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Research Article

On Maximal Subsemigroups of Partial Baer-Levi Semigroups

Boorapa Singha[1] and Jintana Sanwong[1, 2]

[1] *Department of Mathematics, Chiang Mai University, Chiangmai 50200, Thailand*
[2] *Material Science Research Center, Faculty of Science, Chiang Mai University, Chiangmai 50200, Thailand*

Correspondence should be addressed to Jintana Sanwong, scmti004@chiangmai.ac.th

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Suppose that X is an infinite set with $|X| \geq q \geq \aleph_0$ and $I(X)$ is the symmetric inverse semigroup defined on X. In 1984, Levi and Wood determined a class of maximal subsemigroups M_A (using certain subsets A of X) of the Baer-Levi semigroup $BL(q) = \{\alpha \in I(X) : \text{dom}\ \alpha = X$ and $|X \setminus X\alpha| = q\}$. Later, in 1995, Hotzel showed that there are many other classes of maximal subsemigroups of $BL(q)$, but these are far more complicated to describe. It is known that $BL(q)$ is a subsemigroup of the partial Baer-Levi semigroup $PS(q) = \{\alpha \in I(X) : |X \setminus X\alpha| = q\}$. In this paper, we characterize all maximal subsemigroups of $PS(q)$ when $|X| > q$, and we extend M_A to obtain maximal subsemigroups of $PS(q)$ when $|X| = q$.

1. Introduction

Suppose that X is a nonempty set, and let $P(X)$ denote the semigroup (under composition) of all *partial* transformations of X (i.e., all mappings $\alpha : A \to B$, where $A, B \subseteq X$). For any $\alpha \in P(X)$, we let $\text{dom}\,\alpha$ and $\text{ran}\ \alpha$ (or $X\alpha$) denote the *domain* and the *range* of α, respectively. We also write

$$g(\alpha) = |X \setminus \text{dom}\,\alpha|, \qquad d(\alpha) = |X \setminus \text{ran}\ \alpha|, \qquad r(\alpha) = |\text{ran}\ \alpha|, \tag{1.1}$$

and refer to these cardinals as the *gap*, the *defect*, and the *rank* of α, respectively. Let $I(X)$ denote the *symmetric inverse semigroup* on X: that is, the set of all injective mappings in $P(X)$. If $|X| = p \geq q \geq \aleph_0$, we write

$$BL(q) = \{\alpha \in I(X) : g(\alpha) = 0,\ d(\alpha) = q\},\ PS(q) = \{\alpha \in I(X) : d(\alpha) = q\}, \tag{1.2}$$

where $BL(q)$ is the *Baer-Levi semigroup* of type (p,q) defined on X (see [1, 2, vol 2, Section 8.1]). It is wellknown that this semigroup is right simple, right cancellative, and idempotent-free. On the other hand, in [3] the authors showed that $PS(q)$, the *partial Baer-Levi semigroup* on X, does not have these properties but it is *right reductive* in the sense that for every $\alpha, \beta \in PS(q)$, if $\alpha\gamma = \beta\gamma$ for all $\gamma \in PS(q)$, then $\alpha = \beta$. Also, they showed that $PS(q)$ satisfies the dual property, that is, it is *left reductive* (see [1, 2, vol 1, p 9]). The authors also characterized Green's relations and ideals of $PS(q)$ and, in [3, Corollary 1], they proved that $PS(q)$ contains an inverse subsemigroup: namely, the set $R(q)$ defined by

$$R(q) = \{\alpha \in PS(q) : g(\alpha) = q\}. \tag{1.3}$$

This set consists, in fact, of all regular elements of $PS(q)$, as shown in [3, Theorem 4]. Recently, in [4], the authors studied some properties of Mitsch's natural partial order defined on a semigroup (see [5, Theorem 3]) and some other partial orders defined on $PS(q)$. In particular, they described compatibility and the existence of maximal and minimal elements.

For any nonempty subset A of X such that $|X \setminus A| \geq q$, let

$$M_A = \{\alpha \in BL(q) : A \not\subseteq X\alpha \text{ or } (A\alpha \subseteq A \text{ or } |X\alpha \setminus A| < q)\}. \tag{1.4}$$

In other words, given $\alpha \in BL(q)$, we have $\alpha \in M_A$ if and only if $X\alpha$ does not contain A, or $X\alpha$ contains A and either $A\alpha \subseteq A$ or $|X\alpha \setminus A| < q$. In [6], Levi and Wood showed that M_A is a maximal subsemigroup of $BL(q)$. Later, Hotzel [7] showed that there are many other maximal subsemigroups of $BL(q)$.

In this paper, we study maximal subsemigroups of $PS(q)$. In particular, in Section 3 we describe all maximal subsemigroups of $PS(q)$ when $p > q$. We also determine some maximal subsemigroups of a subsemigroup S_r of $PS(q)$ defined by

$$S_r = \{\alpha \in PS(q) : g(\alpha) \leq r\}, \tag{1.5}$$

where $q \leq r \leq p$. Moreover, we extend M_A to determine maximal subsemigroups of $PS(q)$. In Section 4, we determine some maximal subsemigroups of $PS(q)$ when $p = q$.

