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Research Article

Semiconservative Systems of Integral Equations with Two Kernels

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The solvability and the properties of solutions of nonhomogeneous and homogeneous vector integral equation $f(x) = g(x) + \int_0^\infty k(x-t)f(t)dt + \int_{-\infty}^0 T(x-t)f(t)dt$, where K, T are $n \times n$ matrix valued functions, $n \geq 1$, with nonnegative integrable elements, are considered in one semiconservative (singular) case, where the matrix $A = \int_{-\infty}^\infty K(x)dx$ is stochastic one and the matrix $B = \int_{-\infty}^\infty T(x)dx$ is substochastic one. It is shown that in certain conditions the nonhomogeneous equation simultaneously with the corresponding homogeneous one possesses positive solutions.

1. Introduction: Problem Statement

Consider the scalar or vector integral equations on the whole line with two kernels (see [1–4]):

$$f(x) = g(x) + \int_0^\infty K(x-t)f(t)dt + \int_{-\infty}^0 T(x-t)f(t)dt\ , \quad -\infty < x < \infty\ , \tag{1.1}$$

where the kernel-functions $K(x)$, $T(x)$ are matrix-valued functions with nonnegative elements; g and f are the given and sought-for column vectors (vectorfunctions); respectively. Assume that

$$K, T \in L^{n\times n}, \qquad g \in L^n, \qquad K, T, g \geq O. \tag{1.2}$$

Here $L^{n\times n}$ is the space of $n \times n$- ($n \geq 1$) order matrix-valued functions, and L^n is the space of column vectors, with components in Lebesgue space $L \equiv L_1(-\infty, \infty)$. The zero vector or matrix is denoted by O. The inequalities between the matrices or vectors, the operation of integration, and some other operations shall be treated componentwise.

Denote by ς the following n-dimensional row vector:

$$\varsigma = (1, 1, \dots, 1). \tag{1.3}$$

Let $C \geq O$ be an $n \times n$ matrix. If

$$\varsigma C = \varsigma, \tag{1.4}$$

then the matrix C is a stochastic one (accurate within transpose, see [5]). If

$$\varsigma C \leq \varsigma, \tag{1.5}$$

then the matrix C is substochastic to a wide extent. We shall call the matrix C really substochastic, if $\varsigma C \leq \varsigma$, $\varsigma C \neq \varsigma$ and uniform substochastic if there exist $\mu \in [0, 1)$ such that

$$\varsigma C \leq \mu \varsigma, \quad 0 \leq \mu < 1. \tag{1.6}$$

Let us introduce the following $n \times n$ matrices $A, B \geq O$, related with the equation (1.1):

$$A = \int_{-\infty}^{\infty} K(x)dx, \qquad B = \int_{-\infty}^{\infty} T(x)dx. \tag{1.7}$$

We shall call the kernel K conservative, dissipative, or uniform dissipative if the matrix A is stochastic, really substochastic, or uniform substochastic, respectively. We shall use analogous names to the kernel T.

We shall call (1.1) semiconservative, if one of the kernels K,T is conservative and the other is dissipative. Without the loss of generality, one can assume that

$$\varsigma A = \varsigma, \qquad \varsigma B \leq \varsigma, \qquad \varsigma B \neq \varsigma. \tag{1.8}$$

In the uniform semiconservative case of (1.1) we have

$$\varsigma A = \varsigma, \qquad \varsigma B \leq \mu \varsigma, \quad 0 \leq \mu < 1, \tag{1.9}$$

whereas in the conservative case, both of the kernels K, T are assumed to be conservative.

If $T = O$, then (1.1) is reduced to the well-known Wiener-Hopf integral equation:

$$\varphi(x) = h(x) + \int_0^{\infty} K(x - t)\varphi(t)dt, \quad x > 0, \tag{1.10}$$

Here $\varphi = f|_{[0,\infty)}$ and $h = g|_{[0,\infty)}$ are restrictions on $[0, \infty)$ of f and g, respectively.

The theory of the scalar and vector conservative Wiener-Hopf equation (1.10) (where K is the conservative one) passed a long way of development. Many (conservative) physical processes in homogeneous half-space are described by such equations. They are of essential interest in the radiative transfer (RT), kinetic theory of Gases (see [6, 7]), in the mathematical theory of stochastic processes, and so forth.

