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Research Article

Approximate Quartic and Quadratic Mappings in Quasi-Banach Spaces

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we establish the general solution for a mixed type functional equation of aquartic and a quadratic mapping in linear spaces. In addition, we investigate the generalized Hyers-Ulam stability in p-Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1, \cdot)$ be a group, and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : X \to X'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \le \delta, \tag{1.1}$$

for all $x, y \in X$, and for some $\delta > 0$. Then, there exists a unique additive mapping $T : X \to X'$ such that

$$\|f(x) - T(x)\| \leq \delta, \tag{1.2}$$

for all $x \in X$.

The result of Hyers was generalized by Aoki [3] for approximate additive function and by Rassias [4] for approximate linear function by allowing the difference Cauchy equation $\|f(x+y) - f(x) - f(y)\|$ to be controlled by $\varepsilon(\|x\|^p + \|y\|^p)$. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias may be called the Hyers-Ulam-Rassias stability (see [5, 6]). In 1994, a generalization of Rassias theorem was obtained by Găvruta [7], who replaced $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.3}$$

is related to a symmetric biadditive function [8–10]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.3) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B_1 such that $f(x) = B_1(x, x)$ for all x in the vector space. The biadditive function B_1 is given by

$$B_1(x, y) = \frac{1}{4}(f(x+y) - f(x-y)). \tag{1.4}$$

A Hyers-Ulam stability problem for the quadratic functional equation (1.3) was proved by Skof for functions $f : X \to Y$, where X is normed space and Y is Banach space (see [11]). In the paper [12], Czerwik proved the Hyers-Ulam-Rassias stability of (1.3).

Lee et al. [13] considered the following functional equation:

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.5}$$

In fact, they proved that a function f between two real vector spaces X and Y is a solution of (1.5) if and only if there exists a unique symmetric biquadratic function $B_2 : X \times X \to Y$ such that $f(x) = B_2(x, x)$ for all $x \in X$. The biquadratic function B_2 is given by

$$B_2(x, y) = \frac{1}{12}(f(x+y) + f(x-y) - 2f(x) - 2f(y)). \tag{1.6}$$

It is easy to show that the function $f(x) = ax^4$ satisfies the functional equation (1.5), which is called the quartic functional equation (see also [14]).

Jun and Kim [15] have obtained the generalized Hyers-Ulam stability for a mixed type of cubic and additive functional equation. In addition, the generalized Hyers-Ulam stability for a mixed type of cubic, quadratic, and additive functional equation has been investigated by Gordji and Khodaei [16] (see also [17, 18]). The stability problems for several mixed types of functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [19–27].

In this paper, we deal with the following functional equation derived from quartic and quadratic functions:

$$f(kx+y)+f(kx-y) = k^2 f(x+y)+k^2 f(x-y)+\frac{k^2(k^2-1)}{6}(f(2x)-4f(x))-2\left(k^2-1\right)f(y) \tag{1.7}$$

for fixed integers $k \neq 0, \pm 1$. It is easy to see that the function $f(x) = ax^4 + bx^2$ is a solution of the functional equation (1.7). In the sequel, we investigate the general solution of functional equation (1.7) when f is a function between vector spaces, and then we prove the generalized Hyers-Ulam stability of (1.7) in the spirit of Hyers, Ulam, and Rassias using the direct method.

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1 (see [28, 29]). Let X be a real linear space. A quasinorm is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$,

(3) there is a constant $M \geq 1$ such that $\|x+y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasinormed space if $\|\cdot\|$ is a quasinorm on X.

The smallest possible M is called the modulus of concavity of $\|\cdot\|$. A quasi-Banach space is a complete quasinormed space. A quasinorm $\|\cdot\|$ is called a p-norm $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p, \tag{1.8}$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

Given a p-norm, the formula $d(x, y) := \|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz Theorem [29], each quasinorm is equivalent to some p-norm (see also [28]). Since it is much easier to work with p-norms, henceforth we restrict our attention mainly to p-norms.

Lemma 1.2 (see [17]). *Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers. Then, one has*

$$\left(\sum_{i=1}^{n} x_i\right)^p \leq \sum_{i=1}^{n} {x_i}^p, \tag{1.9}$$

for a positive real number p with $p \leq 1$.

2. General Solution

We here present the general solution of (1.7).

Theorem 2.1. *Let both X and Y be real vector spaces. A function $f : X \to Y$ satisfies (1.7) for all $x, y \in X$ if and only if there exists a unique symmetric biquadratic function $B_2 : X \times X \to Y$ and a unique symmetric biadditive function $B_1 : X \times X \to Y$ such that*

$$f(x) = B_2(x,x) + B_1(x,x), \tag{2.1}$$

for all $x \in X$.

Proof. Let f satisfy (1.7) and let $g, h : X \to Y$ be functions defined by

$$g(x) := f(2x) - 16f(x), \qquad h(x) := f(2x) - 4f(x), \tag{2.2}$$

for all $x \in X$. We claim that the functions g and h are quadratic and quartic, respectively.

