

*Research Article*

# Multiple Positive Solutions for a Quasilinear Elliptic System Involving Concave-Convex Nonlinearities and Sign-Changing Weight Functions

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Let  $\Omega \ni 0$  be an-open bounded domain in  $\mathbb{R}^N (N \geq 3)$  and  $p^* = (pN/(N - p))$ . We consider the following quasilinear elliptic system of two equations in  $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ :  $-\Delta_p u = \lambda f(x)|u|^{q-2}u + (\alpha/(\alpha + \beta))h(x)|u|^{\alpha-2}u|v|^\beta, -\Delta_p v = \mu g(x)|v|^{q-2}v + (\beta/(\alpha + \beta))h(x)|u|^\alpha|v|^{\beta-2}v$ , where  $\lambda, \mu > 0$ ,  $\Delta_p$  denotes the  $p$ -Laplacian operator,  $1 \leq q < p < N, \alpha, \beta > 1$  satisfy  $p < \alpha + \beta \leq p^*$ , and  $f, g, h$  are continuous functions on  $\bar{\Omega}$  which are somewhere positive but which may change sign on  $\Omega$ . We establish the existence and multiplicity results of positive solutions to (the above mentioned quasilinear elliptic system equations) by variational methods.

## 1. Introduction and Main Results

Let  $\Omega \ni 0$  be a smooth-bounded domain in  $\mathbb{R}^N$  with  $N \geq 3$ . In this paper, we study the existence and multiplicity of positive solutions for the following quasilinear elliptic system:

$$\begin{aligned} -\Delta_p u &= \lambda f(x)|u|^{q-2}u + \frac{\alpha}{\alpha + \beta}h(x)|u|^{\alpha-2}u|v|^\beta & \text{in } \Omega, \\ -\Delta_p v &= \mu g(x)|v|^{q-2}v + \frac{\beta}{\alpha + \beta}h(x)|u|^\alpha|v|^{\beta-2}v & \text{in } \Omega, \\ u = v &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{S}_{\lambda f, \mu g, h}$$

where  $\lambda, \mu > 0, 1 \leq q < p < N, \Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplacian,  $\alpha > 1, \beta > 1$  satisfy  $p < \alpha + \beta \leq p^*, p^* = (pN)/(N - p)$  denotes the critical Sobolev exponent, and the weight functions  $f, g, h$  are satisfying the following assumptions:

- (A1)  $f, g \in C(\overline{\Omega}), f^+ = \max\{f, 0\} \neq 0, g^+ = \max\{g, 0\} \neq 0$  and  $|f|_\infty = |g|_\infty = 1$ ;  
 (A2)  $h \in C(\overline{\Omega}), h^+ = \max\{h, 0\} \neq 0$  and  $|h|_\infty = 1$ ;  
 (A3) there exist  $a_0, b_0, r_0 > 0$  and  $x_0 \in \Omega$  such that  $B(x_0, 2r_0) \subset \Omega, f(x) \geq a_0$  and  $g(x) \geq b_0$  for all  $x \in B(x_0, 2r_0)$ , without loss of generality, we assume below that  $x_0 = 0$ ;  
 (A4)  $h(x) > 0$  for all  $x \in \Omega, |h|_\infty = h(0)$  and there exists  $\delta_0 > N/(p - 1)$  such that

$$h(x) = h(0) + o(|x|^{\delta_0}) \quad \text{as } x \rightarrow 0. \quad (1.1)$$

System  $(S_{\lambda f, \mu g, h})$  is posed in the framework of the Sobolev space  $W = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$  with the standard norm

$$\|(u, v)\| = \left( \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |\nabla v|^p dx \right)^{1/p}. \quad (1.2)$$

Moreover, a pair of functions  $(u, v) \in W$  is said to be a weak solution of System  $(S_{\lambda f, \mu g, h})$  if

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi_1 dx + \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \varphi_2 dx \\ & - \lambda \int_{\Omega} f(x) |u|^{q-2} u \varphi_1 dx - \mu \int_{\Omega} g(x) |v|^{q-2} v \varphi_2 dx \\ & - \frac{\alpha}{\alpha + \beta} \int_{\Omega} h(x) |u|^{\alpha-2} u |v|^{\beta} \varphi_1 dx - \frac{\beta}{\alpha + \beta} \int_{\Omega} h(x) |u|^{\alpha} |v|^{\beta-2} v \varphi_2 dx = 0 \end{aligned} \quad (1.3)$$

for all  $(\varphi_1, \varphi_2) \in W$ . Thus, the corresponding energy functional of System  $(S_{\lambda f, \mu g, h})$  is defined by

$$I_{\lambda, \mu}(u, v) = \frac{1}{p} \|(u, v)\|^p - \frac{1}{q} \int_{\Omega} (\lambda f(x) |u|^q + \mu g(x) |v|^q) dx - \frac{1}{\alpha + \beta} \int_{\Omega} h(x) |u|^{\alpha} |v|^{\beta} dx. \quad (1.4)$$

Semilinear and quasilinear scalar elliptic equations with concave-convex nonlinearities are widely studied: we refer the reader to Ambrosetti et al. [1], de Figueiredo et al. [2], Azorero and Peral [3], Azorero et al. [4], EL Hamidi [5], Hirano et al. [6], Hsu [7], and Wu [8], and so forth. For the nonlinear elliptic systems, we refer to Adriouch and EL Hamidi [9], Ahammou [10], Alves et al. [11], Bozhkov and Mitidieri [12], Clément et al. [13], de Figueiredo and Felmer [14], EL Hamidi [15], Hsu and Lin [16, 17], Squassina [18], Vélin [19], and Wu [20], and so forth.

Recently, in [8], the author has considered a semilinear scalar elliptic equation involving concave-convex nonlinearities and sign-changing weight functions and showed multiplicity results with respect to the parameter via the extraction of Palais-Smale sequences

in the Nehari manifold, and for the definition of Nehari manifold we refer the reader to see Nehari [21] or Willem [22].

More recently, in [20] the author extends the method of [8] to system  $(S_{\lambda f, \mu g, h})$  in the semilinear case  $p = 2$  with the subcritical case  $2 < \alpha + \beta < 2^*$  and the sign-changing weight functions  $f, g$ . In [16] the author also extends the method of [8] to system  $(S_{\lambda f, \mu g, h})$  in the quasilinear case  $1 < p < N$  with critical case  $\alpha + \beta = p^*$  and the constant weight functions  $f \equiv g \equiv h \equiv 1$ . In the present paper, motivated by [16, 20] we extend and improve the papers by Hsu [16] and Wu [20]. First, we deal with more general weight functions  $f, g, h$  which may be changing sign, and second, we also deal with quasilinear elliptic systems involving subcritical or critical Sobolev exponents.

Let  $S$  be the best Sobolev constant for the embedding of  $W_0^{1,p}(\Omega)$  in  $L^{\alpha+\beta}(\Omega)$  defined by

$$S = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p dx}{\left(\int_{\Omega} |u|^{\alpha+\beta} dx\right)^{p/(\alpha+\beta)}}, \tag{1.5}$$

and set

$$\Lambda_1 = \left(\frac{p-q}{\alpha+\beta-q}\right)^{p/(\alpha+\beta-p)} \left(\frac{\alpha+\beta-q}{\alpha+\beta-p} |\Omega|^{(\alpha+\beta-q)/(\alpha+\beta)}\right)^{-p/(p-q)} S^{p(\alpha+\beta-q)/(p-q)(\alpha+\beta-p)} > 0, \tag{1.6}$$

where  $|\Omega|$  is the Lebesgue measure of  $\Omega$ . Our main results are as follows.

**Theorem 1.1** (existence of one positive solution). *Assume that (A1)-(A2) hold. If  $1 \leq q < p < N$ ,  $p < \alpha + \beta \leq p^*$ , and  $\lambda, \mu > 0$  satisfy  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_1$ , then system  $(S_{\lambda f, \mu g, h})$  has at least one positive solution in  $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ .*

**Theorem 1.2** (second positive solution in the subcritical case). *Assume that (A1)-(A2) hold. If  $1 \leq q < p < N$ ,  $p < \alpha + \beta < p^*$ , and  $\lambda, \mu > 0$  satisfy  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ , then System  $(S_{\lambda f, \mu g, h})$  has at least two positive solutions in  $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ .*

**Theorem 1.3** (second positive solution in the critical case). *Assume that (A1)-(A4) hold. If  $1 \leq q < p < N$ ,  $\alpha + \beta = p^*$ , and  $\lambda, \mu > 0$  satisfy  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ , then System  $(S_{\lambda f, \mu g, h})$  has at least two positive solutions in  $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ .*

This paper is organized as follows. In Section 2, we give some notations and preliminaries. The proofs of Theorems 1.1 and 1.2 are in Section 3. In Section 4, we manage to give the proof of Theorem 1.3. Throughout this paper, (A1) and (A2) will be assumed.

