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Research Article

On Degenerate Parabolic Equations

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The paper deals with the existence of solutions of some generalized Stefan-type equation in the framework of Orlicz spaces.

1. Introduction

In this paper, we deal with the following boundary value problems

$$\begin{aligned} &\frac{\partial u}{\partial t} + A(\theta(u)) = f, \quad \text{in } Q, \\ &u = 0, \quad \text{on } \partial Q = \partial\Omega \times (0,T), \\ &u(x,0) = u_0(x), \quad \text{in } \Omega, \end{aligned} \tag{P}$$

where

$$A(u) = -\operatorname{div}(a(\cdot,t,\nabla u)), \tag{1.1}$$

$Q = \Omega \times [0,T]$, $T > 0$, and Ω is a bounded domain of $\mathbf{R}^N$, with the segment property, f is a smooth function, $u_0 \in L^2(\Omega)$, θ is a positive real function increasing but not necessarily strictly increasing, $\theta(0) = 0$, and $\theta(u_0) \in L^2(\Omega)$. $a : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function (i.e., measurable with respect to x in Ω for every (t,ξ) in $\mathbf{R} \times \mathbf{R} \times \mathbf{R}^N$, and continuous with respect to ξ in $\mathbf{R}^N$ for almost every x in Ω) such that for all $\xi, \xi^* \in \mathbf{R}^N$, $\xi \neq \xi^*$,

$$a(x,t,\xi)\xi \geq \alpha B(|\xi|), \tag{1.2}$$

$$[a(x,t,\xi) - a(x,t,\xi^*)][\xi - \xi^*] > 0, \tag{1.3}$$

$$|a(x,t,\xi)| \leq c(x,t) + k_1 \overline{B}^{-1} B(k_2|\xi|). \tag{1.4}$$

There exist an N-function M such that

$$B(\theta(t)) \ll M(t), \tag{1.5}$$

where $c(x,t)$ belongs to $E_{\overline{B}}(Q), c \geq 0$ and $k_i (i = 1,2)$ to $\mathbf{R}^+$, and α to $\mathbf{R}^+_*$.

Some examples of such operator are in particular the case where

$$a(\cdot, \nabla\theta(u)) = \frac{B(|\nabla\theta(u)|)}{|\nabla\theta(u)|^2} \nabla\theta(u), \tag{1.6}$$

where B is an N-function.

Many physical models in hydrology, infiltration through porous media, heat transport, metallurgy, and so forth lead to the nonlinear equations (systems) of the form

$$\partial_t u = \nabla\psi(\nabla\beta(u)), \tag{E}$$

where β, ψ are monotone, $\psi(s)\, s$ is even and convex for $s \geq s_o > 0$, $|\psi(s)| \to \infty$, $|\beta(s)| \to \infty$ for $|s| \to \infty$ (for the details see [1]). Jäger and Kačur treated the porous medium systems where β is strictly monotone in [2] and Stefan-type problems where β is only monotone. For the last model, there exists a large number of references. Among them, let us mention the earlier works [3–5] for a variational approach and [6] for semigroup.

In [7], a different approach was introduced to study the porous and Stefan problems.The enthalpy formulation and the variational technique are used. Nonstandard semidiscretization in time is used, and Newton-like iterations are applied to solve the corresponding elliptic problems.

Due to the possible jumps of θ, problem (P) enters the class of Stefan problems. In the present paper, we are interested in the parabolic problem with regular data. It is similar in many respects to the so-called porous media equation. However, the equation we consider has a more general structure than that in the references above.

Two main difficulties appear in the study of existence of solutions of problem (P). The first one comes from the diffusion terms in (P) since they do not depend on u but on $\theta(u)$, and, moreover, at the same time, (P) poses big problems, since in general we have not information on u but on $\theta(u)$. For the last reason, the authors in [8] define a new notion of weak solution to overcome this problem.

In the above cited references, the authors have shown the existence of a weak solution when the function $a(x,t,\xi)$ was assumed to satisfy a polynomial growth condition with respect to ∇u. When trying to relax this restriction on the function $a(\cdot,\xi)$, we are led to replace the space $L^p(0,T;W^{1,p}(\Omega))$ by an inhomogeneous Sobolev space $W^{1,x}L_B$ built from an Orlicz space L_B instead of L^p, where the N-function B which defines L_B is related to the actual growth of the Carathéodory's function.