Letting $x = y = 0$ in (1.7), we have $f(0) = 0$. By putting $x = 0$ in (1.7), one leads to the evenness $f(-y) = f(y)$ of f. Replacing y by $x + y$ in (1.7), we have

$$\begin{aligned} &f((k+1)x+y) + f((k-1)x-y) \\ &\quad = k^2 f(2x+y) + k^2 f(-y) + \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) + 2\left(1-k^2\right)f(x+y), \end{aligned} \tag{2.3}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.3), we obtain

$$\begin{aligned} &f((k+1)x-y) + f((k-1)x+y) \\ &\quad = k^2 f(2x-y) + k^2 f(y) + \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) + 2\left(1-k^2\right)f(x-y), \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Adding (2.3) to (2.4), we get by evenness of f,

$$\begin{aligned} &f((k+1)x+y) + f((k+1)x-y) + f((k-1)x+y) + f((k-1)x-y) \\ &\quad = k^2(f(2x+y) + f(2x-y)) + \frac{2k^2(k^2-1)}{6}(f(2x) - 4f(x)) \\ &\qquad + 2\left(1-k^2\right)(f(x+y) + f(x-y)) + 2k^2 f(y), \end{aligned} \tag{2.5}$$

for all $x, y \in X$. From the substitution $y = kx + y$ in (1.7), we have by evenness of f,

$$\begin{aligned} f(2kx+y) + f(y) &= k^2 f((k+1)x+y) + k^2 f((k-1)x+y) \\ &\quad + \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) + 2\left(1-k^2\right)f(kx+y), \end{aligned} \tag{2.6}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.6), we get

$$f(2kx-y)+f(-y)=k^2f((k+1)x-y)+k^2f((k-1)x-y) + \frac{k^2(k^2-1)}{6}(f(2x)-4f(x))+2\left(1-k^2\right)f(kx-y), \tag{2.7}$$

for all $x, y \in X$. Adding (2.6) to (2.7), we get by evenness of f,

$$\begin{aligned} &f(2kx+y)+f(2kx-y) \\ &\quad = k^2(f((k+1)x+y)+f((k+1)x-y)+f((k-1)x+y)+f((k-1)x-y)) \\ &\qquad + \frac{2k^2(k^2-1)}{6}(f(2x)-4f(x))+2\left(1-k^2\right)(f(kx+y)+f(kx-y))-2f(y), \end{aligned} \tag{2.8}$$

for all $x, y \in X$. By using (1.7) and (2.5), it follows from (2.8) that

$$\begin{aligned} &f(2kx+y)+f(2kx-y) \\ &\quad = k^2\Bigg[k^2(f(2x+y)+f(2x-y))+\frac{2k^2(k^2-1)}{6}(f(2x)-4f(x)) \\ &\qquad +2\left(1-k^2\right)(f(x+y)+f(x-y))+2k^2f(y)\Bigg] \\ &\qquad +2\left(1-k^2\right)\left[k^2f(x+y)+k^2f(x-y)+\frac{k^2(k^2-1)}{6}(f(2x)-4f(x))+2\left(1-k^2\right)f(y)\right] \\ &\qquad + \frac{2k^2(k^2-1)}{6}(f(2x)-4f(x))-2f(y), \end{aligned} \tag{2.9}$$

for all $x, y \in X$. If we replace x by $2x$ in (1.7), then we get that

$$\begin{aligned} &f(2kx+y)+f(2kx-y) \\ &\quad = k^2f(2x+y)+k^2f(2x-y)+\frac{k^2(k^2-1)}{6}(f(4x)-4f(2x))+2\left(1-k^2\right)f(y), \end{aligned} \tag{2.10}$$

for all $x, y \in X$. It follows from (2.9) and (2.10) that

$$k^2\Bigg[k^2(f(2x+y)+f(2x-y))+\frac{2k^2(k^2-1)}{6}(f(2x)-4f(x))$$
$$+2\left(1-k^2\right)(f(x+y)+f(x-y))+2k^2f(y)\Bigg]$$
$$+2\left(1-k^2\right)\left[k^2f(x+y)+k^2f(x-y)+\frac{k^2(k^2-1)}{6}(f(2x)-4f(x))+2\left(1-k^2\right)f(y)\right]$$
$$+\frac{2k^2(k^2-1)}{6}(f(2x)-4f(x))-2f(y)$$
$$=k^2f(2x+y)+k^2f(2x-y)+\frac{k^2(k^2-1)}{6}(f(4x)-4f(2x))+2\left(1-k^2\right)f(y), \tag{2.11}$$

for all $x, y \in X$. On the other hand, putting $y = 0$ in (1.7), we get

$$f(kx)=k^2f(x)+\frac{k^2(k^2-1)}{12}(f(2x)-4f(x)), \tag{2.12}$$

for all $x \in X$. Putting $y = x$ in (1.7), we get

$$f((k+1)x)+f((k-1)x)=k^2f(2x)+\frac{k^2(k^2-1)}{6}(f(2x)-4f(x))+2\left(1-k^2\right)f(x), \tag{2.13}$$

for all $x \in X$. Putting $y = kx$ in (1.7) and using the evenness of f, we obtain