## 2. Notations and Preliminaries

In this section, we give some notations and necessary preliminary results.

*Notations.* We make use of the following notation.

$L^s(\Omega)$ ,  $1 \leq s < \infty$ , denote Lebesgue spaces; the norm  $L^s$  is denoted by  $|\cdot|_s$  for  $1 \leq s \leq \infty$ ;

$W = [W_0^{1,p}(\Omega)]^2$ , endowed with norm  $\|z\|^p = \|(u, v)\|^p = |\nabla u|_p^p + |\nabla v|_p^p$ ;

The dual space of a Banach space  $W$  will be denoted by  $W^{-1}$ ;

$tz = t(u, v) = (tu, tv)$  for all  $z = (u, v) \in W$  and  $t \in \mathbb{R}$ ;

$|z| = (|u|, |v|)$  for all  $z = (u, v) \in W$ ;

$z = (u, v)$  is said to be nonnegative in  $\Omega$  if  $u \geq 0$  and  $v \geq 0$  in  $\Omega$ ;

$z = (u, v)$  is said to be positive in  $\Omega$  if  $u > 0$  and  $v > 0$  in  $\Omega$ ;

$|\Omega|$  is the Lebesgue measure of  $\Omega$ ;

$B(x_0, r) = \{x \in \mathbb{R}^N \mid |x - x_0| < r\}$  is the ball in  $\mathbb{R}^N$ ;

$O(\varepsilon^t)$  denotes  $|O(\varepsilon^t)|/\varepsilon^t \leq C$  as  $\varepsilon \rightarrow 0$  for  $t \geq 0$ ;

$o_n(1)$  denotes  $o_n(1) \rightarrow 0$  as  $n \rightarrow \infty$ ;

$C, C_i$  will denote various positive constants, the exact values of which are not important;

$p^* = pN/(N - p)$  ( $1 < p < N$ ) is the critical Sobolev exponent;

Let  $J, K_{\lambda, \mu} : W \rightarrow \mathbb{R}$  be the functionals defined by

$$\begin{aligned} J(z) &= \int_{\Omega} h(x)|u|^\alpha |v|^\beta dx, \\ K_{\lambda, \mu}(z) &= \int_{\Omega} (\lambda f(x)|u|^q + \mu g(x)|v|^q) dx \end{aligned} \quad (2.1)$$

for all  $z = (u, v) \in W$ .

As the energy functional  $I_{\lambda, \mu}$  is not bounded below on  $W$ , it is useful to consider the functional on the Nehari manifold

$$\mathcal{N}_{\lambda, \mu} = \left\{ z \in W \setminus \{0\} \mid \langle I'_{\lambda, \mu}(z), z \rangle = 0 \right\}. \quad (2.2)$$

Thus,  $z = (u, v) \in \mathcal{N}_{\lambda, \mu}$  if and only if

$$\langle I'_{\lambda, \mu}(z), z \rangle = \|z\|^p - K_{\lambda, \mu}(z) - J(z) = 0. \quad (2.3)$$

Note that  $\mathcal{N}_{\lambda, \mu}$  contains every nonzero solution of System  $(S_{\lambda, \mu, g, h})$ . Moreover, we have the following results

**Lemma 2.1.** *The energy functional  $I_{\lambda, \mu}$  is coercive and bounded on  $\mathcal{N}_{\lambda, \mu}$ .*

*Proof.* If  $z = (u, v) \in \mathcal{N}_{\lambda, \mu}$ , then by (2.3), the Hölder inequality, and the Sobolev embedding theorem,

$$I_{\lambda, \mu}(z) = \frac{\alpha + \beta - p}{p(\alpha + \beta)} \|z\|^p - \frac{\alpha + \beta - q}{q(\alpha + \beta)} K_{\lambda, \mu}(z) \quad (2.4)$$

$$\geq \frac{\alpha + \beta - p}{p(\alpha + \beta)} \|z\|^p - \frac{\alpha + \beta - q}{q(\alpha + \beta)} S^{-q/p} |\Omega|^{(\alpha + \beta - q)/(\alpha + \beta)} (\lambda^{p/(p-q)} + \mu^{p/(p-q)})^{(p-q)/p} \|z\|^q. \quad (2.5)$$

Thus,  $I_{\lambda, \mu}$  is coercive and bounded on  $\mathcal{N}_{\lambda, \mu}$ . □

Define

$$\Phi_{\lambda, \mu}(z) = \langle I'_{\lambda, \mu}(z), z \rangle. \quad (2.6)$$

Then for  $u \in \mathcal{N}_{\lambda, \mu}$ ,

$$\begin{aligned} \langle \Phi'_{\lambda, \mu}(z), z \rangle &= p\|z\|^p - qK_{\lambda, \mu}(z) - (\alpha + \beta)J(z) \\ &= (p - q)\|z\|^p - (\alpha + \beta - q)J(z) \end{aligned} \quad (2.7)$$

$$= (\alpha + \beta - q)K_{\lambda, \mu}(z) - (\alpha + \beta - p)\|z\|^p. \quad (2.8)$$

Similar to the method used in Tarantello [23], we split  $\mathcal{N}_{\lambda, \mu}$  into three parts:

$$\begin{aligned} \mathcal{N}_{\lambda, \mu}^+ &= \{z \in \mathcal{N}_{\lambda, \mu} : \langle \Phi'_{\lambda, \mu}(z), z \rangle > 0\}, \\ \mathcal{N}_{\lambda, \mu}^0 &= \{z \in \mathcal{N}_{\lambda, \mu} : \langle \Phi'_{\lambda, \mu}(z), z \rangle = 0\}, \\ \mathcal{N}_{\lambda, \mu}^- &= \{z \in \mathcal{N}_{\lambda, \mu} : \langle \Phi'_{\lambda, \mu}(z), z \rangle < 0\}. \end{aligned} \quad (2.9)$$

Then, we have the following results.

**Lemma 2.2.** Assume that  $z_0$  is a local minimizer for  $I_{\lambda, \mu}$  on  $\mathcal{N}_{\lambda, \mu}$  and  $z_0 \notin \mathcal{N}_{\lambda, \mu}^0$ . Then  $I'_{\lambda, \mu}(z_0) = 0$  in  $W^{-1}$ .

*Proof.* Our proof is almost the same as that in [24, Theorem 2.3]. □

**Lemma 2.3.** One has the following:

- (i) if  $z \in \mathcal{N}_{\lambda, \mu}^+$ , then  $K_{\lambda, \mu}(z) > 0$ ;
- (ii) if  $z \in \mathcal{N}_{\lambda, \mu}^0$ , then  $K_{\lambda, \mu}(z) > 0$  and  $J(z) > 0$ ;
- (iii) if  $z \in \mathcal{N}_{\lambda, \mu}^-$ , then  $J(z) > 0$ .

*Proof.* The proof is immediate from (2.7) and (2.8). □

Moreover, we have the following result.

**Lemma 2.4.** If  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_1$ , then  $\mathcal{N}_{\lambda, \mu}^0 = \emptyset$  where  $\Lambda_1$  is the same as in (1.6).

*Proof.* We argue by contradiction. Assume that there exist  $\lambda, \mu > 0$  with

$$0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_1 \quad (2.10)$$

such that  $\mathcal{N}_{\lambda,\mu}^0 \neq \emptyset$ . Then by (2.7) and (2.8), for  $u \in \mathcal{N}_{\lambda,\mu}^0$ , one has the following:

$$\begin{aligned} \|z\|^p &= \frac{\alpha + \beta - q}{p - q} J(z), \\ \|z\|^p &= \frac{\alpha + \beta - q}{\alpha + \beta - p} K_{\lambda,\mu}(z). \end{aligned} \quad (2.11)$$

By  $|f|_\infty = |g|_\infty = |h|_\infty = 1$ , the Hölder inequality, and the Sobolev embedding theorem, we have

$$\begin{aligned} \|z\| &\geq \left( \frac{p - q}{\alpha + \beta - q} S^{(\alpha+\beta)/p} \right)^{1/(\alpha+\beta-p)}, \\ \|z\| &\leq \left( \frac{\alpha + \beta - q}{\alpha + \beta - p} S^{-q/p} |\Omega|^{(\alpha+\beta-q)/(\alpha+\beta)} \right)^{1/(p-q)} \left( \lambda^{p/(p-q)} + \mu^{p/(p-q)} \right)^{1/p}. \end{aligned} \quad (2.12)$$

This implies

$$\begin{aligned} \lambda^{p/(p-q)} + \mu^{p/(p-q)} \lambda^{p/(p-q)} + \mu^{p/(p-q)} &\geq \left( \frac{p - q}{\alpha + \beta - q} \right)^{p/(\alpha+\beta-p)} \left( \frac{\alpha + \beta - q}{\alpha + \beta - p} |\Omega|^{(\alpha+\beta-q)/(\alpha+\beta)} \right)^{-p/(p-q)} \\ &\quad \times S^{(\alpha+\beta)/(\alpha+\beta-p)+q/(p-q)} = \Lambda_1, \end{aligned} \quad (2.13)$$

which contradicts  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_1$ .  $\square$

By Lemma 2.4, we write  $\mathcal{N}_{\lambda,\mu} = \mathcal{N}_{\lambda,\mu}^+ \cup \mathcal{N}_{\lambda,\mu}^-$  and define

$$\theta_{\lambda,\mu} = \inf_{z \in \mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(z); \quad \theta_{\lambda,\mu}^+ = \inf_{z \in \mathcal{N}_{\lambda,\mu}^+} I_{\lambda,\mu}(z); \quad \theta_{\lambda,\mu}^- = \inf_{z \in \mathcal{N}_{\lambda,\mu}^-} I_{\lambda,\mu}(z). \quad (2.14)$$

Then we get the following result.