Our goal in this paper is, on the one hand, to give a generalization of (E) in the case of one equation in the framework of Leray-Lions operator in Orlicz-Sobolev spaces. on the second hand, we prove the existence of solutions in the $BV(Q)$ space.

2. Preliminaries

Let $M : \mathbf{R}^+ \to \mathbf{R}^+$ be an N-function, that is, M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$, and $M(t)/t \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \sup\{st - M(s) : s > 0\}$.

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, that is, for each $\varepsilon > 0$,

$$\frac{P(t)}{Q(\varepsilon t)} \longrightarrow 0, \quad \text{as } t \longrightarrow \infty. \tag{2.1}$$

Let Ω be an open subset of $\mathbf{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_\Omega M(u(x))dx < +\infty$ (resp., $\int_\Omega M(u(x)/\lambda)dx < +\infty$ for some $\lambda > 0$).

Note that $L_M(\Omega)$ is a Banach space under the norm $\|u\|_{M,\Omega} = \inf\{\lambda > 0 : \int_\Omega M(u(x)/\lambda)dx \le 1\}$ and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. In general, $E_M(\Omega) \neq L_M(\Omega)$ and the dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

We say that u_n converges to u for the modular convergence in $L_M(\Omega)$ if, for some $\lambda > 0$, $\int_\Omega M((u_n - u)/\lambda)dx \to 0$. This implies convergence for $\sigma(L_M, L_{\overline{M}})$.

The inhomogeneous Orlicz-Sobolev spaces are defined as follows: $W^{1,x}L_M(Q) = \{u \in L_M(Q) : D_x^\alpha u \in L_M(Q)$ for all $|\alpha| \le 1\}$. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which have as many copies as there are α-order derivatives, $|\alpha| \le 1$. We define the space $W_0^{1,x}L_M(Q) = \overline{\mathfrak{D}(Q)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}$. (For more details, see [9].)

For $k > 0$, we define the truncation at height $k, T_k : \mathbf{R} \to \mathbf{R}$ by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k, \\ k\dfrac{s}{|s|}, & \text{if } |s| > k. \end{cases} \tag{2.2}$$

3. Main Result

Before giving our main result, we give the following lemma which will be used.

Lemma 3.1 (see [10]). *Under the hypothesis* (1.2)–(1.4), $\theta(s) = s$, *the problem* (P) *admits at least one solution* u *in the following sense:*

$$\begin{gathered} u \in W_0^{1,x}L_B(Q) \cap L^2(Q), \\ \left\langle \frac{\partial u}{\partial t}, v \right\rangle + \int_Q a(\cdot, \nabla u)\nabla v = \int_Q fv\, dxdt, \end{gathered} \tag{3.1}$$

for all $v \in W_0^{1,x}L_B(Q) \cap L^2(Q)$ *and for* $v = u$.

Theorem 3.2. *Under the hypothesis* (1.2)–(1.5)*, the problem* (P_0) *admits at least one solution* u *in the following sense:*

$$u \in BV_{\mathrm{loc}}(Q), \qquad \theta(u) \in W_0^{1,x}L_B(Q) \cap L^2(Q),$$
$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle + \int_Q a(\cdot, \nabla\theta(u))\nabla v = \int_Q f v \, dx dt, \tag{3.2}$$

for all $v \in W_0^{1,x}L_B(Q)$.

Proof.

Step 1 (approximation and a priori estimate). Consider the approximate problem:

$$\frac{\partial u_n}{\partial t} - \operatorname{div}(a(\cdot, \nabla\theta(u_n))) - \frac{1}{n}\Delta_M(u_n) = f, \quad \text{in } Q,$$
$$u_n(x,0) = u_{0n}(x), \quad \text{in } \Omega, \tag{3.3}$$

where $-\Delta_M(u) = -\operatorname{div}((M(|\nabla u_n|)/|\nabla u_n|^2)\nabla u_n)$ is the M-Laplacian operator and (u_{0n}) is a smooth sequence converging strongly to u_0 in $L^2(Q)$.

The approximate problem has a regular solution u_n and in particular $u_n \in W_0^{1,x}L_M(Q)$ (by Lemma 3.1).

Let $\Theta(s) = \int_0^s \theta(t)dt$.

Let $v = \theta(u_n)\chi_{(0,\tau)}$ as test function, one has

$$\int_\Omega \Theta(u_n(\tau))dx + \alpha \int_{Q_\tau} B(|\nabla\theta(u_n)|) + \leq \int_{Q_\tau} f\theta(u_n) + \int_\Omega \Theta(u_n(0))dx, \tag{3.4}$$

then, $(\theta(u_n))_n$ bounded in $W_0^{1,x}L_B(Q)$.