$$f(2kx)=k^2(f((k+1)x)+f((k-1)x))+\frac{k^2(k^2-1)}{6}(f(2x)-4f(x))+2\left(1-k^2\right)f(kx), \tag{2.14}$$

for all $x \in X$. Letting $y = 0$ in (2.10), we have

$$f(2kx)=k^2f(2x)+\frac{k^2(k^2-1)}{12}(f(4x)-4f(2x)), \tag{2.15}$$

for all $x \in X$. It follows from (2.14) and (2.15) that

$$\frac{k^2(k^2-1)}{12}(f(4x)-4f(2x))=k^2(f((k+1)x)+f((k-1)x))+\frac{k^2(k^2-1)}{6}(f(2x)-4f(x))$$
$$+2\left(1-k^2\right)f(kx)-k^2f(2x), \tag{2.16}$$

for all $x \in X$. Now, by using (2.12), (2.13) and (2.16), we lead to

$$\begin{aligned}\frac{k^2(k^2-1)}{12}(f(4x)-4f(2x)) &= k^2\left[k^2 f(2x)+\frac{k^2(k^2-1)}{6}(f(2x)-4f(x))+2\left(1-k^2\right)f(x)\right]\\&\quad+2\left(1-k^2\right)\left[k^2 f(x)+\frac{k^2(k^2-1)}{12}(f(2x)-4f(x))\right]\\&\quad+\frac{k^2(k^2-1)}{6}(f(2x)-4f(x))-k^2 f(2x),\end{aligned} \tag{2.17}$$

for all $x \in X$. Finally, comparing (2.11) with (2.17), then we conclude that

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+2(f(2x)-4f(x))-6f(y), \tag{2.18}$$

for all $x, y \in X$. Replacing y by $2y$ in (2.18), we get

$$f(2x+2y)+f(2x-2y)=4f(x+2y)+4f(x-2y)+2(f(2x)-4f(x))-6f(2y), \tag{2.19}$$

for all $x, y \in X$. Interchanging x with y in (2.18), one gets

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)+2(f(2y)-4f(y))-6f(x), \tag{2.20}$$

for all $x, y \in X$. It follows from (2.19) and (2.20) that

$$\begin{aligned}&f(2(x+y))-16f(x+y)+f(2(x-y))-16f(x-y)\\&\quad=2(f(2x)-16f(x))+2(f(2y)-16f(y)),\end{aligned} \tag{2.21}$$

for all $x, y \in X$. This means that

$$g(x+y)+g(x-y)=2g(x)+2g(y), \tag{2.22}$$

for all $x, y \in X$. So the function $g : X \to Y$ defined by $g(x) := f(2x)-16f(x)$ is quadratic.

To prove that $h : X \to Y$ defined by $h(x) := f(2x)-4f(x)$ is quartic, we need to show that

$$h(2x+y)+h(2x-y)=4h(x+y)+4h(x-y)+24h(x)-6h(y), \tag{2.23}$$

for all $x, y \in X$. Replacing x and y by $2x$ and $2y$ in (2.18), respectively, we obtain

$$f(2(2x+y))+f(2(2x-y))=4f(2(x+y))+4f(2(x-y))+2(f(4x)-4f(2x))-6f(2y), \tag{2.24}$$

for all $x, y \in X$. But, since $g(2x) = 4g(x)$ for all $x \in X$, where $g : X \to Y$ is a quadratic function defined above, we see that

$$f(4x) = 20f(2x) - 64f(x), \tag{2.25}$$

for all $x \in X$. Hence, according to (2.24) and (2.25), we get

$$f(2(2x+y)) + f(2(2x-y)) = 4f(2(x+y)) + 4f(2(x-y)) + 32(f(2x) - 4f(x)) - 6f(2y), \tag{2.26}$$

for all $x, y \in X$. By multiplying 4 on both sides of (2.18), we get that

$$4f(2x+y) + 4f(2x-y) = 16f(x+y) + 16f(x-y) + 8(f(2x) - 4f(x)) - 24f(y), \tag{2.27}$$

for all $x, y \in X$. If we subtract the last equation from (2.26), then we arrive at

$$\begin{aligned} f(2(2x+y)) &- 4f(2x+y) + f(2(2x-y)) - 4f(2x-y) \\ &= 4(f(2(x+y)) - 4f(x+y)) + 4(f(2(x-y)) - 4f(x-y)) \\ &\quad + 24(f(2x) - 4f(x)) - 6(f(2y) - 4f(y)), \end{aligned} \tag{2.28}$$

for all $x, y \in X$. This means that h satisfies (2.23) and, therefore, the function $h : X \to Y$ is quartic. Thus, there exists a unique symmetric biquadratic function $B_2 : X \times X \to Y$ and a unique symmetric biadditive function $B_1 : X \times X \to Y$ such that $h(x) = 12B_2(x, x)$ and $g(x) = -12B_1(x, x)$ for all $x \in X$ (see [8, 13]). Therefore, we obtain from (2.2) that

$$f(x) = \frac{1}{12}h(x) - \frac{1}{12}g(x) = B_2(x, x) + B_1(x, x), \tag{2.29}$$

for all $x \in X$.