**Theorem 2.5.** (i) If  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_1$ , then one has  $\theta_{\lambda,\mu} \leq \theta_{\lambda,\mu}^+ < 0$ ;

(ii) if  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ , then  $\theta_{\lambda,\mu}^- > d_0$  for some positive constant  $d_0$  depending on  $\lambda, \mu, p, q, N, S, |\Omega|, |f|_\infty, |g|_\infty$  and  $|h|_\infty$ .

*Proof.* (i) Let  $z = (u, v) \in \mathcal{N}_{\lambda,\mu}^+$ . By (2.7)

$$\frac{p - q}{\alpha + \beta - q} \|z\|^p > J(z) \quad (2.15)$$

and so

$$\begin{aligned}
 I_{\lambda,\mu}(z) &= \left(\frac{1}{p} - \frac{1}{q}\right) \|z\|^p + \left(\frac{1}{q} - \frac{1}{\alpha + \beta}\right) J(z) \\
 &< \left[\left(\frac{1}{p} - \frac{1}{q}\right) + \left(\frac{1}{q} - \frac{1}{\alpha + \beta}\right) \frac{p - q}{\alpha + \beta - q}\right] \|z\|^p \\
 &= -\frac{(p - q)(\alpha + \beta - p)}{pq(\alpha + \beta)} \|z\|^p < 0.
 \end{aligned}
 \tag{2.16}$$

Therefore, from the definitions of  $\theta_{\lambda,\mu}, \theta_{\lambda,\mu}^+$ , we can deduce that  $\theta_{\lambda,\mu} \leq \theta_{\lambda,\mu}^+ < 0$ .

(ii) Let  $z \in \mathcal{N}_{\lambda,\mu}^-$ . By (2.7)

$$\frac{p - q}{\alpha + \beta - q} \|z\|^p < J(z).
 \tag{2.17}$$

Moreover, by  $|h|_\infty = 1$ , the Hölder inequality, and the Sobolev embedding theorem,

$$J(z) \leq S^{-(\alpha+\beta)/p} \|z\|^{\alpha+\beta}.
 \tag{2.18}$$

This implies

$$\|z\| > \left(\frac{p - q}{\alpha + \beta - q}\right)^{1/(\alpha+\beta-p)} S^{(\alpha+\beta)/p(\alpha+\beta-p)} \quad \forall z \in \mathcal{N}_{\lambda,\mu}^-.
 \tag{2.19}$$

By (2.5), one has the following:

$$\begin{aligned}
 I_{\lambda,\mu}(z) &\geq \|z\|^q \left[ \frac{\alpha + \beta - p}{p(\alpha + \beta)} \|z\|^{p-q} - \frac{\alpha + \beta - q}{q(\alpha + \beta)} S^{-q/p} |\Omega|^{(\alpha+\beta-q)/(\alpha+\beta)} \left( \lambda^{p/(p-q)} + \mu^{p/(p-q)} \right)^{(p-q)/p} \right] \\
 &> \left(\frac{p - q}{\alpha + \beta - q}\right)^{q/(\alpha+\beta-p)} S^{q(\alpha+\beta)/p(\alpha+\beta-p)} \\
 &\quad \times \left[ \frac{\alpha + \beta - p}{p(\alpha + \beta)} S^{(p-q)(\alpha+\beta)/p(\alpha+\beta-p)} \left(\frac{p - q}{\alpha + \beta - q}\right)^{p-q/(\alpha+\beta-p)} \right. \\
 &\quad \left. - \frac{\alpha + \beta - q}{q(\alpha + \beta)} S^{-q/p} |\Omega|^{(\alpha+\beta-q)/(\alpha+\beta)} \left( \lambda^{p/(p-q)} + \mu^{p/(p-q)} \right)^{(p-q)/p} \right].
 \end{aligned}
 \tag{2.20}$$

Thus, if  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ , then

$$I_{\lambda,\mu}(z) > d_0 \quad \forall z \in \mathcal{N}_{\lambda,\mu}^-,
 \tag{2.21}$$

for some positive constant  $d_0 = d_0(\lambda, \mu, p, q, N, S, |\Omega|, |f|_\infty, |g|_\infty, |h|_\infty)$ . This completes the proof.  $\square$

For each  $z \in W$  with  $J(z) > 0$ , we write

$$t_{\max} = \left( \frac{(p-q)\|z\|^p}{(\alpha+\beta-q)J(z)} \right)^{1/(\alpha+\beta-p)} > 0. \quad (2.22)$$

Then the following lemma holds.

**Lemma 2.6.** *Let  $\lambda^{p/(p-q)} + \mu^{p/(p-q)} \in (0, \Lambda_1)$ . For each  $z \in W$  with  $J(z) > 0$ , one has the following:*

(i) *if  $K_{\lambda,\mu}(z) \leq 0$ , then there exists a unique  $t^- > t_{\max}$  such that  $t^-z \in \mathcal{N}_{\lambda,\mu}^-$  and*

$$I_{\lambda,\mu}(t^-z) = \sup_{t \geq 0} I_{\lambda,\mu}(tz); \quad (2.23)$$

(ii) *if  $K_{\lambda,\mu}(z) > 0$ , then there exist unique  $0 < t^+ < t_{\max} < t^-$  such that  $t^+z \in \mathcal{N}_{\lambda,\mu}^+$ ,  $t^-z \in \mathcal{N}_{\lambda,\mu}^-$  and*

$$I_{\lambda,\mu}(t^+z) = \inf_{0 \leq t \leq t_{\max}} I_{\lambda,\mu}(tz); \quad I_{\lambda,\mu}(t^-z) = \sup_{t \geq 0} I_{\lambda,\mu}(tz). \quad (2.24)$$

*Proof.* The proof is almost the same as that in [25, Lemma 2.6] and is omitted here.  $\square$

For each  $z \in W$  with  $K_{\lambda,\mu}(z) > 0$ , we write

$$\bar{t}_{\max} = \left( \frac{(\alpha+\beta-q)K_{\lambda,\mu}(z)}{(\alpha+\beta-p)\|z\|^p} \right)^{1/(p-q)} > 0. \quad (2.25)$$

Then we have the following lemma.

**Lemma 2.7.** *Let  $\lambda^{p/(p-q)} + \mu^{p/(p-q)} \in (0, \Lambda_1)$ . For each  $z \in W$  with  $K_{\lambda,\mu}(z) > 0$ , one has the following:*

(i) *if  $J(z) \leq 0$ , then there exists a unique  $0 < t^+ < \bar{t}_{\max}$  such that  $t^+z \in \mathcal{N}_{\lambda,\mu}^+$  and*

$$I_{\lambda,\mu}(t^+z) = \inf_{t \geq 0} I_{\lambda,\mu}(tz); \quad (2.26)$$

(ii) *if  $J(z) > 0$ , then there exist unique  $0 < t^+ < \bar{t}_{\max} < t^-$  such that  $t^+z \in \mathcal{N}_{\lambda,\mu}^+$ ,  $t^-z \in \mathcal{N}_{\lambda,\mu}^-$  and*

$$I_{\lambda,\mu}(t^+z) = \inf_{0 \leq t \leq \bar{t}_{\max}} I_{\lambda,\mu}(tz); \quad I_{\lambda,\mu}(t^-z) = \sup_{t \geq 0} I_{\lambda,\mu}(tz). \quad (2.27)$$

*Proof.* The proof is almost the same as that in [25, Lemma 2.7] and is omitted here.  $\square$



### 3. Proofs of Theorems 1.1 and 1.2

First, we give the following definitions about  $(PS)_c$ -sequence.