2. Preliminaries

In this paper, $Y = A \dot\cup B$ means Y is a *disjoint* union of sets A and B. As usual, $\emptyset$ denotes the empty (one-to-one) mapping which acts as a zero for $P(X)$. For each nonempty $A \subseteq X$, we write id_A for the identity transformation on A: these mappings constitute all the idempotents in $I(X)$ and belong to $PS(q)$ precisely when $|X \setminus A| = q$.

We modify the convention introduced in [1, 2, vol 2, p 241]: namely, if $\alpha \in I(X)$ is non-zero, then we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix} \tag{2.1}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $X\alpha = \operatorname{ran}\alpha = \{x_i\}$, $a_i\alpha = x_i$ for each i and $\operatorname{dom}\alpha = \{a_i\}$. To simplify notation, if $A \subseteq X$, we sometimes write $A\alpha$ in place of $(A \cap \operatorname{dom}\alpha)\alpha$.

Let S be a semigroup and $\emptyset \neq A \subseteq S$. Then $\langle A \rangle$ denotes the subsemigroup of S generated by A. Recall that a proper subsemigroup M of S is *maximal* in S if, whenever $M \subseteq N \subsetneq S$ and N is a subsemigroup of S, then $M = N$. Note that this is equivalent to each one of the following:

(a) $\langle M \cup \{a\} \rangle = S$ for all $a \in S \setminus M$;

(b) for any $a, b \in S \setminus M$, a can be written as a finite product of elements of $M \cup \{b\}$ (note that a is not expressible as a product of elements of M since $a \notin M$).

Throughout this paper, we will use this fact to show the maximality of subsemigroups of $PS(q)$.

3. Maximal Subsemigroups of $PS(q)$ When $p > q$

The characterisation of maximal subsemigroups of a given semigroup is a natural topic to consider when studying its structure. Sometimes, it is difficult to describe all of them (see [6, 7], e.g.), but for a semigroup with some special properties, we can easily describe some of its maximal subsemigroups.

Lemma 3.1. *Let S be a semigroup and suppose that S is a disjoint union of a subsemigroup T and an ideal I of S. Then,*

(a) *for any maximal subsemigroup M of T, $M \cup I$ is a maximal subsemigroup of S;*

(b) *for any maximal subsemigroup N of S such that $T \setminus N \neq \emptyset$ and $T \cap N \neq \emptyset$, the set $T \cap N$ is a maximal subsemigroup of T.*

Proof. To see that (a) holds, let M be a maximal subsemigroup of T. Since I is an ideal, we have $M \cup I$ is a subsemigroup of S. Clearly, $M \cup I \subsetneq T \cup I = S$. If $a \in S \setminus (M \cup I)$, then $a \in T \setminus M$ and thus $T = \langle M \cup \{a\} \rangle \subseteq \langle M \cup I \cup \{a\} \rangle$. Since $\langle M \cup I \cup \{a\} \rangle$ contains I, we have $S = T \cup I = \langle M \cup I \cup \{a\} \rangle$ and so $M \cup I$ is maximal in S as required.

To prove (b), let N be a maximal subsemigroup of S, where $T \setminus N \neq \emptyset$ and $T \cap N \neq \emptyset$, and let $a \in T \setminus N$. Since N is maximal in S, we have $\langle N \cup \{a\} \rangle = S$. Thus, for each $b \in T \setminus N$, $b = c_1 c_2 \cdots c_n$ for some natural n and some $c_i \in N \cup \{a\}$ for all $i = 1, 2, \ldots, n$. Since $b \notin N$, we have $c_i = a$ for some i. Moreover, since $b \notin I$, we have $c_j \in T \cap N$ for all $j \neq i$. It follows that $T \setminus N \subseteq \langle (T \cap N) \cup \{a\} \rangle$, therefore

$$T = (T \setminus N) \cup (T \cap N) \subseteq \langle (T \cap N) \cup \{a\} \rangle, \tag{3.1}$$

that is, $T = \langle (T \cap N) \cup \{a\} \rangle$ and thus $T \cap N$ is maximal in T. □

Let u be a cardinal number. The *successor* of u, denoted by u', is defined as

$$u' = \min\{v : v > u\}. \tag{3.2}$$

Note that u' always exists since the cardinals are wellordered, and when u is finite we have $u' = u + 1$.

From [3, p 95], for $\aleph_0 \leq k \leq p$,

$$S_k = \{\alpha \in PS(q) : g(\alpha) \leq k\} \tag{3.3}$$

is a subsemigroup of $PS(q)$. Also, when $p > q$, the proper ideals of $PS(q)$ are precisely the sets:

$$T_s = \{\alpha \in PS(q) : g(\alpha) \geq s\}, \tag{3.4}$$

where $q < s \leq p$ (see [3, Theorem 13]). Thus, for any $q \leq r < p$, it is clear that

$$PS(q) = S_r \,\dot{\cup}\, T_{r'}, \tag{3.5}$$

that is, $PS(q)$ can be written as a disjoint union of the semigroup S_r and the ideal $T_{r'}$. Hence, the next result follows directly from Lemma 3.1(a).