The proof of the converse is trivial. □

3. Generalized Hyers-Ulam Stability

From this point on, assume that X is a quasinormed space with quasinorm $\|\cdot\|_X$ and that Y is a p-Banach space with p-norm $\|\cdot\|_Y$. Let M be the modulus of concavity of $\|\cdot\|_Y$.

Before taking up the main subject, given a mapping $f : X \to Y$, we define the difference operator $D_f : X \times X \to Y$ by

$$\begin{aligned} D_f(x, y) := f(kx+y) &+ f(kx-y) - k^2 f(x+y) - k^2 f(x-y) \\ &- \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) + 2\left(k^2-1\right)f(y), \end{aligned} \tag{3.1}$$

for all $x, y \in X$. Let $\varphi^p(x, y) := (\varphi(x, y))^p$ for notational convenience.

Theorem 3.1. *Let $j \in \{-1,1\}$ be fixed and let $\varphi_q : X \times X \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} 4^{nj} \varphi_q\left(\frac{x}{2^{nj}}, \frac{y}{2^{nj}}\right) = 0, \tag{3.2}$$

for all $x, y \in X$ and

$$\sum_{i=(1+j)/2}^{\infty} 4^{ipj} \varphi_q^p\left(\frac{u}{2^{ij}}, \frac{y}{2^{ij}}\right) < \infty, \tag{3.3}$$

for all $(u,y) \in \{(x,0),(2x,0),(x,x),(x,kx) : x \in X\}$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x,y)\|_Y \le \varphi_q(x,y), \tag{3.4}$$

for all $x, y \in X$. Then, there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(2x) - 16f(x) - Q(x)\|_Y \le \frac{M^2}{4}\left(\widetilde{\varphi}_q(x)\right)^{1/p}, \tag{3.5}$$

for all $x \in X$, where

$$\begin{aligned}\widetilde{\varphi}_q(x) := \sum_{i=(1+j)/2}^{\infty} \frac{4^{ipj}}{k^{2p}(k^2-1)^p} \bigg[\left(12k^2\right)^p \varphi_q^p\left(\frac{x}{2^{ij}}, \frac{x}{2^{ij}}\right) + \left(12\left(k^2-1\right)\right)^p \varphi_q^p\left(\frac{x}{2^{ij}}, 0\right) \\ +6^p \varphi_q^p\left(\frac{2x}{2^{ij}}, 0\right) + 12^p \varphi_q^p\left(\frac{x}{2^{ij}}, \frac{kx}{2^{ij}}\right)\bigg].\end{aligned} \tag{3.6}$$

Proof. Let $j = 1$. Setting $y = 0$ in (3.4), we have

$$\left\|2f(kx) - 2k^2 f(x) - \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x))\right\|_Y \le \varphi_q(x,0), \tag{3.7}$$

for all $x \in X$. Putting $y = x$ in (3.4), we obtain

$$\begin{aligned}&\left\|f((k+1)x) + f((k-1)x) - k^2 f(2x) - \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) + 2\left(k^2-1\right)f(x)\right\|_Y \\ &\quad \le \varphi_q(x,x),\end{aligned} \tag{3.8}$$

for all $x \in X$. Replacing x by $2x$ in (3.7), we see that

$$\left\|2f(2kx) - 2k^2 f(2x) - \frac{k^2(k^2-1)}{6}(f(4x) - 4f(2x))\right\|_Y \le \varphi_q(2x,0), \tag{3.9}$$

for all $x \in X$. Setting y by kx in (3.4) and using the evenness of f, we get

$$\left\| f(2kx) - k^2 f((k+1)x) - k^2 f((k-1)x) + 2\left(k^2-1\right) f(kx) - \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) \right\|_Y \leq \varphi_q(x, kx), \tag{3.10}$$

for all $x \in X$. It follows from (3.9) and (3.10) that

$$\begin{aligned} &\left\| k^2 f(2x) + \frac{k^2(k^2-1)}{12}(f(4x) - 4f(2x)) - k^2 f((k+1)x) - k^2 f((k-1)x) \right. \\ &\left. +2\left(k^2-1\right) f(kx) - \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) \right\|_Y \leq M\left[\frac{1}{2}\varphi_q(2x, 0) + \varphi_q(x, kx)\right], \end{aligned} \tag{3.11}$$

for all $x \in X$. Also, it follows from (3.7) and (3.8) that

$$\begin{aligned} &\left\| k^2 f((k+1)x) + k^2 f((k-1)x) - 2\left(k^2-1\right) f(kx) - k^4 f(2x) \right. \\ &\left. -\frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) + 4k^2\left(k^2-1\right) f(x) \right\|_Y \leq M\left[k^2\varphi_q(x, x) + \left(k^2-1\right)\varphi_q(x, 0)\right], \end{aligned} \tag{3.12}$$