In the RT, the conservative equation (1.10) corresponds to the absence of losses of the radiation inside media (case of pure scattering). However, such losses occur as a result of an exit of radiation from media. In case of the dissipative one, there are losses inside media as well.

Equation (1.1) with two kernels arises in some more general (and more complicated) problems, where the physical processes occur in the infinite media, consisting of two adjacent homogeneous half-spaces (see [7]). In each of these half-spaces, the processes may be dissipative or conservative. Another area of applications is connected with the RT in the atmosphere-ocean system.

In the theory of RT, the free term g in (1.1) plays the role of initial sources of radiation. The conservative and semiconservative cases belong to the singular cases of (1.1). In these cases, the unique solvability of (1.1) in the "standard" functional spaces L_p^n $(1 \le p \le \infty)$ is violated.

A number of results concerning to the scalar conservative equation (1.1) have been obtained by Arabadzhyan [3]. The systems of conservative or semiconservative equations with two kernels have not ever been treated.

The present paper is devoted to the solvability and the properties of the solutions of the nonhomogeneous and homogeneous vector equation (1.1). The main attention will be paid to the uniform semiconservative case (1.9). It will be shown that in certain conditions both the nonhomogeneous equation (1.1) and the corresponding homogeneous equation possess positive locally integrable solutions.

2. Auxiliary Propositions

2.1. Integral Operators

Let $(a,b) \subset (-\infty,\infty)$. Consider Banach space (B-space) $L(a,b) \equiv L_1(a,b)$ and the corresponding B-space $L^n(a,b)$ of vector-valued functions (vector columns) $f = (f_1,\dots,f_n)^T$. Here T is a sign of the transpose. The norm in $L^n(a,b)$ is defined by

$$\|f\| = \sum_{k=1}^{n} \|f_k\|_{L(a,b)} = \varsigma \int_a^b |f(x)|dx. \tag{2.1}$$

Consider the linear topological space $L_{\text{Loc}}[0,\infty)$ of the functions, which are integrable on each finite interval $(0,r)$, $r < \infty$.The space $L_{\text{Loc}}^n[0,\infty)$ possesses the topology of the componentwise convergence.

The unit operator in each of spaces introduced above is denoted by I. Let Ω_n be the following class of matrix convolution operators on the whole line: if $\widehat{U} \in \Omega_n$, then

$$\varphi(x) = \widehat{U}f(x) = \int_{-\infty}^{\infty} U(x-t)f(t)dt, \quad U \in L^{n\times n}. \tag{2.2}$$

The operator $\widehat{U} \in \Omega_n$ acts in the spaces L^n, L_p^n $(1 \le p \le \infty)$, and in some other spaces of vector-valued functions.

The class Ω_n is an algebra where the kernel function of the operator product is the convolution of the kernel functions of the factors.

Let us estimate the norm of operator $\widehat{U} \in \Omega_n$ in the B-space L^n. Let $C \geq O$ be the following $n \times n$ matrix: $C = (c_{km}) = \int_{-\infty}^{\infty} |U(x)| dx$. Taking the (componentwise) modulus in (2.2) and integrating on $(-\infty, \infty)$, we come to the following inequality:

$$\int_{-\infty}^{\infty} |\varphi(x)| dx \leq C \int_{-\infty}^{\infty} |f(t)| dt. \tag{2.3}$$

Multiplying this inequality on the left by the vector ς, we come to the following inequality:

$$\left\| \widehat{U} f \right\| \leq \gamma \|f\|, \quad \text{where } \gamma = \max_k \sum_{m=1}^{n} c_{km}. \tag{2.4}$$

From here the estimate follows:

$$\left\| \widehat{U} \right\| \leq \gamma. \tag{2.5}$$

Let us introduce the projectors (projection operators) $P_\pm$, acting in the spaces of summable or locally summable functions on $(-\infty, \infty)$ by the equalities:

$$P_+ f(x) = f(x) \vartheta(x), \qquad P_- f(x) = f(x) \vartheta(-x). \tag{2.6}$$

Here ϑ is the Heaviside function of the unit jump. In each of the spaces L_p $(1 \leq p \leq \infty)$, we have

$$\|P_\pm\| = 1. \tag{2.7}$$

Denote by $\widehat{K}, \widehat{T} \in \Omega_n$ the following operators, whose kernel functions K, T participate in (1.1):

$$\widehat{K} f(x) = \int_{-\infty}^{\infty} K(x-t) f(t) dt, \qquad \widehat{T} f(x) = \int_{-\infty}^{\infty} T(x-t) f(t) dt. \tag{2.8}$$

Equation (1.1) admits the following operator entry

$$f = g + \widehat{W} f, \tag{2.9}$$

where $\widehat{W} = \widehat{K} \widehat{P}_+ + \widehat{T} \widehat{P}_-$.