*Definition 3.1.* Let  $c \in \mathbb{R}$ ,  $W$  be a Banach space and  $I \in C^1(W, \mathbb{R})$ .

- (i)  $\{z_n\}$  is a  $(PS)_c$ -sequence in  $W$  for  $I$  if  $I(z_n) = c + o_n(1)$  and  $I'(z_n) = o_n(1)$  strongly in  $W^{-1}$  as  $n \rightarrow \infty$ .
- (ii) We say that  $I$  satisfies the  $(PS)_c$  condition if any  $(PS)_c$ -sequence  $\{z_n\}$  in  $W$  for  $I$  has a convergent subsequence.

Now, we use the idea of Tarantello [23] to get the following results.

**Proposition 3.2.** Let  $1 \leq q < p < N$  and  $p < \alpha + \beta \leq p^*$ , we have

- (i) if  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_1$ , then there exists a  $(PS)_{\theta_{\lambda,\mu}}$ -sequence  $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}$  in  $W$  for  $I_{\lambda,\mu}$ ;
- (ii) if  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ , then there exists a  $(PS)_{\theta_{\lambda,\mu}^-}$ -sequence  $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}$  in  $W$  for  $I_{\lambda,\mu}$ ,

where  $\Lambda_1$  is the positive constant given in (1.6).

*Proof.* The proof is almost the same as that in [8, Proposition 9]. □

Now, we establish the existence of a local minimum for  $I_{\lambda,\mu}$  on  $\mathcal{N}_{\lambda,\mu}^+$ .

**Theorem 3.3.** Let  $\Lambda_1$  be the same positive constant as in (1.6). If  $1 \leq q < p < N$ ,  $p < \alpha + \beta \leq p^*$ , and  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_1$ , then  $I_{\lambda,\mu}$  has a minimizer  $z_{\lambda,\mu}^1$  in  $\mathcal{N}_{\lambda,\mu}^+$  and it satisfies the following:

- (i)  $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu} = \theta_{\lambda,\mu}^+ < 0$ ;
- (ii)  $z_{\lambda,\mu}^1$  is a positive solution of System  $(S_{\lambda f, \mu g, h})$ ;
- (iii)  $I_{\lambda,\mu}(z_{\lambda,\mu}^1) \rightarrow 0$  as  $\lambda \rightarrow 0^+$ ,  $\mu \rightarrow 0^+$ .

*Proof.* By Proposition 3.2 (i), there exists a minimizing sequence  $\{z_n\}$  for  $I_{\lambda,\mu}$  on  $\mathcal{N}_{\lambda,\mu}$  such that

$$I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu} + o_n(1), \quad I'_{\lambda,\mu}(z_n) = o_n(1) \quad \text{in } W^{-1}. \tag{3.1}$$

Since  $I_{\lambda,\mu}$  is coercive on  $\mathcal{N}_{\lambda,\mu}$  (see Lemma 2.1), we get that  $\{z_n\}$  is bounded in  $W$ . Then there exist a subsequence  $\{z_n = (u_n, v_n)\}$  and  $z_{\lambda,\mu}^1 = (u_{\lambda,\mu}^1, v_{\lambda,\mu}^1) \in W$  such that

$$\begin{aligned} u_n &\rightharpoonup u_{\lambda,\mu}^1, & v_n &\rightharpoonup v_{\lambda,\mu}^1 && \text{weakly in } W_0^{1,p}(\Omega), \\ u_n &\rightarrow u_{\lambda,\mu}^1, & v_n &\rightarrow v_{\lambda,\mu}^1 && \text{almost everywhere in } \Omega, \\ u_n &\rightarrow u_{\lambda,\mu}^1, & v_n &\rightarrow v_{\lambda,\mu}^1 && \text{strongly in } L^s(\Omega) \quad \forall 1 \leq s < p^*. \end{aligned} \tag{3.2}$$

This implies

$$K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z_{\lambda,\mu}^1) + o_n(1) \quad \text{as } n \rightarrow \infty. \quad (3.3)$$

First, we claim that  $z_{\lambda,\mu}^1$  is a nontrivial solution of System  $(S_{\lambda f, \mu g, h})$ . By (3.1) and (3.2), it is easy to verify that  $z_{\lambda,\mu}^1$  is a weak solution of System  $(S_{\lambda f, \mu g, h})$ . From  $z_n \in \mathcal{N}_{\lambda,\mu}$  and (2.4), we deduce that

$$K_{\lambda,\mu}(z_n) = \frac{q(\alpha + \beta - p)}{p(\alpha + \beta - q)} \|z_n\|^p - \frac{q(\alpha + \beta)}{\alpha + \beta - q} I_{\lambda,\mu}(z_n). \quad (3.4)$$

Let  $n \rightarrow \infty$  in (3.4), by (3.1), (3.3), and  $\theta_{\lambda,\mu} < 0$ , we get

$$K_{\lambda,\mu}(z_{\lambda,\mu}^1) \geq -\frac{q(\alpha + \beta)}{\alpha + \beta - q} \theta_{\lambda,\mu} > 0. \quad (3.5)$$

Thus,  $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}$  is a nontrivial solution of System  $(S_{\lambda f, \mu g, h})$ . Now we prove that  $z_n \rightarrow z_{\lambda,\mu}^1$  strongly in  $W$  and  $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu}$ . By (3.4), if  $z \in \mathcal{N}_{\lambda,\mu}$ , then

$$I_{\lambda,\mu}(z) = \frac{\alpha + \beta - p}{p(\alpha + \beta)} \|z\|^p - \frac{\alpha + \beta - q}{q(\alpha + \beta)} K_{\lambda,\mu}(z). \quad (3.6)$$

In order to prove that  $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu}$ , it suffices to recall that  $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}$ , by (3.6), and apply Fatou's lemma to get

$$\begin{aligned} \theta_{\lambda,\mu} &\leq I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \frac{\alpha + \beta - p}{p(\alpha + \beta)} \|z_{\lambda,\mu}^1\|^p - \frac{\alpha + \beta - q}{q(\alpha + \beta)} K_{\lambda,\mu}(z_{\lambda,\mu}^1) \\ &\leq \liminf_{n \rightarrow \infty} \left( \frac{\alpha + \beta - p}{p(\alpha + \beta)} \|z_n\|^p - \frac{\alpha + \beta - q}{q(\alpha + \beta)} K_{\lambda,\mu}(z_n) \right) \\ &\leq \liminf_{n \rightarrow \infty} I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}. \end{aligned} \quad (3.7)$$

This implies that  $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu}$  and  $\lim_{n \rightarrow \infty} \|z_n\|^p = \|z_{\lambda,\mu}^1\|^p$ . Let  $\tilde{z}_n = z_n - z_{\lambda,\mu}^1$ , then by Brézis and Lieb lemma [26] implies

$$\|\tilde{z}_n\|^p = \|z_n\|^p - \|z_{\lambda,\mu}^1\|^p. \quad (3.8)$$

Therefore,  $z_n \rightarrow z_{\lambda,\mu}^1$  strongly in  $W$ . Moreover, we have  $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+$ . Thus  $\theta_{\lambda,\mu} = \theta_{\lambda,\mu}^+$ . On the contrary, if  $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^-$ , then by (2.17), (3.5), we have that  $J(z_{\lambda,\mu}^1) > 0$  and  $K_{\lambda,\mu}(z_{\lambda,\mu}^1) > 0$ . Thus,

from Lemma 2.6 (ii), there exist unique  $t_1^+$  and  $t_1^-$  such that  $t_1^+ z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+$  and  $t_1^- z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^-$ . In particular, we have  $t_1^+ < t_1^- = 1$ . Since

$$\frac{d}{dt} I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) = 0, \quad \frac{d^2}{dt^2} I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) > 0, \tag{3.9}$$

there exists  $t_1^+ < \bar{t} \leq t_1^-$  such that  $I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) < I_{\lambda,\mu}(\bar{t} z_{\lambda,\mu}^1)$ . By Lemma 2.6,

$$I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) < I_{\lambda,\mu}(\bar{t} z_{\lambda,\mu}^1) \leq I_{\lambda,\mu}(t_1^- z_{\lambda,\mu}^1) = I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu}, \tag{3.10}$$

which is a contradiction. Since  $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = I_{\lambda,\mu}(|z_{\lambda,\mu}^1|)$  and  $|z_{\lambda,\mu}^1| \in \mathcal{N}_{\lambda,\mu}^+$ , by Lemma 2.2 we may assume that  $z_{\lambda,\mu}^1$  is a nontrivial nonnegative solution of System  $(S_{\lambda f, \mu g, h})$ .