Corollary 3.2. *Suppose that $p > r > q \geq \aleph_0$. If M is a maximal subsemigroup of S_r, then $M \cup T_{r'}$ is a maximal subsemigroup of $PS(q)$.*

Lemma 3.3. *Let $p > q \geq \aleph_0$ and suppose that M is a maximal subsemigroup of $PS(q)$. Then,*

(a) *$S_r \cap M \neq \emptyset$ for all $q \leq r < p$;*

(b) *if there exists $\alpha \notin M$ with $g(\alpha) < p$, then $S_k \setminus M \neq \emptyset$ for some $q \leq k < p$.*

Proof. To show that (a) holds, we first note that S_q is contained in S_r for all $q \leq r < p$. If $S_q \cap M = \emptyset$, then $M \subseteq T_{q'} \subsetneq PS(q)$ and thus $M = T_{q'}$ by the maximality of M. But $T_{q'} \subsetneq T_{q'} \cup BL(q) \subsetneq PS(q)$ where $T_{q'} \cup BL(q)$ is a subsemigroup of $PS(q)$ (since $T_{q'}$ is an ideal), so we get a contradiction. Therefore, $\emptyset \neq S_q \cap M \subseteq S_r \cap M$ for all $q \leq r < p$.

To show that (b) holds, suppose there is $\alpha \notin M$ with $g(\alpha) = k < p$. If $k < q$, then $\alpha \in S_r \setminus M$ for all $q \leq r \leq p$. Otherwise, if $q \leq k$, then $\alpha \in S_k \setminus M$. Hence (b) holds. □

For what follows, for any cardinal $r \leq p$, we let

$$G_r = \{\alpha \in PS(q) : g(\alpha) = r\}. \tag{3.6}$$

Then $G_0 = BL(q)$ and $G_q = R(q)$. Moreover, if $p > q$ and $r > q$, then $G_r = S_r \cap T_r$, and so G_r is a subsemigroup of S_r (since it is the intersection of two semigroups). Also, G_r is bisimple and idempotent-free, when $p > q$ and $r > q$ (see [3, Corollary 3]).

From [3, Theorem 5], if $p \geq q$, then $S_q = \alpha \cdot R(q)$ for each $\alpha \in BL(q)$, and by [3, Theorem 6], $S_q = BL(q) \cdot \mu \cdot BL(q)$ for each $\mu \in R(q)$ when $p \neq q$.

This motivates the following result.

Lemma 3.4. *Suppose that $p \geq r > q \geq \aleph_0$. Then $G_r = BL(q) \cdot \alpha \cdot BL(q)$ for each $\alpha \in G_r$.*

Proof. Let $\alpha \in G_r$ and $\beta, \gamma \in BL(q)$. Since

$$X \setminus \operatorname{dom} \alpha = [X\beta \cap (X \setminus \operatorname{dom} \alpha)] \cup [(X \setminus X\beta) \cap (X \setminus \operatorname{dom} \alpha)], \tag{3.7}$$

where $g(\alpha) = |X \setminus \operatorname{dom}\alpha| = r > q$ and the second intersection on the right has cardinal at most q (since $|X \setminus X\beta| = q$), we have $|X\beta \cap (X \setminus \operatorname{dom}\alpha)| = r$. This means that

$$\begin{aligned} r &= \left|\left[X\beta \cap (X \setminus \operatorname{dom}\alpha)\right]\beta^{-1}\right| = \left|(X\beta \setminus \operatorname{dom}\alpha)\beta^{-1}\right| = \left|\operatorname{dom}\beta \setminus \operatorname{dom}(\beta\alpha)\right| \\ &= \left|X \setminus \operatorname{dom}(\beta\alpha)\right| = g(\beta\alpha). \end{aligned} \tag{3.8}$$

Since $\operatorname{dom}\gamma = X$, we have $\operatorname{dom}(\beta\alpha\gamma) = \operatorname{dom}(\beta\alpha)$, and so $g(\beta\alpha\gamma) = g(\beta\alpha) = r$. Hence $\beta\alpha\gamma \in G_r$ and therefore $BL(q) \cdot \alpha \cdot BL(q) \subseteq G_r$.

For the converse, if $\alpha, \beta \in G_r$, then $|X \setminus \operatorname{dom}\alpha| = r = |X \setminus \operatorname{dom}\beta|$. Since $p > q$, every element in $PS(q)$ has rank p, so we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} b_i \\ y_i \end{pmatrix} \quad \text{where } |I| = p. \tag{3.9}$$

Now write $X \setminus \{y_i\} = A \,\dot\cup\, B$ and $X \setminus \{a_i\} = C \,\dot\cup\, D$ where $|A| = |B| = |C| = q$ and $|D| = r$ (note that this is possible since $d(\beta) = q \geq \aleph_0$ and $g(\alpha) = r > q \geq \aleph_0$). Define

$$\delta = \begin{pmatrix} b_i & X \setminus \{b_i\} \\ a_i & D \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} x_i & X \setminus \{x_i\} \\ y_i & A \end{pmatrix}, \tag{3.10}$$

where $\delta|(X \setminus \{b_i\})$ and $\epsilon|(X \setminus \{x_i\})$ are bijections. Then $\delta, \epsilon \in BL(q)$ and $\beta = \delta\alpha\epsilon$, that is, $G_r \subseteq BL(q) \cdot \alpha \cdot BL(q)$ and equality follows. □

Now we can describe all maximal subsemigroups of $PS(q)$ when $p > q$.