The projectors $\widehat{P}_\pm$ are the diagonal matrices of the operators with the diagonal elements $P_\pm$.

The operator $\widehat{W}$ is an Integral operator:

$$\widehat{W}f(x)=\int_{-\infty}^{\infty}W(x,t)f(t)dt,\qquad \text{where } W(x,t)=K(x-t)\tilde{\vartheta}(t)+T(x-t)\tilde{\vartheta}(-t). \tag{2.10}$$

Here $\tilde{\vartheta}(x)$ is the diagonal matrix with the diagonal elements $\vartheta(x)$.

2.2. On the Invertibility of the Operator $I-\widehat{W}$ in L^n

Let us estimate the norm of $\widehat{W}$ in L^n. Assume at first that the kernel functions K,T are arbitrary elements of $L^{n\times n}$. Let $\varphi=\widehat{W}f$, $f\in L^n$. One can obtain the following inequality (similar to (2.3)):

$$\int_{-\infty}^{\infty}|\varphi(x)|dx\le A\int_{0}^{\infty}|f(t)|dt+B\int_{-\infty}^{0}|f(t)|dt. \tag{2.11}$$

Here $A=(a_{km})=\int_{-\infty}^{\infty}|K(x)|dx$, $B=(b_{km})=\int_{-\infty}^{\infty}|T(x)|dx$.

We have $\varsigma A\le\lambda\varsigma$, $\varsigma B\le\mu\varsigma$, where

$$\lambda=\max_m\sum_{k=1}^{n}a_{km},\qquad \mu=\max_m\sum_{k=1}^{n}b_{km}. \tag{2.12}$$

Multiplying (2.11) on the left by the vector ς, we get

$$\|\varphi\|\le\lambda\varsigma\int_{0}^{\infty}|f(t)|dt+\mu\varsigma\int_{-\infty}^{0}|f(t)|dt\le\max(\lambda,\mu)\|f\|. \tag{2.13}$$

Thus, we proved the following.

Lemma 2.1. *The following estimate for the norm of the operator $\widehat{W}$ in L^n is valid:*

$$\left\|\widehat{W}\right\|\le q=\max(\lambda,\mu). \tag{2.14}$$

If $q<1$, then the operator $\widehat{W}$ is contracting in L^n, hence the operator $I-\widehat{W}$ is invertible, and (1.1) with $g\in L^n$ has a unique solution $f\in L^n$. If therewith $K,T,g\ge O$, then $f\ge O$.

In accordance with the general theory of the integral equations with two kernels (see [1, 2]), for the invertibility of the operator $I-\widehat{W}$ in L^n, it is necessary the fulfilment of the following conditions of nondegeneration:

$$\det\left[J-\overline{K}(s)\right]\ne0,\quad \det\left[J-\overline{T}(s)\right]\ne0,\quad -\infty<s<+\infty. \tag{2.15}$$

Here J is the unit $n\times n$ matrix; the matrices $\overline{K}(s)$ and $\overline{T}(s)$ are the (elementwise) Fourier transforms of K and T, respectively. For example, $\overline{K}(s)=\int_{-\infty}^{\infty}K(x)e^{isx}dx$, $-\infty<s<+\infty$.

In the semiconservative case (1.9), we have: $\varsigma[J-\overline{K}(0)]=\varsigma[J-A]=O$. Hence $\det(J-A)=0$, that is, the symbol $J-\overline{K}(s)$, is degenerated in the point $s=0$. In the conservative case (where A and B are stochastic matrices), both of the conditions (2.15) are violated. Thus, the operator $I-\widehat{W}$ is noninvertible in L^n in the semiconservative and conservative cases.

3. Semiconservative Nonhomogeneous Equation

In this section, we shall consider the question of the solvability of the uniform semiconservative nonhomogeneous equation (1.1), (1.9) under the following additional assumption: there exists a strong positive vector-column η such that $A\eta=\eta,\ \eta>O$. In accordance with Perron-Frobenius theorem (see [8]), the existence of such vector η is secured if the stochastic matrix A is an irreducible one.