There exist a measurable function v and a subsequence, also denoted (u_n), such that,

$$\theta(u_n) \rightharpoonup v, \quad \text{a.e in } Q \text{ and weakly in } W_0^{1,x}L_B(Q). \tag{3.5}$$

Let us consider the C^2 function defined by

$$\eta_k(s) = \begin{cases} s & |s| \leq \dfrac{k}{2}, \\ k \operatorname{sign}(s) & |s| \geq k. \end{cases} \tag{3.6}$$

Multiplying the approximating equation by $\eta_k'(u_n)$, we get

$$\frac{\partial \eta_k(u_n)}{\partial t} - \operatorname{div}\big(a(\cdot,\nabla\theta(u_n))\eta_k'(u_n)\big) + a(\cdot,\nabla\theta(u_n))\eta_k''(u_n),$$
$$-\frac{1}{n}\operatorname{div}\left(\frac{M(|\nabla u_n|)}{|\nabla u_n|^2}\nabla u_n \eta_k'(u_n)\right) + \frac{1}{n}\frac{M(|\nabla u_n|)}{|\nabla u_n|^2}\nabla u_n \eta_k''(u_n) = f\eta_k'(u_n) \tag{3.7}$$

in the distributions sense. We deduce, then, $\eta_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$ and $\partial\eta_k(u_n)/\partial t$ in $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$. Then, $\eta_k(u_n)$ is compact in $L^1(Q)$.

Following the same way as in [11], we obtain $\theta(u_n) \rightharpoonup \theta(u)$, weakly in $W_0^{1,x}L_B(Q)$ for $\sigma(\Pi L_B, \Pi E_{\overline{B}})$, strongly in $L^1(Q)$ and a.e in Q.

Step 2 (passage to the limit). Let set $b(\cdot,\nabla u) = (M(|\nabla u|)/|\nabla u|^2)\nabla u$.

Let $v \in W_0^{1,x}L_B(Q)$, one has

$$\left\langle \frac{\partial u_n}{\partial t}, \theta(u_n) - v\right\rangle + \int_Q a(\cdot,\nabla\theta(u_n))\nabla(\theta(u_n)-v) + \int_Q \frac{1}{n} b(\cdot,\nabla u_n)\nabla(\theta(u_n)-v)dx$$
$$= \int_Q f(\theta(u_n)-v)dxdt. \tag{3.8}$$

By using the following decomposition:

$$\begin{aligned} a(\cdot,\nabla\theta(u_n))\nabla(\theta(u_n)-v) &= a(\cdot,\nabla\theta(u_n)) - a(\cdot,\theta(\nabla v))\nabla(\theta(u_n)-v) \\ &\quad + a(\cdot,\theta(\nabla v))\nabla(\theta(u_n)-v), \\ b(\cdot,\nabla u_n)\nabla(\theta(u_n)-v) &= b(\cdot,\nabla u_n) - b(\cdot,\nabla v)\nabla(\theta(u_n)-v) + b(\cdot,\nabla v)\nabla(\theta(u_n)-v), \\ \nabla(\theta(u_n)-v) &= \big(\theta'(u_n)\nabla u_n - \nabla v\big) = \theta'(u_n)\nabla u_n - \theta'(u_n)\nabla v + \theta'(u_n)\nabla v - \nabla v, \end{aligned} \tag{3.9}$$

and by the monotonicity of the operator defined by a and b, we obtain

$$\left\langle \frac{\partial u_n}{\partial t}, \theta(u_n) - v\right\rangle + \int_Q a(\cdot,\theta(\nabla v))(\nabla\theta(u_n) - \nabla v) + \int_Q \frac{1}{n} b(\cdot,\nabla v)(\nabla\theta(u_n) - \nabla v)$$
$$+ \int_Q \frac{1}{n}(b(\cdot,\nabla u_n) - b(\cdot,\nabla v))\big(\theta'(u_n) - 1\big)\nabla v \le \int_Q f(\theta(u_n)-v), \tag{3.10}$$

by passage to the limit with a standard argument as in [10, 11], and using the above convergence of $\theta(u_n)$, we have

$$\left\langle \frac{\partial u}{\partial t}, \theta(u) - v\right\rangle + \int_Q a(\cdot,\nabla\theta(v))(\nabla\theta(u) - \nabla v)dx \le \int_Q f(\theta(u)-v). \tag{3.11}$$

Taking now $v = \theta(u) - t\varphi$, with $\varphi \in W_0^{1,x} L_B(Q)$ and $t \in (-1,1)$, we deduce that u is solution of the problem (1.2).