for all $x \in X$. Finally, using (3.11) and (3.12), we obtain that

$$\begin{aligned} &\|f(4x) - 20f(2x) + 64f(x)\|_Y \\ &\quad\leq \frac{M^2}{k^2(k^2-1)}\left[12k^2\varphi_q(x, x) + 12\left(k^2-1\right)\varphi_q(x, 0) + 6\varphi_q(2x, 0) + 12\varphi_q(x, kx)\right] \\ &\quad= M^2\psi_q(x), \end{aligned} \tag{3.13}$$

where

$$\psi_q(x) := \frac{1}{k^2(k^2-1)}\left[12k^2\varphi_q(x, x) + 12\left(k^2-1\right)\varphi_q(x, 0) + 6\varphi_q(2x, 0) + 12\varphi_q(x, kx)\right], \tag{3.14}$$

for all $x \in X$. Let $g : X \to Y$ be a function defined by $g(x) := f(2x) - 16f(x)$ for all $x \in X$. From (3.13), we conclude that

$$\|g(2x) - 4g(x)\|_Y \leq M^2\psi_q(x), \tag{3.15}$$

for all $x \in X$. If we replace x in (3.15) by $x/2^{n+1}$ and multiply both sides of (3.15) by 4^n, then we get

$$\left\| 4^{n+1} g\left(\frac{x}{2^{n+1}}\right) - 4^n g\left(\frac{x}{2^n}\right) \right\|_Y \leq M^2 4^n \varphi_q\left(\frac{x}{2^{n+1}}\right), \tag{3.16}$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space, the inequality (3.16) gives

$$\left\| 4^{n+1} g\left(\frac{x}{2^{n+1}}\right) - 4^m g\left(\frac{x}{2^m}\right) \right\|_Y^p \leq \sum_{i=m}^{n} \left\| 4^{i+1} g\left(\frac{x}{2^{i+1}}\right) - 4^i g\left(\frac{x}{2^i}\right) \right\|_Y^p \leq M^{2p} \sum_{i=m}^{n} 4^{ip} \varphi_q^p\left(\frac{x}{2^{i+1}}\right), \tag{3.17}$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in X$. Since $0 < p \leq 1$, by Lemma 1.2 and (3.14), we conclude that

$$\varphi_q^p(x) \leq \frac{1}{k^{2p}(k^2-1)^p} \left[\left(12k^2\right)^p \varphi_q^p(x,x) + \left(12\left(k^2-1\right)\right)^p \varphi_q^p(x,0) + 6^p \varphi_q^p(2x,0) + 12^p \varphi_q^p(x,kx) \right], \tag{3.18}$$

for all $x \in X$. Therefore, it follows from (3.3) and (3.18) that

$$\sum_{i=1}^{\infty} 4^{ip} \varphi_q^p\left(\frac{x}{2^i}\right) < \infty, \tag{3.19}$$

for all $x \in X$. It follows from (3.17) and (3.19) that the sequence $\{4^n g(x/2^n)\}$ is a Cauchy for all $x \in X$. Since Y is complete, the sequence $\{4^n g(x/2^n)\}$ converges for all $x \in X$. So one can define a function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} 4^n g\left(\frac{x}{2^n}\right), \tag{3.20}$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.17), we get

$$\|g(x) - Q(x)\|_Y^p \leq M^{2p} \sum_{i=0}^{\infty} 4^{ip} \varphi_q^p\left(\frac{x}{2^{i+1}}\right) = \frac{M^{2p}}{4^p} \sum_{i=1}^{\infty} 4^{ip} \varphi_q^p\left(\frac{x}{2^i}\right), \tag{3.21}$$

for all $x \in X$. Thus (3.5) follows from (3.18) and (3.21). Now we show that Q is quadratic. It follows from (3.16), (3.19) and (3.20) that

$$\begin{aligned} \|Q(2x) - 4Q(x)\|_Y &= \lim_{n\to\infty} \left\| 4^n g\left(\frac{x}{2^{n-1}}\right) - 4^{n+1} g\left(\frac{x}{2^n}\right) \right\|_Y = 4 \lim_{n\to\infty} \left\| 4^{n-1} g\left(\frac{x}{2^{n-1}}\right) - 4^n g\left(\frac{x}{2^n}\right) \right\|_Y \\ &\leq M^2 \lim_{n\to\infty} 4^n \varphi_q\left(\frac{x}{2^n}\right) = 0, \end{aligned} \tag{3.22}$$

for all $x \in X$. So,

$$Q(2x) = 4Q(x), \tag{3.23}$$

for all $x \in X$. On the other hand, it follows from (3.2), (3.4) and (3.20) that

$$\begin{aligned} \|D_Q(x,y)\|_Y &= \lim_{n\to\infty} 4^n \left\| D_g\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\|_Y = \lim_{n\to\infty} 4^n \left\| D_f\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) - 16 D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\|_Y \\ &\le M \lim_{n\to\infty} 4^n \left\{ \left\| D_f\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right)\right\|_Y + 16 \left\| D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\|_Y \right\} \\ &\le M \lim_{n\to\infty} 4^n \left\{ \varphi_q\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) + 16 \varphi_q\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\} = 0, \end{aligned} \tag{3.24}$$

for all $x, y \in X$. Hence the function Q satisfies (1.7). Thus, by Theorem 2.1, the function $x \rightsquigarrow Q(2x) - 16Q(x)$ is quadratic. Therefore, (3.23) implies that the function Q is quadratic.