In particular  $u_{\lambda,\mu}^1 \not\equiv 0, v_{\lambda,\mu}^1 \not\equiv 0$ . Indeed, without loss of generality, we may assume that  $v_{\lambda,\mu}^1 \equiv 0$ . Then as  $u_{\lambda,\mu}^1$  is a nontrivial nonnegative solution of

$$\begin{aligned} -\Delta_p u &= \lambda f(x) |u|^{q-2} u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.11}$$

By the standard regularity theory, we have  $u_{\lambda,\mu}^1 > 0$  in  $\Omega$  and

$$\|(u_{\lambda,\mu}^1, 0)\|^p = K_{\lambda,\mu}(u_{\lambda,\mu}^1, 0) > 0. \tag{3.12}$$

Moreover, by conditions (A1), (A2) and  $u_{\lambda,\mu}^1 > 0$  in  $\Omega$ , we may choose  $w \in W_0^{1,p}(\Omega) \setminus \{0\}$  such that

$$\begin{aligned} \|(0, w)\|^p &= K_{\lambda,\mu}(0, w) > 0, \\ J(u_{\lambda,\mu}^1, w) &\geq 0. \end{aligned} \tag{3.13}$$

Now

$$K_{\lambda,\mu}(u_{\lambda,\mu}^1, w) = K_{\lambda,\mu}(u_{\lambda,\mu}^1, 0) + K_{\lambda,\mu}(0, w) > 0, \tag{3.14}$$

and so by Lemma 2.7 there is unique  $0 < t^+ < \bar{t}_{\max}$  such that  $(t^+ u_{\lambda,\mu}^1, t^+ w) \in \mathcal{N}_{\lambda,\mu}^+$ . Moreover,

$$\begin{aligned} \bar{t}_{\max} &= \left( \frac{(\alpha + \beta - q) K_{\lambda,\mu}(u_{\lambda,\mu}^1, w)}{(\alpha + \beta - p) \|(u_{\lambda,\mu}^1, w)\|^p} \right) = \left( \frac{\alpha + \beta - q}{\alpha + \beta - p} \right) > 1, \\ I_{\lambda,\mu}(t^+ u_{\lambda,\mu}^1, t^+ w) &= \inf_{0 \leq t \leq \bar{t}_{\max}} I_{\lambda,\mu}(t u_{\lambda,\mu}^1, t w). \end{aligned} \tag{3.15}$$

This implies

$$\theta_{\lambda,\mu}^+ \leq I_{\lambda,\mu}(t^+ u_{\lambda,\mu}^1, t^+ w) \leq I_{\lambda,\mu}(u_{\lambda,\mu}^1, w) < I_{\lambda,\mu}(u_{\lambda,\mu}^1, 0) = \theta_{\lambda,\mu}^+ \quad (3.16)$$

which is a contradiction.

By a standard bootstrap argument, it is proved that a weak solution for System  $(S_{\lambda f, \mu g, h})$  is in the space  $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ , and it is really a classical solution. Finally, by the Harnack inequality [27] we deduce that  $z_{\lambda,\mu}^1$  is a positive solution of System  $(S_{\lambda f, \mu g, h})$ . Moreover, by Theorem 2.5 (i) and (2.5) we have

$$0 > \theta_{\lambda,\mu}^+ = I_{\lambda,\mu}(z_{\lambda,\mu}^1) > -\frac{\alpha + \beta - q}{q(\alpha + \beta)} S^{-q/p} |\Omega|^{(\alpha + \beta - q)/(\alpha + \beta)} (\lambda^{p/(p-q)} + \mu^{p/(p-q)})^{(p-q)/p} \|z_{\lambda,\mu}^1\|^q. \quad (3.17)$$

This implies that  $I_{\lambda,\mu}(z_{\lambda,\mu}^1) \rightarrow 0$  as  $\lambda \rightarrow 0^+$ ,  $\mu \rightarrow 0^+$ . □

Next, we establish the existence of a local minimum for  $I_{\lambda,\mu}$  on  $\mathcal{N}_{\lambda,\mu}^-$  in the subcritical case  $p < \alpha + \beta < p^*$ . This implies that there exists the second positive solution in the subcritical case  $p < \alpha + \beta < p^*$ .

**Theorem 3.4.** *Let  $\Lambda_1$  be the same positive constant as in (1.6). If  $1 \leq q < p < N$ ,  $p < \alpha + \beta < p^*$ , and  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ , then  $I_{\lambda,\mu}$  has a minimizer  $z_{\lambda,\mu}^2$  in  $\mathcal{N}_{\lambda,\mu}^-$  and it satisfies the following:*

- (i)  $I_{\lambda,\mu}(z_{\lambda,\mu}^2) = \theta_{\lambda,\mu}^-$ ;
- (ii)  $z_{\lambda,\mu}^2$  is a positive solution of System  $(S_{\lambda f, \mu g, h})$ ;

*Proof.* Let  $\{z_n\}$  be a minimizing sequence for  $I_{\lambda,\mu}$  on  $\mathcal{N}_{\lambda,\mu}^-$ . Then by  $I_{\lambda,\mu}$  coercive on  $\mathcal{N}_{\lambda,\mu}$  and the compact imbedding theorem, there exist a subsequence  $\{z_n = (u_n, v_n)\}$  and  $z_{\lambda,\mu}^2 = (u_{\lambda,\mu}^2, v_{\lambda,\mu}^2) \in W$  such that

$$\begin{aligned} u_n &\rightharpoonup u_{\lambda,\mu}^2, \quad v_n \rightharpoonup v_{\lambda,\mu}^2 \quad \text{weakly in } W_0^{1,p}(\Omega), \\ u_n &\rightarrow u_{\lambda,\mu}^2, \quad v_n \rightarrow v_{\lambda,\mu}^2 \quad \text{strongly in } L^q(\Omega), L^{\alpha+\beta}(\Omega). \end{aligned} \quad (3.18)$$

This implies

$$K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z_{\lambda,\mu}^2) + o_n(1), \quad J(z_n) = J(z_{\lambda,\mu}^2) + o_n(1). \quad (3.19)$$

By (2.17) and (2.19) there exists a positive number  $\overline{C}$  such that

$$J(z_n) > \overline{C}. \quad (3.20)$$

This implies

$$J(z_{\lambda,\mu}^2) \geq \bar{c}. \tag{3.21}$$

Now, we prove that  $z_n \rightarrow z_{\lambda,\mu}^2$  strongly in  $W$ . Suppose otherwise, then  $\|z_{\lambda,\mu}^2\| < \liminf_{n \rightarrow \infty} \|z_n\|$ . By Lemma 2.6, there exists a unique  $t_2^-$  such that  $t_2^- z_{\lambda,\mu}^2 \in \mathcal{N}_{\lambda,\mu}^-$ . Since  $z_n \in \mathcal{N}_{\lambda,\mu}^-$ ,  $I_{\lambda,\mu}(z_n) \geq I_{\lambda,\mu}(t z_n)$  for all  $t \geq 0$ , we have

$$\theta_{\lambda,\mu}^- \leq I_{\lambda,\mu}(t^- z_{\lambda,\mu}^2) < \lim_{n \rightarrow \infty} I_{\lambda,\mu}(t^- z_n) \leq \lim_{n \rightarrow \infty} I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}^-, \tag{3.22}$$

and this is a contradiction. Hence

$$z_n \rightarrow z_{\lambda,\mu}^2 \text{ strongly in } W. \tag{3.23}$$

This implies

$$I_{\lambda,\mu}(z_{\lambda,\mu}^2) = \lim_{n \rightarrow \infty} I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}^-. \tag{3.24}$$

Since  $I_{\lambda,\mu}(z_{\lambda,\mu}^2) = I_{\lambda,\mu}(|z_{\lambda,\mu}^2|)$  and  $|z_{\lambda,\mu}^2| \in \mathcal{N}_{\lambda,\mu}^-$ , by Lemma 2.2 and (3.21) we may assume  $z_{\lambda,\mu}^2$  is a nontrivial nonnegative solution of System  $(S_{\lambda f, \mu g, h})$ . Finally, by using the same arguments as in the proof of Theorem 3.3, for all  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ , we have that  $z_{\lambda,\mu}^2$  is a positive solution of System  $(S_{\lambda f, \mu g, h})$ .  $\square$

Now, we complete the proofs of Theorems 1.1 and 1.2: Theorem 1.1 follows from Theorem 3.3. By Theorems 3.3 and 3.4, we obtain that for all  $1 \leq q < p < N, p < \alpha + \beta < p^*$ ,  $\lambda, \mu > 0$ , and  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ . System  $(S_{\lambda f, \mu g, h})$  has two positive solutions  $z_{\lambda,\mu}^1, z_{\lambda,\mu}^2$  with  $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+, z_{\lambda,\mu}^2 \in \mathcal{N}_{\lambda,\mu}^-$ . Since  $\mathcal{N}_{\lambda,\mu}^+ \cap \mathcal{N}_{\lambda,\mu}^- = \emptyset$ , this implies that  $z_{\lambda,\mu}^1$  and  $z_{\lambda,\mu}^2$  are distinct. This completes the proofs of Theorems 1.1 and 1.2.