Theorem 3.5. *Suppose that $p > q \geq \aleph_0$. Then M is a maximal subsemigroup of $PS(q)$ if and only if M equals one of the following sets:*

(a) $PS(q) \setminus G_p = \{\alpha \in PS(q) : g(\alpha) < p\}$;

(b) $N \cup T_{r'}$, *where $q \leq r < p$ and N is a maximal subsemigroup of S_r.*

Proof. Let $\alpha, \beta \in PS(q)$ be such that $g(\alpha) < p$ and $g(\beta) < p$. Clearly $|X\alpha \setminus \operatorname{dom}\beta| \leq |X \setminus \operatorname{dom}\beta| = g(\beta) < p$. Then

$$\begin{aligned} \left|\operatorname{dom}\alpha \setminus \operatorname{dom}(\alpha\beta)\right| &= \left|\left[X\alpha \setminus (X\alpha \cap \operatorname{dom}\beta)\right]\alpha^{-1}\right| \\ &= \left|(X\alpha \setminus \operatorname{dom}\beta)\alpha^{-1}\right| \\ &= \left|X\alpha \setminus \operatorname{dom}\beta\right| < p. \end{aligned} \tag{3.11}$$

Hence,

$$\left|X \setminus \operatorname{dom}(\alpha\beta)\right| = |X \setminus \operatorname{dom}\alpha| + \left|\operatorname{dom}\alpha \setminus \operatorname{dom}(\alpha\beta)\right| < p, \tag{3.12}$$

and this shows that $PS(q)\backslash G_p$ is a subsemigroup of $PS(q)$. To show that $PS(q)\backslash G_p$ is maximal in $PS(q)$, we let $\alpha, \beta \in PS(q) \setminus (PS(q) \setminus G_p) = G_p$. By Lemma 3.4, $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in BL(q) \subseteq PS(q)\backslash G_p$. Thus, α can be written as a finite product of elements of $(PS(q)\backslash G_p) \cup \{\beta\}$, and hence $PS(q)\backslash G_p$ is maximal in $PS(q)$. Also, if $q \le r < p$ and N is a maximal subsemigroup of S_r, then $N \cup T_{r'}$ is maximal in $PS(q)$ by Corollary 3.2.

We now suppose that M is a maximal subsemigroup of $PS(q)$ such that $M \neq PS(q) \setminus G_p$. Then there exists $\alpha \notin M$ with $g(\alpha) < p$. Thus, Lemma 3.3 implies that $S_k \setminus M \neq \emptyset$ and $S_k \cap M \neq \emptyset$ for some k, where $q \le k < p$. Since $PS(q) = S_k \dot\cup T_{k'}$, Lemma 3.1(b) implies that $S_k \cap M$ is maximal in S_k. We also see that

$$M = (S_k \cap M) \cup (T_{k'} \cap M) \subseteq (S_k \cap M) \cup T_{k'}, \tag{3.13}$$

where $(S_k \cap M) \cup T_{k'}$ is maximal in $PS(q)$ by Corollary 3.2. This means that $M = (S_k \cap M) \cup T_{k'}$ by the maximality of M. □

By the previous theorem, when $p > q$, most of the maximal subsemigroups of $PS(q)$ are induced by maximal subsemigroups of S_r where $q \le r < p$. Hence we now determine some maximal subsemigroups of S_r.

As mentioned in Section 1, for every nonempty subset A of X with $|X \setminus A| \ge q$, M_A is a maximal subsemigroup of $BL(q)$. Here we extend the definition of M_A and consider the set $\overline{M}_A$ defined as

$$\overline{M}_A = \{\alpha \in PS(q) : A \nsubseteq X\alpha \text{ or } (A\alpha \subseteq A \subseteq \operatorname{dom}\alpha \text{ or } |X\alpha \setminus A| < q)\}, \tag{3.14}$$

that is, α in $PS(q)$ belongs to $\overline{M}_A$ if and only if

(a) $A \nsubseteq X\alpha$, or

(b) $A \subseteq X\alpha$ and either $A\alpha \subseteq A \subseteq \operatorname{dom}\alpha$, or $|X\alpha \setminus A| < q$.

The next result gives more detail on $\overline{M}_A$.

Lemma 3.6. *Suppose that $p \ge q \ge \aleph_0$, and let A be a nonempty subset of X such that $|X \setminus A| \ge q$. Then,*

(a) *for any cardinal k such that $0 \le k \le p$, there exist $\alpha, \beta \in PS(q)$ such that $g(\alpha) = k = g(\beta)$ and $\alpha \in \overline{M}_A$, $\beta \notin \overline{M}_A$;*

(b) *for each $\gamma \notin \overline{M}_A$, $|\operatorname{dom}\gamma \setminus A\gamma^{-1}| = |X \setminus A| = |X\gamma \setminus A|$ and $|A\gamma^{-1}| = |A|$.*

Proof. To show that (a) holds, let $|X \setminus A| = r \ge q$, and let k be a cardinal such that $0 \le k \le p$. We write $X \setminus A = R \dot\cup Q$ where $|R| = r$ and $|Q| = q$. If $r = p$, then $|A \cup R| \ge r = p$; if not, then $|X \setminus A| < p$, and this implies $|A| = p$, and so $|A \cup R| = p$. Fix $a \in A$ and let $B = (A \setminus \{a\}) \cup R$. Then, $|B| = p$ and $|X \setminus B| = |Q \cup \{a\}| = q$. We write $X = K \dot\cup L$ where $|K| = k$ and $|L| = p$. Then there exists a bijection $\alpha : L \to B$ and so $g(\alpha) = k$, $d(\alpha) = q$. Also, since $A \nsubseteq B = X\alpha$, we have $\alpha \in \overline{M}_A$.