Now, to prove the uniqueness property of Q, let $Q' : X \to Y$ be another quadratic function satisfying (3.5). It follows from (3.3) that

$$\lim_{n\to\infty} 4^{np} \sum_{i=1}^{\infty} 4^{ip} \varphi_q^p\left(\frac{u}{2^{n+i}}, \frac{y}{2^{n+i}}\right) = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} 4^{ip} \varphi_q^p\left(\frac{u}{2^i}, \frac{y}{2^i}\right) = 0, \tag{3.25}$$

for all $(u, y) \in \{(x,0), (2x,0), (x,x), (x,kx) : x \in X\}$. Hence,

$$\lim_{n\to\infty} 4^{np} \widetilde{\varphi}_q\left(\frac{x}{2^n}\right) = 0, \tag{3.26}$$

for all $x \in X$. It follows from (3.5), (3.20) and (3.26) that

$$\|Q(x) - Q'(x)\|_Y^p = \lim_{n\to\infty} 4^{np} \left\| g\left(\frac{x}{2^n}\right) - Q'\left(\frac{x}{2^n}\right)\right\|_Y^p \le \frac{M^{2p}}{4^p} \lim_{n\to\infty} 4^{np} \widetilde{\varphi}_q\left(\frac{x}{2^n}\right) = 0, \tag{3.27}$$

for all $x \in X$. So $Q = Q'$.

For $j = -1$, we can prove the theorem by a similar argument. □

Corollary 3.2. *Let θ, r, s be nonnegative real numbers such that $r, s > 2$ or $r, s < 2$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_f(x,y)\|_Y \le \theta\left(\|x\|_X^r + \|y\|_X^s\right), \tag{3.28}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\|f(2x) - 16f(x) - Q(x)\|_Y \le \frac{M^2\theta}{k^2(k^2-1)} \gamma_q(x), \tag{3.29}$$

for all $x \in X$, *where*

$$\gamma_q(x) = \left(\frac{12^p\left[k^{2p} + (k^2-1)^p + \left(2^{r-1}\right)^p + 1\right]}{|4^p - 2^{rp}|} \|x\|_X^{rp} + \frac{12^p\left(k^{2p} + k^{sp}\right)}{|4^p - 2^{sp}|} \|x\|_X^{sp} \right)^{1/p}. \tag{3.30}$$

Proof. In Theorem 3.1, putting $\varphi_q(x,y) := \theta(\|x\|_X^r + \|y\|_X^s)$ for all $x, y \in X$, we get the desired result. □

Corollary 3.3. *Let* $\theta \geq 0$ *and* $r, s > 0$ *be real numbers such that* $\lambda := r + s \neq 2$. *Suppose that an even function* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\left\|D_f(x,y)\right\|_Y \leq \theta \|x\|_X^r \|y\|_X^s, \tag{3.31}$$

for all $x, y \in X$. *Then there exists a unique quadratic function* $Q : X \to Y$ *satisfying*

$$\left\|f(2x) - 16f(x) - Q(x)\right\|_Y \leq \frac{M^2\theta}{k^2(k^2-1)} \left(\frac{12^p\left(k^{2p} + k^{sp}\right)}{\left|4^p - 2^{\lambda p}\right|} \right)^{1/p} \|x\|_X^\lambda, \tag{3.32}$$

for all $x \in X$.

Proof. In Theorem 3.1, taking $\varphi_q(x,y) := \theta\|x\|_X^r \|y\|_X^s$, for all $x, y \in X$, we arrive at the desired result. □

Theorem 3.4. *Let* $j \in \{-1, 1\}$ *be fixed and let* $\varphi_v : X \times X \to [0, \infty)$ *be a function such that*

$$\lim_{n \to \infty} 16^{nj} \varphi_v\left(\frac{x}{2^{nj}}, \frac{y}{2^{nj}}\right) = 0, \tag{3.33}$$

for all $x, y \in X$ *and*

$$\sum_{i=(1+j)/2}^{\infty} 16^{ipj} \varphi_v^p\left(\frac{u}{2^{ij}}, \frac{y}{2^{ij}}\right) < \infty, \tag{3.34}$$

for all $(u, y) \in \{(x,0), (2x,0), (x,x), (x,kx) : x \in X\}$. *Suppose that an even function* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\left\|D_f(x,y)\right\|_Y \leq \varphi_v(x,y), \tag{3.35}$$

for all $x, y \in X$. *Then there exists a unique quartic function* $V : X \to Y$ *such that*

$$\left\|f(2x) - 4f(x) - V(x)\right\|_Y \leq \frac{M^2}{16} \left(\widetilde{\varphi}_v(x)\right)^{1/p}, \tag{3.36}$$

for all $x \in X$, *where*

$$\widetilde{\varphi}_v(x) := \sum_{i=(1+j)/2}^{\infty} \frac{16^{ipj}}{k^{2p}(k^2-1)^p} \left\{ \left(12k^2\right)^p \varphi_v^p\left(\frac{x}{2^{ij}}, \frac{x}{2^{ij}}\right) + \left(12\left(k^2-1\right)\right)^p \varphi_v^p\left(\frac{x}{2^{ij}}, 0\right) \right. \\ \left. +6^p \varphi_v^p\left(\frac{2x}{2^{ij}}, 0\right) + 12^p \varphi_v^p\left(\frac{x}{2^{ij}}, \frac{kx}{2^{ij}}\right) \right\}. \tag{3.37}$$