### 4. Proof of Theorem 1.3

For the existence of a second positive solution of System  $(S_{\lambda f, \mu g, h})$  in the critical case  $\alpha + \beta = p^*$ , we will however need here a stronger restriction on  $h(x)$ , namely,  $h(x) > 0$  in  $\Omega$  but  $f(x)$  and  $g(x)$  may be also allowed to change sign in  $\Omega$ . Now, we will establish the existence of a local minimum for  $I_{\lambda,\mu}$  on  $\mathcal{N}_{\lambda,\mu}^-$  in the critical case  $\alpha + \beta = p^*$  to obtain a second positive solution of System  $(S_{\lambda f, \mu g, h})$ .

**Lemma 4.1.** *If  $\{z_n\} \subset W$  is a  $(PS)_c$ -sequence for  $I_{\lambda,\mu}$ , then  $\{z_n\}$  is bounded in  $W$ .*

*Proof.* Let  $z_n = (u_n, v_n)$ . We argue by contradiction. Assume that  $\|z_n\| \rightarrow \infty$ . Let

$$\hat{z}_n = (\hat{u}_n, \hat{v}_n) = \frac{z_n}{\|z_n\|} = \left( \frac{u_n}{\|z_n\|}, \frac{v_n}{\|z_n\|} \right). \tag{4.1}$$

We may assume that  $\widehat{z}_n \rightarrow \widehat{z} = (\widehat{u}, \widehat{v})$  in  $W$ . This implies that  $\widehat{u}_n \rightarrow \widehat{u}$ ,  $\widehat{v}_n \rightarrow \widehat{v}$  strongly in  $L^s(\Omega)$  for all  $1 \leq s < p^*$  and

$$K_{\lambda,\mu}(\widehat{z}_n) = K_{\lambda,\mu}(\widehat{z}) + o_n(1). \quad (4.2)$$

Since  $\{z_n\}$  is a  $(PS)_c$ -sequence for  $I_{\lambda,\mu}$  and  $\|z_n\| \rightarrow \infty$ , there hold

$$\frac{1}{p}\|\widehat{z}_n\|^p - \frac{\|z_n\|^{q-p}}{q}K_{\lambda,\mu}(\widehat{z}_n) - \frac{\|z_n\|^{\alpha+\beta-p}}{\alpha+\beta}J(\widehat{z}_n) = o_n(1), \quad (4.3)$$

$$\|\widehat{z}_n\|^p - \|z_n\|^{q-p}K_{\lambda,\mu}(\widehat{z}_n) - \|z_n\|^{\alpha+\beta-p}J(\widehat{z}_n) = o_n(1). \quad (4.4)$$

From (4.2)–(4.4), we can deduce that

$$\|\widehat{z}_n\|^p = \frac{p(\alpha+\beta-q)}{q(\alpha+\beta-p)}\|z_n\|^{q-p}K_{\lambda,\mu}(\widehat{z}_n) + o_n(1). \quad (4.5)$$

Since  $1 \leq q < p$  and  $\|z_n\| \rightarrow \infty$ , (4.5) implies

$$\|\widehat{z}_n\|^p \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (4.6)$$

which is contrary to the fact  $\|\widehat{z}_n\| = 1$ . □

Denote

$$S_{\alpha,\beta} = \inf_{u,v \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (|\nabla u|^p + |\nabla v|^p) dx}{\left(\int_{\Omega} |u|^\alpha |v|^\beta dx\right)^{p/(\alpha+\beta)}}. \quad (4.7)$$

Modifying the proof of Alves et al. [11, Theorem 5], we can easily deduce that

$$S_{\alpha,\beta} = \left( \left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)} \right) S, \quad (4.8)$$

where  $S$  is the best constant defined by

$$S = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p dx}{\left(\int_{\Omega} |u|^{\alpha+\beta} dx\right)^{p/(\alpha+\beta)}}. \quad (4.9)$$

When  $\alpha + \beta = p^*$ , it is well known that  $S$  is achieved if and only if  $\Omega = \mathbb{R}^N$  by the function

$$U_\varepsilon(x) = C_N \left( \frac{\varepsilon^{1/p}}{\varepsilon + |x|^{p/(p-1)}} \right)^{(N-p)/p}, \quad \varepsilon > 0, \tag{4.10}$$

where  $C_N = [((N - p)/(p - 1))^{p-1} N]^{(N-p)/p^2}$ .

**Lemma 4.2.** *Let  $c_\infty = (1/N) S_{\alpha,\beta}^{N/p} |h|_\infty^{-(N-p)/p}$ . If  $\{z_n\} \subset W$  is a  $(PS)_c$ -sequence for  $I_{\lambda,\mu}$  with  $c \in (0, c_\infty)$ , then there exists a subsequence of  $\{z_n\}$  converging weakly to a nontrivial solution of System  $(S_{\lambda f, \mu g, h})$ .*

*Proof.* Let  $\{z_n\} \subset W$  be a  $(PS)_c$ -sequence for  $I_{\lambda,\mu}$  with  $c \in (0, c_\infty)$ . Write  $z_n = (u_n, v_n)$ . We know from Lemma 4.1 that  $\{z_n\}$  is bounded in  $W$ , and then  $z_n \rightharpoonup z = (u, v)$  up to a subsequence, when  $z$  is a critical point of  $I_{\lambda,\mu}$ . Furthermore, we may assume  $u_n \rightharpoonup u, v_n \rightharpoonup v$  in  $W_0^{1,p}(\Omega)$  and  $u_n \rightarrow u, v_n \rightarrow v$  in  $L^s(\Omega)$  for all  $1 \leq s < p^*$  and  $u_n \rightarrow u, v_n \rightarrow v$  a.e. on  $\Omega$ . Hence we have that  $I'_{\lambda,\mu}(z) = 0$  and

$$K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z) + o_n(1). \tag{4.11}$$

Next we verify that  $u \neq 0$  or  $v \neq 0$ . Arguing by contradiction, we assume  $u \equiv 0$  and  $v \equiv 0$ . Set

$$l = \lim_{n \rightarrow \infty} J(z_n). \tag{4.12}$$

Since  $I'_{\lambda,\mu}(z_n) = o_n(1)$  and  $\{z_n\}$  is bounded in  $W$ , then by (4.11), we can deduce that

$$0 = \left\langle \lim_{n \rightarrow \infty} I'_{\lambda,\mu}(z_n), z_n \right\rangle = \lim_{n \rightarrow \infty} (\|z_n\|^p - J(z_n)) = \lim_{n \rightarrow \infty} \|z_n\|^p - l, \tag{4.13}$$

that is,

$$\lim_{n \rightarrow \infty} \|z_n\|^p = l. \tag{4.14}$$

If  $l = 0$ , then we get  $c = \lim_{n \rightarrow \infty} I_{\lambda,\mu}(z_n) = 0$ , which contradicts with  $c > 0$ . Thus we conclude that  $l > 0$ . By (4.7) and (4.12), we obtain

$$\begin{aligned} S_{\alpha,\beta} l^{p/p^*} &= S_{\alpha,\beta} \lim_{n \rightarrow \infty} \left( \int_{\Omega} h(x) |u_n|^\alpha |v_n|^\beta dx \right)^{p/p^*} \\ &\leq S_{\alpha,\beta} \lim_{n \rightarrow \infty} \left( \int_{\Omega} |h|_\infty |u_n|^\alpha |v_n|^\beta dx \right)^{p/p^*} \\ &\leq |h|_\infty^{p/p^*} \lim_{n \rightarrow \infty} \|z_n\|^p \\ &= |h|_\infty^{p/p^*} l, \end{aligned} \tag{4.15}$$

which implies that

$$l \geq S_{\alpha,\beta}^{N/p} |h|_{\infty}^{-(N-p)/p}. \quad (4.16)$$

Hence, from (4.11)–(4.16) we get

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} I_{\lambda,\mu}(z_n) \\ &= \lim_{n \rightarrow \infty} \left( \frac{1}{p} \|z_n\|^p - \frac{1}{q} K_{\lambda,\mu}(z_n) - \frac{1}{\alpha + \beta} J(z_n) \right) \\ &= \left( \frac{1}{p} - \frac{1}{\alpha + \beta} \right) l \\ &\geq \frac{1}{N} S_{\alpha,\beta}^{N/p} |h|_{\infty}^{-(N-p)/p} = c_{\infty}. \end{aligned} \quad (4.17)$$

This is a contradiction to  $c < c_{\infty}$ . Therefore  $z$  is a nontrivial solution of System  $(S_{\lambda f, \mu g, h})$ .  $\square$

**Lemma 4.3.** *Assume that (A1)–(A4) hold. Then for any  $\lambda, \mu > 0$ , there exist a nonnegative function  $z_{\lambda,\mu} \in W \setminus \{0\}$  such that*

$$\sup_{t \geq 0} I_{\lambda,\mu}(tz_{\lambda,\mu}) < c_{\infty}, \quad (4.18)$$

where  $c_{\infty}$  is the constant given in Lemma 4.2.