To find $\beta \in PS(q) \setminus \overline{M}_A$ with $g(\beta) = k$, we consider two cases. First, if $r = p$, we write $X \setminus A = P \,\dot{\cup}\, Q \,\dot{\cup}\, K$ where $|P| = p,\ |Q| = q,\ |K| = k$. Fix $a \in A$ and define

$$\beta = \begin{pmatrix} P \cup Q \cup \{a\} & A \setminus \{a\} \\ P \cup K \cup \{a\} & A \setminus \{a\} \end{pmatrix}, \tag{3.15}$$

where $\beta|(P \cup Q \cup \{a\})$ and $\beta|(A \setminus \{a\})$ are bijections and $a\beta \neq a$. On the other hand, if $r < p$, then $|A| = p$. In this case we write $A = A' \,\dot{\cup}\, K'$ and $X \setminus A = R \,\dot{\cup}\, Q$ where $|A'| = p,\ |K'| = k,\ |R| = r$ and $|Q| = q$. Fix $a \in A'$ and redefine

$$\beta = \begin{pmatrix} (X \setminus A) \cup \{a\} & A' \setminus \{a\} \\ R \cup \{a\} & A \setminus \{a\} \end{pmatrix}, \tag{3.16}$$

where $\beta|((X \setminus A) \cup \{a\})$ and $\beta|(A' \setminus \{a\})$ are bijections and $a\beta \neq a$. In both cases, we have $d(\beta) = q$, $g(\beta) = k$, $A \subseteq X\beta$, $A\beta \nsubseteq A$, and $|X\beta \setminus A| \geq q$, that is $\beta \in PS(q) \setminus \overline{M}_A$.

To see that (b) holds, suppose that there is $\gamma \notin \overline{M}_A$, then $A \subseteq X\gamma$ and $|X\gamma \setminus A| \geq q$. So $|A\gamma^{-1}| = |A|$ since γ is injective. Also,

$$X \setminus A = (X \setminus X\gamma) \,\dot{\cup}\, (X\gamma \setminus A), \tag{3.17}$$

where $|X \setminus X\gamma| = q$. Since $|X \setminus A| \geq q$ and by our assumption $|X\gamma \setminus A| \geq q$, we have $|X \setminus A| = |X\gamma \setminus A| = |(X\gamma \setminus A)\gamma^{-1}| = |\operatorname{dom}\gamma \setminus A\gamma^{-1}|$ as required. □

In [6, Theorem 1], the authors proved that M_A is a maximal subsemigroup of $BL(q)$ for every nonempty subset A of X such that $|X \setminus A| \geq q$. Using a similar argument, we show that $\overline{M}_A$ is a subsemigroup of $PS(q)$.

Lemma 3.7. *Suppose that $p \geq q \geq \aleph_0$, and let A be a nonempty subset of X such that $|X \setminus A| \geq q$. Then $\overline{M}_A$ is a proper subsemigroup of $PS(q)$.*

Proof. Let $\alpha, \beta \in \overline{M}_A$. If $A \nsubseteq X\alpha\beta$, then $\alpha\beta \in \overline{M}_A$. Now we suppose that $A \subseteq X\alpha\beta$. Then, $A \subseteq X\beta$ and since $\beta \in \overline{M}_A$, we either have $A\beta \subseteq A \subseteq \operatorname{dom}\beta$, or $|X\beta \setminus A| < q$. If $|X\beta \setminus A| < q$, then

$$\left|X\alpha\beta \setminus A\right| \leq \left|X\beta \setminus A\right| < q \tag{3.18}$$

and so $\alpha\beta \in \overline{M}_A$. Otherwise, we have $A\beta \subseteq A \subseteq X\alpha\beta$ and hence $A \subseteq X\alpha$ since β is injective. Since $\alpha \in \overline{M}_A$, we either have $A\alpha \subseteq A \subseteq \operatorname{dom}\alpha$, or $|X\alpha \setminus A| < q$. If the latter occurs, then

$$\left|X\alpha\beta \setminus A\right| \leq \left|X\alpha\beta \setminus A\beta\right| = \left|(X\alpha \setminus A)\beta\right| \leq |X\alpha \setminus A| < q, \tag{3.19}$$

therefore $\alpha\beta \in \overline{M}_A$. On the other hand, if $A\alpha \subseteq A \subseteq \operatorname{dom}\alpha$, we have $A\alpha\beta \subseteq A\beta \subseteq A$. Moreover, $A\alpha \subseteq X\alpha \cap \operatorname{dom}\beta$, that is, $A \subseteq (X\alpha \cap \operatorname{dom}\beta)\alpha^{-1} = \operatorname{dom}(\alpha\beta)$. Therefore $\alpha\beta \in \overline{M}_A$, and hence

$\overline{M}_A$ is a subsemigroup of $PS(q)$. Finally, this subsemigroup is properly contained in $PS(q)$ by Lemma 3.6(a). □