Proof. Being similar to the proof of Theorem 3.1, we omit its proof. □

Corollary 3.5. *Let* θ, r, s *be nonnegative real numbers such that* $r, s > 4$ *or* $r, s < 4$. *Suppose that an even function* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality* (3.28) *for all* $x, y \in X$. *Then there exists a unique quartic function* $V : X \to Y$ *satisfying*

$$\left\| f(2x) - 4f(x) - V(x) \right\|_Y \leq \frac{M^2\theta}{k^2(k^2-1)} \gamma_v(x), \tag{3.38}$$

for all $x \in X$, *where*

$$\gamma_v(x) = \left(\frac{12^p\left[k^{2p} + (k^2-1)^p + (2^{r-1})^p + 1\right]}{|16^p - 2^{rp}|} \|x\|_X^{rp} + \frac{12^p\left(k^{2p} + k^{sp}\right)}{|16^p - 2^{sp}|} \|x\|_X^{sp} \right)^{1/p}, \tag{3.39}$$

for all $x \in X$.

Corollary 3.6. *Let* $\theta \geq 0$ *and* $r, s > 0$ *be real numbers such that* $\lambda := r + s \neq 4$. *Suppose that an even function* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality* (3.31) *for all* $x, y \in X$. *Then, there exists a unique quartic function* $V : X \to Y$ *satisfying*

$$\left\| f(2x) - 4f(x) - V(x) \right\|_Y \leq \frac{M^2\theta}{k^2(k^2-1)} \left(\frac{12^p\left(k^{2p} + k^{sp}\right)}{\left|16^p - 2^{\lambda p}\right|} \right)^{1/p} \|x\|_X^{\lambda}, \tag{3.40}$$

for all $x \in X$.

Now, we are ready to prove the main theorem concerning the stability problem for (1.7).

Theorem 3.7. *Let* $j \in \{-1, 1\}$ *be fixed and let* $\varphi : X \times X \to [0, \infty)$ *be a function such that*

$$\lim_{n\to\infty} \left(\left(\frac{1-j}{2}\right) 4^{nj} \varphi\left(\frac{x}{2^{nj}}, \frac{y}{2^{nj}}\right) + \left(\frac{1+j}{2}\right) 16^{nj} \varphi\left(\frac{x}{2^{nj}}, \frac{y}{2^{nj}}\right) \right) = 0, \tag{3.41}$$

for all $x, y \in X$ *and*

$$\sum_{i=(1+j)/2}^{\infty} \left(\left(\frac{1-j}{2}\right) 4^{ipj} \varphi^p\left(\frac{u}{2^{ij}}, \frac{y}{2^{ij}}\right) + \left(\frac{1+j}{2}\right) 16^{ipj} \varphi^p\left(\frac{u}{2^{ij}}, \frac{y}{2^{ij}}\right) \right) < \infty, \tag{3.42}$$

for all $(u,y) \in \{(x,0),(2x,0),(x,x),(x,kx) : x \in X\}$. *Suppose that an even function* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|D_f(x,y)\|_Y \le \varphi(x,y), \tag{3.43}$$

for all $x,y \in X$. *Then, there exists a unique quadratic function* $Q : X \to Y$ *and a unique quartic function* $V : X \to Y$ *such that*

$$\|f(x) - Q(x) - V(x)\|_Y \le \frac{M^3}{192}\left\{4\left(\widetilde{\varphi}_q(x)\right)^{1/p} + \left(\widetilde{\varphi}_v(x)\right)^{1/p}\right\}, \tag{3.44}$$

for all $x \in X$, *where*

$$\begin{aligned}\widetilde{\varphi}_q(x) &:= \sum_{i=(1+j)/2}^{\infty}\frac{4^{ipj}}{k^{2p}(k^2-1)^p}\left\{\left(12k^2\right)^p\varphi^p\left(\frac{x}{2^{ij}},\frac{x}{2^{ij}}\right) + \left(12\left(k^2-1\right)\right)^p\varphi^p\left(\frac{x}{2^{ij}},0\right)\right.\\ &\qquad\left. +6^p\varphi^p\left(\frac{2x}{2^{ij}},0\right) + 12^p\varphi^p\left(\frac{x}{2^{ij}},\frac{kx}{2^{ij}}\right)\right\},\\ \widetilde{\varphi}_v(x) &:= \sum_{i=(1+j)/2}^{\infty}\frac{16^{ipj}}{k^{2p}(k^2-1)^p}\left\{\left(12k^2\right)^p\varphi^p\left(\frac{x}{2^{ij}},\frac{x}{2^{ij}}\right) + \left(12\left(k^2-1\right)\right)^p\varphi^p\left(\frac{x}{2^{ij}},0\right)\right.\\ &\qquad\left. +6^p\varphi^p\left(\frac{2x}{2^{ij}},0\right) + 12^p\varphi^p\left(\frac{x}{2^{ij}},\frac{kx}{2^{ij}}\right)\right\}.\end{aligned} \tag{3.45}$$