*In particular,  $\theta_{\lambda,\mu}^- < c_{\infty}$  for all  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_1$ , where  $\Lambda_1$  is as in (1.6).*

*Proof.* From (A4), we know that there exists  $0 < \rho_0 \leq r_0$  such that, for all  $x \in B(0, 2\rho_0)$ ,

$$h(x) = h(0) + o(|x|^{\delta_0}) \quad \text{for some } \delta_0 > \frac{N}{p-1}. \quad (4.19)$$

Now, we consider the functional  $Q : W \rightarrow \mathbb{R}$  defined by

$$Q(z) = \frac{1}{p} \|z\|^p - \frac{1}{\alpha + \beta} J(z) \quad \forall z = (u, v) \in W \quad (4.20)$$

and define a cut-off function  $\eta(x) \in C_0^{\infty}(\Omega)$  such that  $\eta(x) = 1$  for  $|x| < \rho_0$ ,  $\eta(x) = 0$  for  $|x| > 2\rho_0$ ,  $0 \leq \eta \leq 1$ , and  $|\nabla \eta| \leq C$ . For  $\varepsilon > 0$ , let

$$u_{\varepsilon}(x) = \frac{\varepsilon^{(N-p)/p^2} \eta(x)}{\left( \varepsilon + |x|^{p/(p-1)} \right)^{(N-p)/p}}. \quad (4.21)$$

From Hsu [7, Lemma 4.3], we have

$$|\nabla u_{\varepsilon}|_p^p = \int_{\mathbb{R}^N} |\nabla U|^p dx + O\left(\varepsilon^{(N-p)/p}\right) \quad (4.22)$$



$$\frac{\int_{\Omega} |\nabla u_{\varepsilon}|^p dx}{\left(\int_{\Omega} h(x)|u_{\varepsilon}|^{p^*}\right)^{p/p^*}} = S + O\left(\varepsilon^{(N-p)/p}\right), \tag{4.23}$$

where  $U(x) = (1 + |x|^{p/(p-1)})^{-(N-p)/p}$ .

Set  $u_0 = \sqrt[p]{\alpha}u_{\varepsilon}$ ,  $v_0 = \sqrt[p]{\beta}u_{\varepsilon}$ , and  $z_0 = (u_0, v_0) \in W$ . Then, from (4.8), (4.23), and  $|h|_{\infty} = 1$ , we conclude that

$$\begin{aligned} \sup_{t \geq 0} Q(tz_0) &\leq \frac{1}{N} \left( \frac{(\alpha + \beta) \int_{\Omega} |\nabla u_{\varepsilon}|^p dx}{\left(\alpha^{\alpha/p} \beta^{\beta/p} \int_{\Omega} h(x)|u_{\varepsilon}|^{p^*}\right)^{p/p^*}} \right)^{N/p} \\ &\leq \frac{1}{N} \left( \left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)} \right)^{N/p} \left(S + O\left(\varepsilon^{(N-p)/p}\right)\right)^{N/p} \\ &= \frac{1}{N} \left( \left(\frac{\alpha}{\beta}\right)^{\alpha/(\alpha+\beta)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)} \right)^{N/p} \left(S^{N/p} + O\left(\varepsilon^{(N-p)/p}\right)\right) \\ &= \frac{1}{N} S_{\alpha, \beta}^{N/p} + O\left(\varepsilon^{(N-p)/p}\right), \end{aligned} \tag{4.24}$$

where the following fact has been used:

$$\sup_{t \geq 0} \left( \frac{t^p}{p} A - \frac{t^{\alpha+\beta}}{\alpha + \beta} B \right) = \frac{1}{N} A \left(\frac{A}{B}\right)^{(N-p)/p} = \frac{1}{N} \left(\frac{A}{B^{p/p^*}}\right)^{N/p}, \quad A, B > 0. \tag{4.25}$$

Using the definitions of  $I_{\lambda, \mu}$ ,  $z_0$  and by (A3) and (A4), we get

$$I_{\lambda, \mu}(tz_0) \leq \frac{t^p}{p} \|z_0\|^p = \frac{\alpha + \beta}{p} t^p |\nabla u_{\varepsilon}|_p^p \quad \forall t \geq 0, \lambda, \mu > 0. \tag{4.26}$$

Combining this with (4.22), let  $\varepsilon \in (0, 1)$ , then there exists  $t_0 \in (0, 1)$  independent of  $\varepsilon$  such that

$$\sup_{0 \leq t \leq t_0} I_{\lambda, \mu}(tz_0) < c_{\infty}, \quad \forall \lambda, \mu > 0, \forall \varepsilon \in (0, 1). \tag{4.27}$$

Using the definitions of  $I_{\lambda, \mu}$ ,  $z_0$  and by  $\alpha, \beta > 1$ , (4.24) and (A3), we have

$$\begin{aligned}
\sup_{t \geq t_0} I_{\lambda, \mu}(tz_0) &= \sup_{t \geq t_0} \left( Q(tz_0) - \frac{t^q}{q} K_{\lambda, \mu}(tz_0) \right) \\
&\leq \frac{1}{N} S_{\alpha, \beta}^{N/p} + O\left(\varepsilon^{(N-p)/p}\right) - \frac{t_0^q}{q} \left( a_0 \alpha^{q/p} \lambda + b_0 \beta^{q/p} \mu \right) \int_{B(0, \rho_0)} |u_\varepsilon|^q dx \\
&\leq \frac{1}{N} S_{\alpha, \beta}^{N/p} + O\left(\varepsilon^{(N-p)/p}\right) - \frac{t_0^q}{q} \gamma_0 (\lambda + \mu) \int_{B(0, \rho_0)} |u_\varepsilon|^q dx,
\end{aligned} \tag{4.28}$$

where  $\gamma_0 = \min\{a_0 \alpha^{q/p}, b_0 \beta^{q/p}\}$ .

Let  $0 < \varepsilon \leq \rho_0^{p/(p-1)}$ ; we have

$$\begin{aligned}
\int_{B(0, \rho_0)} |u_\varepsilon|^q dx &= \int_{B(0, \rho_0)} \frac{\varepsilon^{q(N-p)/p^2}}{\left(\varepsilon + |x|^{p/(p-1)}\right)^{((N-p)/p)q}} dx \\
&\geq \int_{B(0, \rho_0)} \frac{\varepsilon^{q(N-p)/p^2}}{\left(2\rho_0^{p/(p-1)}\right)^{((N-p)/p)q}} dx \\
&= C_1(N, p, q, \rho_0) \varepsilon^{q(N-p)/p^2}.
\end{aligned} \tag{4.29}$$

Combining with (4.28) and (4.29), for all  $\varepsilon \in (0, \rho_0^{p/(p-1)})$ , we get

$$\sup_{t \geq t_0} I_{\lambda, \mu}(tz_0) \leq \frac{1}{N} S_{\alpha, \beta}^{N/p} + O\left(\varepsilon^{(N-p)/p}\right) - \frac{t_0^q}{q} C_1 \gamma_0 (\lambda + \mu) \varepsilon^{q(N-p)/p^2}. \tag{4.30}$$

Hence, for any  $\lambda, \mu > 0$ , we can choose small positive constant  $\varepsilon_{\lambda, \mu} < \min\{1, \rho_0^{p/(p-1)}\}$  such that

$$O\left(\varepsilon_{\lambda, \mu}^{(N-p)/p}\right) - \frac{t_0^q}{q} C_1 \gamma_0 (\lambda + \mu) \varepsilon_{\lambda, \mu}^{q(N-p)/p^2} < 0. \tag{4.31}$$

Now, we fix  $\varepsilon_{\lambda, \mu}$  and let  $z_{\lambda, \mu} = (\sqrt[p]{\alpha} u_{\varepsilon_{\lambda, \mu}}, \sqrt[p]{\beta} u_{\varepsilon_{\lambda, \mu}})$ . From (4.27), (4.30), (4.31), we can deduce that, for any  $\lambda, \mu > 0$ , there exists a nonnegative function  $z_{\lambda, \mu} \in W \setminus \{0\}$  such that

$$\sup_{t \geq 0} I_{\lambda, \mu}(tz_{\lambda, \mu}) < c_\infty. \tag{4.32}$$

Finally, we prove that  $\theta_{\lambda, \mu}^- < c_\infty$  for all  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_1$ . Recall that  $z_{\lambda, \mu} = (u_{\lambda, \mu}, v_{\lambda, \mu}) = (\sqrt[p]{\alpha} u_{\varepsilon_{\lambda, \mu}}, \sqrt[p]{\beta} u_{\varepsilon_{\lambda, \mu}})$ . By (A3), (A4), and the definition of  $u_{\varepsilon_{\lambda, \mu}}$ , we have

$$J(z_{\lambda, \mu}) > 0, \quad K_{\lambda, \mu}(z_{\lambda, \mu}) > 0. \tag{4.33}$$