Remark 3.8. For any cardinal r such that $q \le r \le p$, $S_r \cap \overline{M}_A$ is a proper subsemigroup of S_r but it is not maximal when $q < r$. To see this, suppose $S_r \cap \overline{M}_A$ is maximal and choose $\alpha, \beta \notin \overline{M}_A$ such that $g(\alpha) = r$ and $g(\beta) = 0$ (possible by Lemma 3.6(a)). Then $\alpha,\ \beta \in S_r \setminus \overline{M}_A$ where $\operatorname{dom}\beta = X$. Moreover $\langle (S_r \cap \overline{M}_A) \cup \{\alpha\} \rangle = S_r$, and so

$$\beta = \gamma_1\gamma_2\cdots\gamma_n\alpha\lambda_1\lambda_2\cdots\lambda_m \tag{3.20}$$

for some $n, m \in \mathbb{N}_0$ and $\gamma_i, \lambda_j \in (S_r \cap \overline{M}_A) \cup \{\alpha\}$, $i = 1,\dots,n$, $j = 1,\dots,m$. If $n = 0$ or $\gamma_1 = \alpha$, then $\operatorname{dom}\beta \subseteq \operatorname{dom}\alpha$ and so $g(\alpha) = 0$, a contradiction. Thus, $n \ne 0$ and $\gamma_1 \ne \alpha$. Since $X = \operatorname{dom}\beta \subseteq \operatorname{dom}(\gamma_1\gamma_2\cdots\gamma_n)$, it follows that $\gamma = \gamma_1\gamma_2\cdots\gamma_n \in BL(q)$. Moreover, $X\gamma \subseteq \operatorname{dom}\alpha$, and this implies,

$$q \le r = |X \setminus \operatorname{dom}\alpha| \le \left|X \setminus X\gamma\right| = q, \tag{3.21}$$

and hence $r = q$.

Since M_A is maximal in $BL(q)$, a subsemigroup of $PS(q)$, it is natural to think that $\overline{M}_A$ is maximal in $PS(q)$. But when $p > q$, by taking $r = p$, the above observation shows that this claim is false since $S_p = PS(q)$. Thus, $\overline{M}_A$ is not always a maximal subsemigroup of $PS(q)$.

The proof of the next result follows some ideas from [6, Theorem 1].

Theorem 3.9. *Suppose that $p \ge r \ge q \ge \aleph_0$, and let A be a nonempty subset of X such that $|X \setminus A| \ge q$. Then $S_r \cap \overline{M}_A$ is a maximal subsemigroup of S_r precisely when $r = q$.*

Proof. In Remark 3.8, we have shown that $S_r \cap \overline{M}_A$ is not maximal in S_r when $r > q$. It remains to show $S_q \cap \overline{M}_A$ is maximal in S_q. Let $\alpha, \beta \in S_q \setminus \overline{M}_A$. Then $g(\alpha), g(\beta) \le q$ and Lemma 3.6(b) implies that

$$\begin{gathered} \left|A\alpha^{-1}\right| = |A| = \left|A\beta^{-1}\right|, \\ \left|\operatorname{dom}\alpha \setminus A\alpha^{-1}\right| = \left|\operatorname{dom}\beta \setminus A\beta^{-1}\right| = \left|X\beta \setminus A\right| = |X\alpha \setminus A| = |X \setminus A| = s \quad (\text{say}) \ge q. \end{gathered} \tag{3.22}$$

We also have $A\beta \not\subseteq A$ or $A \not\subseteq \operatorname{dom}\beta$. In the case that $A\beta \not\subseteq A$, we have $A\beta \cap (X \setminus A) \ne \emptyset$. Thus, there exists $y \in A \cap (X \setminus A)\beta^{-1}$, so $y \notin A\beta^{-1}$. Since $|\operatorname{dom}\beta \setminus (A\beta^{-1} \cup \{y\})| = s$, we can write

$$\operatorname{dom}\beta \setminus \left(A\beta^{-1} \cup \{y\}\right) = \{c_j\} \mathbin{\dot\cup} \{d_k\}, \tag{3.23}$$

where $|J| = s$ and $|K| = q$. Also, since $\alpha, \beta \notin \overline{M}_A$, we have $A \subseteq X\alpha$ and $A \subseteq X\beta$. Thus, for convenience, write $A = \{a_i\}$, let $y_i, z_i \in X$ be such that $y_i\alpha = a_i = z_i\beta$ for each i, and let $\operatorname{dom}\alpha \setminus A\alpha^{-1} = \{b_j\}$. Hence, we can write

$$\beta = \begin{pmatrix} z_i & c_j & d_k & y \\ a_i & c_j\beta & d_k\beta & y\beta \end{pmatrix}. \tag{3.24}$$

Now define $\gamma \in P(X)$ by

$$\gamma = \begin{pmatrix} y_i & b_j \\ z_i & c_j \end{pmatrix}. \tag{3.25}$$

Then, $d(\gamma) = |\{d_k\} \cup \{y\}| + g(\beta) = q$, that is, $\gamma \in PS(q)$. Also, since $\operatorname{dom}\gamma = \operatorname{dom}\alpha$, we have $g(\gamma) = g(\alpha) \le q$ and so $\gamma \in S_q$. Moreover, since $y \in A$ and $y \notin X\gamma$, we have $A \nsubseteq X\gamma$, that is, $\gamma \in \overline{M}_A$. Also, since $d(\alpha) = q$, we can write $X \setminus X\alpha = \{m_k\} \,\dot\cup\, \{n_k\} \,\dot\cup\, \{z\}$ and define μ in $P(X)$ by

$$\mu = \begin{pmatrix} a_i & c_j\beta & d_k\beta & y\beta \\ a_i & b_j\alpha & m_k & z \end{pmatrix}. \tag{3.26}$$

Then $d(\mu) = |\{n_k\}| = q = d(\beta) = g(\mu)$, that is, $\mu \in S_q$. Moreover, $\mu \in \overline{M}_A$ since $A\mu = A \subseteq \operatorname{dom}\mu$. Finally, we can see that $\alpha = \gamma\beta\mu$ where $\gamma, \mu \in S_q \cap \overline{M}_A$.

