times \det(\lambda I_m - A_m) = \prod_{a \in P} \chi(G(V(a)), \lambda)$, where I_j is a $m_j \times m_j$ identity matrix, for all $j \in \{1, 2, \ldots, m\}$. □

Corollary 4.2. *Let X be a finite BCI-algebra and t is the number of elements $a \in P$ such that $|V(a)| = 1$.*

(i) $\chi(G(X), \lambda) = \lambda^t \times f(\lambda)$, *for some polynomial $f(\lambda)$.*

(ii) *X is a p-semisimple BCI-algebra if and only if $\chi(G(X), \lambda) = \lambda^n$, for some $n \in \mathbb{N}$.*

Proof. (i) Let $|X| = n$ and $\{a_1, \ldots, a_t\}$ be the set of all elements of P such that $|V(a_i)| = 1$, for all $i \in \{1, 2, \ldots, t\}$. Then by using the proof of Theorem 4.1, the adjacent matrix of $G(X)$ is of the form

$$\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}_{n \times n}, \tag{4.2}$$

where B is an $(n-t \times n-t)$ matrix. Hence, by properties of determinant, we have $\chi(G(X), \lambda) = \det(\lambda I_t) \times \det(\lambda I_{n-t} - B) = \lambda^t \times \det(\lambda I_{n-t} - B)$. Let $f(\lambda) = \det(\lambda I_{n-t} - B)$, then $\chi(G(X), \lambda) = \lambda^t \times f(\lambda)$.

(ii) Since X is a p-semisimple BCI-algebra, then $|V(a)| = 1$, for all $a \in X$. Therefore, by (i), $\chi(G(X), \lambda) = \lambda^n$, where $|X| = n$. Conversely, let $\chi(G(X), \lambda) = \lambda^n$, for some $n \in \mathbb{N}$. Then by Proposition 2.11, $G(X)$ is an empty graph. Therefore, by Theorem 3.9, X is a p-semisimple algebra. $\qquad \square$

Theorem 4.3. *Let X be a finite BCI-algebra and $a \in P$. If $\chi(G(V(a)), \lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$, for some $n \in \mathbb{N}$, then*

(i) *$c_1 = 0$ and $2c_2 = -\sum_{x \in X}|Z_x| + |P|$,*

(ii) *let A be the set of all $\{x, y, z\} \subseteq X$ such that x, y, z are distinct elements of X and $u * v = u * a$, for all $u, v \in \{x, y, z\}$ and some $a \in P$. If X is a commutative BCI-algebra, then $c_3 = -2|A|$.*

Proof. (i) By Proposition 2.11, $c_1 = 0$. If $x \in X - P$, then by Lemma 3.2,

$$\deg(x) = \left|\{y \in X - \{x\} \mid L\{x, y\} = a, \text{ for some } a \in P\}\right| = |Z_x|. \qquad (4.3)$$

Now, if $x \in P$, then by Lemma 3.2, $\deg(x) = |\{y \in X - \{x\}|L\{x, y\} = a, \text{ for some } a \in P\}| = |Z_x| - 1$. Hence, by Proposition 2.11, we have

$$2c_2 = -2|E(G(X))| = -\deg(G) = -\left(\sum_{x \in X-P}|Z_x| + \sum_{x \in P}(|Z_x| - 1)\right) = -\sum_{x \in X}|Z_x| + |P|. \qquad (4.4)$$

(ii) By Proposition 2.11, $c_3 = -2t$, where t is the number of triangles of $G(X)$. Since by Proposition 3.8, $G(X) = \bigcup_{a \in P} G(V(a))$ and $V(a) \cap V(b) = \emptyset$, for any distinct elements $a, b \in P$, then $t = \sum_{a \in P} t_a$, where t_a is the number of triangles of $G(V(a))$, for all $a \in P$. Now, let $a \in P$ and x, y, z be three vertexes of a triangles of $G(V(a))$. Then x, y, z are distinct elements of X and $L\{x, y\} = L\{x, z\} = L\{y, z\} = \{a\}$. Since $x * (x * y) \in L\{x, y\}$, then $x * (x * y) = a$ and so by $(BCI4)$ and $(BCI6)$, $x * a \le x * y$. Moreover, $a \le y$ and $(BCI7)$, imply $x * y \le x * a$ and so $x * y = x * a$. By the similar way, we can prove that $x * z = x * a$, $y * z = y * a = y * x$ and $z * x = z * a = z * y$. Hence, $\{x, y, z\} \in A$. Conversely, let $\{x, y, z\} \in A$. Then there exists $a \in P$ such that $u * v = u * a$, for any $u, v \in \{x, y, z\}$. Let $u, v \in \{x, y, z\}$ and $w \in L\{u, v\}$. Then by $(P2)$, $w \in V(a)$. Since X is commutative, then by Lemma 2.2, $(BCI6)$ and $(BCI7)$, $w = w * 0 = w * (w * u) = u * (u * w) \le u * (u * v) = u * (u * a) \le a$. Hence, $w = a$ (since $a \in P$), and so $L\{u, v\} = \{a\}$. Therefore, $xy, xz, yz \in E(G(V(a)))$ and so x, y, z are three vertexes of a triangle of $G(V(a))$, for some $a \in P$. Hence, $t = \sum_{a \in P} t_a = |A|$. This complete the proof. $\qquad \square$

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