Combining this with Lemma 2.6 (ii), from the definition of  $\theta_{\lambda,\mu}^-$  and (4.32), for all  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_1$ , we obtain that there exists  $t_{\lambda,\mu} > 0$  such that  $t_{\lambda,\mu} z_{\lambda,\mu} \in \mathcal{N}_{\lambda,\mu}^-$  and

$$\theta_{\lambda,\mu}^- \leq I_{\lambda,\mu}(t_{\lambda,\mu} z_{\lambda,\mu}) \leq \sup_{t \geq 0} I_{\lambda,\mu}(t z_{\lambda,\mu}) < c_\infty. \tag{4.34}$$

This completes the proof. □

**Theorem 4.4.** *Assume that (A1)–(A4) hold. If  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ , then  $I_{\lambda,\mu}$  satisfies the (PS) $_{\theta_{\lambda,\mu}^-}$  condition. Moreover,  $I_{\lambda,\mu}$  has a minimizer  $z_{\lambda,\mu}^2$  in  $\mathcal{N}_{\lambda,\mu}^-$  and satisfies the following:*

- (i)  $I_{\lambda,\mu}(z_{\lambda,\mu}^2) = \theta_{\lambda,\mu}^-$ ;
- (ii)  $z_{\lambda,\mu}^2$  is a positive solution of System  $(S_{\lambda f, \mu g, h})$ , where  $\Lambda_1$  is the same as in (1.6).

*Proof.* If  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ , then by Theorem 2.5 (ii), Proposition 3.2 (ii), and Lemma 4.3, there exists a (PS) $_{\theta_{\lambda,\mu}^-}$ -sequence  $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}^-$  in  $W$  for  $I_{\lambda,\mu}$  with  $\theta_{\lambda,\mu}^- \in (0, c_\infty)$ . From Lemma 4.2, there exist a subsequence still denoted by  $\{z_n\}$  and a nontrivial solution  $z_{\lambda,\mu}^2 = (u_{\lambda,\mu}^2, v_{\lambda,\mu}^2) \in W$  of System  $(S_{\lambda f, \mu g, h})$  such that  $z_n \rightharpoonup z_{\lambda,\mu}^2$  weakly in  $W$ . Now we prove that  $z_n \rightarrow z_{\lambda,\mu}^2$  strongly in  $W$  and  $I_{\lambda,\mu}(z_{\lambda,\mu}^2) = \theta_{\lambda,\mu}^-$ . By (3.4), if  $z \in \mathcal{N}_{\lambda,\mu}$ , then

$$I_{\lambda,\mu}(z) = \frac{p^* - p}{p^* p} \|z\|^p - \frac{p^* - q}{p^* q} K_{\lambda,\mu}(z). \tag{4.35}$$

First, we prove that  $z_{\lambda,\mu}^2 \in \mathcal{N}_{\lambda,\mu}^-$ . On the contrary, if  $z_{\lambda,\mu}^2 \in \mathcal{N}_{\lambda,\mu}^+$ , then by  $\mathcal{N}_{\lambda,\mu}^-$  closed in  $W$ , we have  $\|z_{\lambda,\mu}^2\| < \lim_{n \rightarrow \infty} \|z_n\|$ . By Lemma 2.3 (i) and (A4), we obtain that

$$K_{\lambda,\mu}(z_{\lambda,\mu}^2) > 0, \quad J(z_{\lambda,\mu}^2) > 0. \tag{4.36}$$

By Lemma 2.7, there exists a unique  $t^-$  such that  $t^- z_{\lambda,\mu}^2 \in \mathcal{N}_{\lambda,\mu}^-$ . Since  $z_n \in \mathcal{N}_{\lambda,\mu}^-$ ,  $I_{\lambda,\mu}(z_n) \geq I_{\lambda,\mu}(t z_n)$  for all  $t \geq 0$  and by (4.35), we have

$$\theta_{\lambda,\mu}^- \leq I_{\lambda,\mu}(t^- z_{\lambda,\mu}^2) < \lim_{n \rightarrow \infty} I_{\lambda,\mu}(t^- z_n) \leq \lim_{n \rightarrow \infty} I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}^-, \tag{4.37}$$

and this is a contradiction.

In order to prove that  $I_{\lambda,\mu}(z_{\lambda,\mu}^2) = \theta_{\lambda,\mu}^-$ , it suffices to recall that  $z_n, z_{\lambda,\mu}^2 \in \mathcal{N}_{\lambda,\mu}^-$  for all  $n$ , by (4.35). and apply Fatou's lemma to get

$$\begin{aligned} \theta_{\lambda,\mu}^- &\leq I_{\lambda,\mu}(z_{\lambda,\mu}^2) = \frac{p^* - p}{p^* p} \|z_{\lambda,\mu}^2\|^p - \frac{p^* - q}{p^* q} K_{\lambda,\mu}(z_{\lambda,\mu}^2) \\ &\leq \liminf_{n \rightarrow \infty} \left( \frac{p^* - p}{p^* p} \|z_n\|^p - \frac{p^* - q}{p^* q} K_{\lambda,\mu}(z_n) \right) \\ &\leq \liminf_{n \rightarrow \infty} I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}^-. \end{aligned} \tag{4.38}$$

This implies that  $I_{\lambda,\mu}(z_{\lambda,\mu}^2) = \theta_{\lambda,\mu}^-$  and  $\lim_{n \rightarrow \infty} \|z_n\|^p = \|z_{\lambda,\mu}^2\|^p$ . Let  $\tilde{z}_n = z_n - z_{\lambda,\mu}^2$ , then by Brézis and Lieb lemma [26] implies

$$\|\tilde{z}_n\|^p = \|z_n\|^p - \|z_{\lambda,\mu}^2\|^p. \quad (4.39)$$

Therefore,  $z_n \rightarrow z_{\lambda,\mu}^2$  strongly in  $W$ .

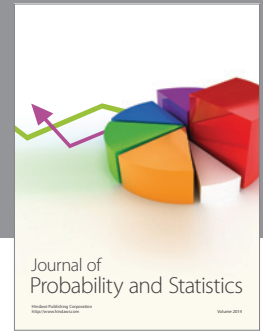
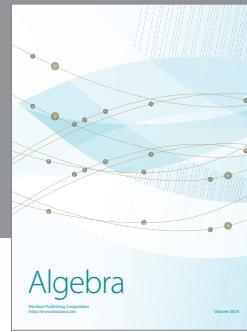
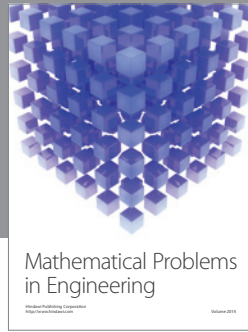
Since  $I_{\lambda,\mu}(z_{\lambda,\mu}^2) = I_{\lambda,\mu}(|z_{\lambda,\mu}^2|)$  and  $|z_{\lambda,\mu}^2| \in \mathcal{N}_{\lambda,\mu}^-$ , by Lemmas 2.2, and 2.3 (iii), we may assume that  $z_{\lambda,\mu}^2$  is a nontrivial nonnegative solution of System  $(S_{\lambda f, \mu g, h})$ . Finally, by using the same arguments as in the proof of Theorem 3.3, for all  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ , we have that  $z_{\lambda,\mu}^2$  is a positive solution of System  $(S_{\lambda f, \mu g, h})$ .  $\square$

*Proof of Theorem 1.3.* By Theorems 3.3, and 4.4, we obtain that for all  $\lambda, \mu > 0$  and  $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < (q/p)^{p/(p-q)} \Lambda_1$ ,  $(S_{\lambda f, \mu g, h})$  has two positive solutions  $z_{\lambda,\mu}^1, z_{\lambda,\mu}^2$  with  $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+, z_{\lambda,\mu}^2 \in \mathcal{N}_{\lambda,\mu}^-$ . Since  $\mathcal{N}_{\lambda,\mu}^+ \cap \mathcal{N}_{\lambda,\mu}^- = \emptyset$ , this implies that  $z_{\lambda,\mu}^1$  and  $z_{\lambda,\mu}^2$  are distinct. This completes the proof of Theorem 1.3.  $\square$

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