On the other hand, if $A \nsubseteq \operatorname{dom}\beta$, then there exists $w \in A \cap (X \setminus \operatorname{dom}\beta)$. In this case, we rewrite $\operatorname{dom}\beta \setminus A\beta^{-1} = \{c_j\} \,\dot\cup\, \{d_k\}$ and $X \setminus X\alpha = \{m_k\} \,\dot\cup\, \{n_k\}$ where $|J| = s$, $|K| = q$. Like before, we write $A = \{a_i\}$ and $\operatorname{dom}\alpha = \{y_i\} \,\dot\cup\, \{b_j\}$ where $\{b_j\} = \operatorname{dom}\alpha \setminus A\alpha^{-1}$, then

$$\beta = \begin{pmatrix} z_i & c_j & d_k \\ a_i & c_j\beta & d_k\beta \end{pmatrix}. \tag{3.27}$$

Define $\gamma, \mu \in P(X)$ by

$$\gamma = \begin{pmatrix} y_i & b_j \\ z_i & c_j \end{pmatrix}, \qquad \mu = \begin{pmatrix} a_i & c_j\beta & d_k\beta \\ a_i & b_j\alpha & m_k \end{pmatrix}. \tag{3.28}$$

Then, $d(\gamma) = |\{d_k\}| + g(\beta) = q$, $g(\gamma) = g(\alpha) \le q$, $d(\mu) = |\{n_k\}| = q = d(\beta) = g(\mu)$, and so $\gamma, \mu \in S_q$. Also, $\gamma, \mu \in \overline{M}_A$ since $A \nsubseteq X\gamma$ (note that $w \in A \setminus \operatorname{dom}\beta \subseteq A \setminus X\gamma$) and $A\mu = A \subseteq \operatorname{dom}\mu$. Moreover, $\alpha = \gamma\beta\mu$. In other words, we have shown that for every $\alpha, \beta \in S_q \setminus \overline{M}_A$, α can be written as a finite product of elements of $(S_q \cap \overline{M}_A) \cup \{\beta\}$. Therefore, $S_q \cap \overline{M}_A$ is maximal in S_q. □

We now determine some other classes of maximal subsemigroups of S_r.

Lemma 3.10. *Suppose that $p \geq r \geq q \geq \aleph_0$. Let k be a cardinal such that $k = 0$ or $q \leq k \leq r$. Then*

$$S_r \setminus G_k = \{\alpha \in PS(q) : k \neq g(\alpha) \leq r\} \tag{3.29}$$

is a proper subsemigroup of S_r.

Proof. Since $k \leq r$, we have $S_r \setminus G_k \subsetneq S_r$. If $k = 0$, then $S_r \setminus G_0 = S_r \setminus BL(q)$, and this is a subsemigroup of S_r since, for $\alpha, \beta \in S_r \setminus BL(q)$, $\operatorname{dom}(\alpha\beta) \subseteq \operatorname{dom}\alpha \subsetneq X$, and this implies $\alpha\beta \in S_r \setminus BL(q)$. Now suppose $q \leq k \leq r$ and let $\alpha, \beta \in S_r$ be such that $g(\alpha\beta) = k$. We claim that $g(\alpha) = k$ or $g(\beta) = k$. To see this, assume that $g(\alpha) \neq k$. Since

$$k = \left|X \setminus \operatorname{dom}(\alpha\beta)\right| = |X \setminus \operatorname{dom}\alpha| + \left|\operatorname{dom}\alpha \setminus \operatorname{dom}(\alpha\beta)\right|, \tag{3.30}$$

we have $|X \setminus \operatorname{dom}\alpha| < k$, thus

$$\begin{aligned} k &= \left|\operatorname{dom}\alpha \setminus \operatorname{dom}(\alpha\beta)\right| = \left|\left[X\alpha \setminus (X\alpha \cap \operatorname{dom}\beta)\right]\alpha^{-1}\right| \\ &= \left|(X\alpha \setminus \operatorname{dom}\beta)\alpha^{-1}\right| = \left|X\alpha \setminus \operatorname{dom}\beta\right|. \end{aligned} \tag{3.31}$$

Note that

$$X \setminus \operatorname{dom}\beta = \left[X\alpha \setminus \operatorname{dom}\beta\right] \,\dot\cup\, \left[(X \setminus X\alpha) \cap (X \setminus \operatorname{dom}\beta)\right], \tag{3.32}$$

where the intersection on the right has cardinal at most q. Hence, $g(\beta) = |X \setminus \operatorname{dom}\beta| = k$ and we have shown that $S_r \setminus G_k$ is a subsemigroup of S_r. □