3.1. One Auxiliary Equation

At the outset, consider the auxiliary conservative Wiener-Hopf equation (1.10), where

$$O\le h\in L^n(0,\infty),\qquad x>0,\qquad \varsigma A=\varsigma,\qquad A\eta=\eta \tag{3.1}$$

with the conservative kernel K participating in (1.1).

The following lemma follows from the results [9]:

Lemma A. *Equation* (1.10), (3.1) *possesses the minimal solution $\varphi\ge O$ which is locally integrable on* $[0,\infty)$ *(see [9]). The following asymptotics holds*

$$\int_0^x\varphi(t)dt=o\left(x^2\right),\quad x\longrightarrow\infty. \tag{3.2}$$

This asymptotics admits an adjustment subject to additional assumptions on kernel K and free term h (see [9]).

Denote by ν the following matrix the first moments of matrix-function K:

$$\nu=\int_{-\infty}^{\infty}xK(x)dx, \tag{3.3}$$

with the assumption of componentwise absolute convergence of this integral. Let

$$\sigma=\varsigma\nu\eta,\quad -\infty<\sigma<+\infty. \tag{3.4}$$

The number σ plays a principal role in the classification of the conservative equation (1.10) (see [9]). If $\sigma<0$, then

$$\int_0^x\varphi(t)dt=o(x),\quad x\longrightarrow\infty. \tag{3.5}$$

If therewith the free term h has a finite first moment: $\int_0^\infty th(t)dt < \infty$, then $\varphi \in L^n(0,\infty)$. Consider the simple iterations for (1.10):

$$\varphi^{(m+1)}(x) = h(x) + \int_0^\infty K(x-t)\varphi^{(m)}(t)dt, \quad \varphi^{(0)} = O, \; m = 0,1,2,\ldots. \tag{3.6}$$

The sequence $\varphi^{(m)}$ possesses the following properties: $O \le \varphi^{(m)} \in L^n(0,\infty)$. It is easy to show that the sequence $\varphi^{(m)}$ is monotonic. Indeed we have

$$\varphi^{(m+1)}(x) - \varphi^{(m)}(x) = \int_0^\infty K(x-t)\left(\varphi^{(m)}(t) - \varphi^{(m-1)}(t)\right)dt, \quad \varphi^{(0)} = O, \; m = 0,1,2,\ldots. \tag{3.7}$$

Using the induction by m, we obtain that $\varphi^{(m+1)}(x) - \varphi^{(m)}(x) \ge O$, which implies the monotonicity of the sequence $\varphi^{(m)}$. The sequence $\varphi^{(m)}$ converges monotonically by the topology of $L^n_{\text{Loc}}[0,\infty)$ to the minimal solution φ of (1.10):

$$\varphi^{(m)} \uparrow \varphi \quad \text{in } L^n_{\text{Loc}}[0,\infty). \tag{3.8}$$

3.2. *One Existence Theorem for* (1.1)

Consider now (1.1) under conditions

$$\varsigma A = \varsigma, \qquad A\eta = \eta, \qquad \varsigma B \le \mu\varsigma, \quad 0 \le \mu < 1. \tag{3.9}$$

Let us consider the following iterations for (1.1):

$$f^{(m+1)}(x) = g(x) + \int_0^\infty K(x-t)f^{(m)}(t)dt + \int_{-\infty}^0 T(x-t)f^{(m)}(t)dt, \tag{3.10}$$

$$f^{(0)} = O, \qquad m = 0,1,2,\ldots. \tag{3.11}$$

We have

$$f^{(m)} \in L^n, \quad m = 0,1,\ldots, \qquad O \le f^{(m)}(x) \uparrow \text{ by } m. \tag{3.12}$$

Let $\tilde{f} \ge O$ be any positive solution of (1.1), (3.9):

$$\tilde{f}(x) = g(x) + \int_0^\infty K(x-t)\tilde{f}(t)dt + \int_{-\infty}^0 T(x-t)\tilde{f}(t)dt. \tag{3.13}$$

It is easy to verify by induction that $f^{(m)} \le \tilde{f}$, for each $m \ge 0$. Hence, if the sequence $f^{(m)} \to f$ converges by the topology of $L^n_{\text{Loc}}(-\infty,\infty)$, then $f \le \tilde{f}$.