Step 3 ($u \in BV_{\mathrm{loc}}(Q)$). Let K be a compact in Q, and let $\varphi \in D(Q)$ with $K \subset \operatorname{supp}(\varphi)$ such that

$$\varphi = 1, \quad \text{on } K, |\nabla\varphi| \leq 1. \tag{3.12}$$

Using φ as test function in (3.7), we get

$$\begin{aligned}
&\int_Q \frac{\partial \eta_k(u_n)}{\partial t}\varphi\,dx\,dt + \int_Q a(\cdot, \nabla\theta(u_n))\nabla\varphi \cdot \eta_k'(u_n)\,dx\,dt + \int_Q a(\cdot, \nabla\theta(u_n))\varphi \cdot \eta_k''(u_n)\,dx\,dt \\
&\quad + \frac{1}{n}\int_Q \frac{M(|\nabla u_n|)}{|\nabla u_n|^2}\nabla u_n \nabla\varphi \cdot \eta_k'(u_n)\,dx\,dt + \frac{1}{n}\int_Q \frac{M(|\nabla u_n|)}{|\nabla u_n|^2}\nabla u_n \varphi \cdot \eta_k''(u_n)\,dx\,dt \\
&\quad = I_1 + I_2 + I_3 + I_4 + I_5 \\
&\quad = \int_Q f\eta_k'(u_n)\varphi\,dx\,dt.
\end{aligned} \tag{3.13}$$

The terms I_2, I_3, I_4, I_5 are bounded, so

$$\int_K \left|\frac{\partial \eta_k(u_n)}{\partial t}\right| dx\,dt \leq C. \tag{3.14}$$

Letting k tend to infinity, we have

$$\int_K \left|\frac{\partial u_n}{\partial t}(x,t)\right| dx\,dt \leq C. \tag{3.15}$$

We deal now with the following estimation which ends the proof.

For all compact $K \subset Q$,

$$\int_K |Du_n(x,t)|\,dx\,dt \leq C. \tag{3.16}$$

Indeed, we differentiate the approximate problem with respect to x_i, we multiply the obtained equation by $\eta_k'(\partial_{x_i} u_n)$, and one has the following equality in the distributions sense

$$\begin{aligned}
&\frac{\partial(\partial_{x_i} u_n)}{\partial t}\eta_k'(\partial_{x_i} u_n) - \operatorname{div}(\partial_{x_i} a(\cdot, \nabla\theta(u_n)))\eta_k'(\partial_{x_i} u_n) - \frac{1}{n}\partial_{x_i}\Delta_M(u_n)\eta_k'(\partial_{x_i} u_n) \\
&\quad = \partial_{x_i} f\eta_k'(\partial_{x_i} u_n),
\end{aligned} \tag{3.17}$$

which is equivalent to

$$\begin{aligned}&\frac{\partial\eta_k(\partial_{x_i}u_n)}{\partial t}-\operatorname{div}(\partial_{x_i}a(\cdot,\nabla\theta(u_n)))\eta_k'(\partial_{x_i}u_n)-\frac{1}{n}\operatorname{div}(\partial_{x_i}b(\cdot,\nabla u_n))\eta_k'(\partial_{x_i}u_n)\\&\quad=\partial_{x_i}f\eta_k'(\partial_{x_i}u_n).\end{aligned}\tag{3.18}$$

We recall that η_k,η_k', and η_k'' are bounded on $\mathbf{R}$, $(\theta(u_n))$ is bounded in $W_0^{1,x}L_B(Q)$, and $(a(\cdot,\nabla\theta(u_n)))$ is bounded in $L_{\overline{B}}(Q)$.

Using now the test function φ (defined below), we obtain, as for (3.14),

$$\int_K\left|\frac{\partial\eta_k}{\partial t}\left(\frac{\partial u_n}{\partial x_i}\right)\right|dxdt\le C.\tag{3.19}$$

With the same way as above, we conclude the result, $u\in BV_{\mathrm{loc}}(Q)$. □

Remark 3.3. As in Theorem 3.2, one can prove the same result in the case where we replace the initial condition in the problem (P) by $\theta(u(x,0))=\theta(u_0(x))$ and $\theta(u_0(x))\in L^2(Q)$.

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