Proof. By Theorems 3.1 and 3.4, there exists a quadratic function $Q_0 : X \to Y$ and a quartic function $V_0 : X \to Y$ such that

$$\|f(2x) - 16f(x) - Q_0(x)\|_Y \le \frac{M^2}{4}\left[\widetilde{\varphi}_q(x)\right]^{1/p}, \qquad \|f(2x) - 4f(x) - V_0(x)\|_Y \le \frac{M^2}{16}\left[\widetilde{\varphi}_v(x)\right]^{1/p}, \tag{3.46}$$

for all $x \in X$. Therefore, it follows from (3.46) that

$$\left\|f(x) + \frac{1}{12}Q_0(x) - \frac{1}{12}V_0(x)\right\|_Y \le \frac{M^3}{192}\left(4\left[\widetilde{\varphi}_q(x)\right]^{1/p} + \left[\widetilde{\varphi}_v(x)\right]^{1/p}\right), \tag{3.47}$$

for all $x \in X$. Thus we obtain (3.44) by letting $Q(x) = -(1/12)Q_0(x)$ and $V(x) = (1/12)V_0(x)$ for all $x \in X$.

To prove the uniqueness property of Q and V, let $Q', V' : X \to Y$ be another quadratic and quartic functions satisfying (3.44). Let $\overline{Q} = Q - Q'$ and $\overline{V} = V - V'$. Hence,

$$\begin{aligned}\left\|\overline{Q}(x) + \overline{V}(x)\right\|_Y &\le M\{\|f(x) - Q(x) - V(x)\|_Y + \|f(x) - Q'(x) - V'(x)\|_Y\}\\ &\le \frac{M^4}{96}\left(4\left[\widetilde{\varphi}_q(x)\right]^{1/p} + \left[\widetilde{\varphi}_v(x)\right]^{1/p}\right),\end{aligned} \tag{3.48}$$

for all $x \in X$. Since $\lim_{n\to\infty} 4^{npj}\widetilde{\varphi}_q(x/2^n) = \lim_{n\to\infty} 16^{npj}\widetilde{\varphi}_v(x/2^n) = 0$, for all $x \in X$, we figure out that

$$\lim_{n\to\infty} 16^n \left\| \overline{Q}\left(\frac{x}{2^n}\right) + \overline{V}\left(\frac{x}{2^n}\right) \right\|_Y = 0, \tag{3.49}$$

for all $x \in X$. Therefore, we get $\overline{V} = 0$ and then $\overline{Q} = 0$. □

Corollary 3.8. *Let θ, r, s be nonnegative real numbers such that $r, s > 4$ or $2 < r,\ s < 4$ or $r, s < 2$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.28)*, for all $x, y \in X$. Then, there exists a unique quadratic function $Q : X \to Y$ and a unique quartic function $V : X \to Y$ such that*

$$\|f(x) - Q(x) - V(x)\|_Y \le \frac{M^3\theta}{12k^2(k^2-1)}\left(\gamma_q(x) + \gamma_v(x)\right), \tag{3.50}$$

for all $x \in X$, where $\gamma_q(x)$ and $\gamma_v(x)$ are defined as in Corollaries 3.2 and 3.5.

Corollary 3.9. *Let $\theta \ge 0$ and $r, s > 0$ be non-negative real numbers such that $\lambda := r + s \in (0,2) \cup (2,4) \cup (4,\infty)$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.31) *for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique quartic function $V : X \to Y$ such that*

$$\begin{aligned} &\|f(x) - Q(x) - V(x)\|_Y \\ &\quad \le \frac{M^3\theta}{12k^2(k^2-1)}\left\{\left(\frac{12^p(k^{2p}+k^{sp})}{|4^p - 2^{\lambda p}|}\right)^{1/p} + \left(\frac{12^p(k^{2p}+k^{sp})}{|16^p - 2^{\lambda p}|}\right)^{1/p}\right\}\|x\|_X^{\lambda}, \end{aligned} \tag{3.51}$$

for all $x \in X$.

Corollary 3.10. *Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_f(x,y)\|_X \le \varepsilon, \tag{3.52}$$

for all $x, y \in X$ where $\varepsilon > 0$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique quartic function $V : X \to Y$ such that

$$\begin{aligned} &\|f(x) - Q(x) - V(x)\|_Y \\ &\quad \le \frac{M^3\varepsilon}{k^2(k^2-1)}\left\{\left(\frac{k^{2p} + (k^2-1)^p + 2^{-p} + 1}{4^p - 1}\right)^{1/p} + \left(\frac{k^{2p} + (k^2-1)^p + 2^{-p} + 1}{16^p - 1}\right)^{1/p}\right\}, \end{aligned} \tag{3.53}$$

for all $x \in X$.

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