Remark 3.1. If the sequence $f^{(m)} \to f$ converges by the topology of $L^n_{\text{Loc}}(-\infty,\infty)$, then one can take the limit in (3.10), and $f \geq O$ will be the minimal positive solution of (1.1).

This fact is proved using the monotonicity of $f^{(m)}$ and the two-sided inequalities (see [10] Item 2).

Let us introduce the restrictions of the functions $f^{(m)}$ on $(0,\infty)$ and $(-\infty,0)$:

$$\omega^{(m)} = f^{(m)}\big|_{[0,\infty)}, \qquad \varphi^{(m)} = f^{(m)}\big|_{(-\infty,0)} \in L^n(-\infty,0). \tag{3.14}$$

Theorem 3.2. *Let the conditions* (3.9) *hold. Then* (1.1) *has the minimal positive solution* $f \in L^n_{\text{Loc}}(-\infty,\infty)$ *with* $f|_{(-\infty,0)} \in L^n(-\infty,0)$ *and*

$$\int_0^x f(t)dt = o\left(x^2\right), \qquad x \longrightarrow +\infty. \tag{3.15}$$

If $\exists\, \sigma < 0$, *then* $\int_0^x f(t)dt = o(x),\ x \to \infty$.

Proof. After the integration of (3.10) over x on $(-\infty,\infty)$, we shall have

$$a^{(m+1)} + b^{(m+1)} = \gamma + Aa^{(m)} + Bb^{(m)}, \tag{3.16}$$

where $a^{(m)} = \int_0^\infty \omega^{(m)}(x)dx,\ b^{(m)} = \int_{-\infty}^0 \varphi^{(m)}(x)dx, \gamma = \int_{-\infty}^\infty g(x)dx$.

Multiplying (3.16) on the left by the vector ς and taking into account (3.9), we obtain the inequality

$$\varsigma a^{(m+1)} + \varsigma b^{(m+1)} \leq \varsigma\gamma + \varsigma a^{(m)} + \mu\varsigma b^{(m)}, \tag{3.17}$$

whence it follows, with due regard for the monotony of sequences $a^{(m)}$, $b^{(m)}$, that $(1-\mu)\ \varsigma b^{(m)} \leq \varsigma\gamma$. We arrive at the following estimate:

$$\varsigma b^{(m)} = \left\|\varphi^{(m)}\right\| \leq (1-\mu)^{-1}\varsigma\gamma. \tag{3.18}$$

It follows from B. Levy well-known theorem that the monotonous and bounded by norm sequence $\varphi^{(m)}$ converges in $L^n(-\infty,0)$:

$$O \leq\ \varphi^{(m)} \uparrow \varphi \in L^n(-\infty,0). \tag{3.19}$$

Now compare relations (3.10) for $x > 0$ with iterations (3.6), in which $h(x) = g(x) + \int_{-\infty}^0 T(x-t)\varphi(t)dt,\ x > 0$ (φ is determined according to (3.19)). In virtue of $\varphi^{(m)} \leq \varphi$, we have the inequality $\omega^{(m)}(x) \leq \varphi^{(m)}(x),\ x > 0,\ m = 0,1,\ldots$. Hence $\omega^{(m)}(x) \leq \varphi(x),\ x > 0,\ m = 0,1,\ldots$. According to the Lebesgue theorem, the monotonic sequence $\omega^{(m)}$ converges by the topology $L^n_{\text{Loc}}[0,\infty)$:

$$O \leq \omega^{(m)} \uparrow \omega \in L^n_{\text{Loc}}[0,\infty). \tag{3.20}$$

We have obtained that the narrowing of the monotonic iteration sequence $f^{(m)}$ to the negative semiaxis is convergent in $L^n(-\infty,0)$, and the narrowing of $f^{(m)}$ to the positive semiaxis is convergent in $L^n_{\mathrm{Loc}}[0,\infty)$. If we denote $f(x) = \begin{cases} \omega(x), & x>0 \\ \psi(x), & x<0 \end{cases}$, then $f^{(m)} \to f$ in $L^n_{\mathrm{Loc}}[-\infty,\infty)$ (i.e., in $L^n(-\infty,r)$, for all $r < +\infty$). Taking limit in (3.10) (see Remark 3.1), we obtain that the vector function f satisfies (1.1),(3.9), and thereby, it is its minimal solution. The Theorem is proved. □

Observe that, under the assumptions of Theorem 3.2, the existence of the locally integrable solution of (1.1) could be proved using the fixed point principle of the paper [10]. Anyway, with this method, one cannot obtain the properties $f|_{(-\infty,0)} \in L^n(-\infty,0)$ and (3.15).

4. The Homogeneous Semiconservative Equation

The homogeneous system (1.1) under the conditions (3.9) will be considered in the present section:

$$G(x) = \int_0^\infty K(x-t)G(t)dt + \int_{-\infty}^0 T(x-t)G(t)dt. \tag{4.1}$$

Consider at first the corresponding conservative homogeneous system of Wiener-Hopf equations:

$$S(x) = \int_0^\infty K(x-t)S(t)dt. \tag{4.2}$$

Let us formulate some results on the existence of positive solutions of the system (4.1) (see [9]).

Theorem A. *Let K satisfy the conditions $\varsigma\, A = \varsigma$, $A\eta = \eta$ (see (3.9)), and one of the following conditions (a) or (b):*

(a) *the property of symmetry (here T is the sign of transpose):*

$$K(-x) = K^T(x), \tag{4.3}$$

(b) *the kernel K has a finite first moment ν (see (3.3)) and that*

$$\sigma \leq 0, \tag{4.4}$$

where σ is determined by (3.4).

Then the equation (4.2) has a positive solution $S(x) > O$. The vector function S is absolutely continuous and monotone increasing. The following asymptotics holds

$$S(x) = O(x), \quad x \longrightarrow \infty. \tag{4.5}$$

Let us (in conditions of the Theorem A) continue the vector function S to all the real axis in accordance with the equality (4.2). Then the equality (4.2) takes place on the whole real axis.

The convergence of the following integral is necessary and sufficient in order that S has a integrable extension on the negative semiaxis

$$\int_0^{\infty} S(t)dt \int_{-\infty}^{-t} K(x)dx < +\infty. \tag{4.6}$$

If (4.6) holds, then we will have $S \in L^n_{\text{Loc}}[-\infty, \infty)$, $S(x) > \mathrm{O}$.

It follows from the asymptotics (4.5) that for the fulfilment of the requirement (4.6), it is sufficient that the kernel function K has the (componentwise) finite second moment on the negative semiaxis, that is,

$$\int_0^{\infty} K(-x)x^2 dx < +\infty. \tag{4.7}$$

Now consider, uniform semiconservative (4.1).

Theorem 4.1. *Let the homogeneous equation* (4.2) *satisfy the conditions* (3.9), (4.7) *and either of the conditions* (4.3) *or* (4.4). *Then there exists a solution $G > O$, $G \in L^n_{\text{Loc}}[-\infty, \infty)$ of this equation. The following asymptotics hold:*

$$\int_{-\infty}^{x} G(t)dt = O\left(x^2\right), \quad x \longrightarrow \infty. \tag{4.8}$$

Proof. In accordance with Theorem A, there exists a solution $S > O$ of (4.1). The inequality (4.6) follows from the condition (4.7) and from the asymptotics (4.5); hence, $S \in L^n(-\infty, 0)$.

Let us introduce a new sought-for vector function $f \geq O$ in (4.1) by means of the relation:

$$G(x) = f(x) + S(x), \quad -\infty < x < +\infty. \tag{4.9}$$

Substituting (4.9) into (4.1) with due regard for (4.2), we obtain an inhomogeneous equation of the type (1.1) with respect to f, in which

$$g(x) = \int_{-\infty}^{0} T(x-t)S(t)dt, \quad x \in R. \tag{4.10}$$

Because of $S \in L^n(-\infty, 0)$, we have $g \in L^n$. In accordance with Theorem 3.2, there exists a (minimal) solution of (1.1) with a free term (4.10) that implies the existence of the strong positive solution of the form (4.9) of the homogeneous equation (4.1).

The asymptotics (4.8) follow immediately from the properties of f and S, included in Theorem 3.2 and Theorem A. The Theorem is proved. □

It is remarkable that under the conditions of Theorem 4.1, both the nonhomogeneous equation (1.1) (with $g \in L^n$) and the homogeneous equation (4.1) simultaneously have positive solutions.

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