Remark 3.11. Observe that, if $0 < k < q$ then $S_r \setminus G_k$ is not a semigroup for all $q \leq r \leq p$. To see this, let $\alpha \in BL(q)$ and $\beta = \operatorname{id}_{X\alpha \setminus K}$ for some subset K of $X\alpha$ such that $|K| = k$ (possible since $|X\alpha| = p > k$), then $\alpha, \beta \in PS(q)$ since $d(\beta) = d(\alpha) + k = q$. Moreover, since $g(\alpha) = 0$ and $g(\beta) = q \neq k$, we have $\alpha, \beta \in S_r \setminus G_k$. But

$$\operatorname{dom}(\alpha\beta) = (X\alpha \cap \operatorname{dom}\beta)\alpha^{-1} = (X\alpha \setminus K)\alpha^{-1} = X \setminus K\alpha^{-1}, \tag{3.33}$$

thus $g(\alpha\beta) = |K\alpha^{-1}| = k$, that is, $\alpha\beta \in G_k$.

Theorem 3.12. *Suppose that $p \geq r \geq q \geq \aleph_0$. Then the following statements hold:*

(a) *$S_r \setminus G_0$ is a maximal subsemigroup of S_r;*

(b) *if $p > q$, then for each cardinal k such that $q \leq k \leq r$, $S_r \setminus G_k$ is a maximal subsemigroup of S_r.*

Proof. By Lemma 3.10, $S_r \setminus G_0$ is a subsemigroup of S_r. To see that it is maximal, let $\alpha, \beta \in G_0 = BL(q) \subseteq S_q$. By [3, Theorem 5], $S_q = \beta \cdot R(q)$, and this implies that $\alpha = \beta\gamma$ for some $\gamma \in R(q) \subseteq S_r \setminus G_0$. Hence $S_r \setminus G_0$ is maximal in S_r.

Now suppose that $p > q$ and let $q \leq k \leq r$. Let $\alpha, \beta \in G_k$. If $k = q$, then $G_k = R(q) \subseteq S_q$ and, by [3, Theorem 6], $S_q = BL(q) \cdot \beta \cdot BL(q)$. If $k > q$, then $G_k = BL(q) \cdot \beta \cdot BL(q)$ (by

Lemma 3.4). Therefore, $\alpha = \gamma\beta\mu$ for some $\gamma, \mu \in BL(q) \subseteq S_r \setminus G_k$, and so $S_r \setminus G_k$ is maximal in S_r. □

Corollary 3.13. *Suppose that $p > q \geq \aleph_0$ and let A be a nonempty subset of X such that $|X \setminus A| \geq q$. Then the following sets are maximal subsemigroups of $PS(q)$:*

(a) $\overline{M}_A \cup T_{q'}$;

(b) $N_k = \{\alpha \in PS(q) : g(\alpha) \neq k\}$ *where* $k = 0$ *or* $q \leq k \leq p$.

Proof. By Theorem 3.9, $S_q \cap \overline{M}_A$ is maximal in S_q. Then Corollary 3.2 implies that $(S_q \cap \overline{M}_A) \cup T_{q'}$ is maximal in $PS(q)$. But

$$\left(S_q \cap \overline{M}_A\right) \cup T_{q'} = (S_q \cup T_{q'}) \cap \left(\overline{M}_A \cup T_{q'}\right) = PS(q) \cap \left(\overline{M}_A \cup T_{q'}\right) = \overline{M}_A \cup T_{q'}, \tag{3.34}$$

and so (a) holds. To show that (b) holds, let $r = p$ in Theorem 3.12. Then $S_p = PS(q)$ and thus $N_k = S_p \setminus G_k$ is maximal in $PS(q)$. □

Theorem 3.14. *Suppose that $p > q \geq \aleph_0$ and k equals 0 or q. Let A be a nonempty subset of X such that $|X \setminus A| \geq q$. Then the two classes of maximal subsemigroups $S_q \cap \overline{M}_A$ and $S_q \setminus G_k$ of S_q are always disjoint.*

Proof. By Theorems 3.9 and 3.12, $S_q \cap \overline{M}_A$ and $S_q \setminus G_k$ are maximal subsemigroups of S_q. By Lemma 3.6(a), there exists $\alpha \in \overline{M}_A$ with $g(\alpha) = k$. Then $\alpha \in S_k \cap \overline{M}_A \subseteq S_q \cap \overline{M}_A$ but $\alpha \notin S_q \setminus G_k$, that is, $S_q \cap \overline{M}_A \nsubseteq S_q \setminus G_k$. Also, $S_q \setminus G_k \nsubseteq S_q \cap \overline{M}_A$ by the maximality of $S_q \cap \overline{M}_A$ and $S_q \setminus G_k$. Therefore, $S_q \cap \overline{M}_A$ is not equal to $S_q \setminus G_k$. □

4. Maximal Subsemigroups of $PS(q)$ When $p = q$

We first recall that, when $p = q$, the empty transformation $\emptyset$ belongs to $PS(q)$ since $d(\emptyset) = p = q$. In this case, the ideals of $PS(q)$ are precisely the sets:

$$J_r = \{\alpha \in PS(q) : r(\alpha) < r\}, \tag{4.1}$$

where $1 \leq r \leq p'$ (see [3, Theorem 14]). Clearly, $J_{p'} = PS(q)$ and $J_p = \{\alpha \in PS(q) : r(\alpha) < p\}$ is the largest proper ideal. In this case, the complement of each J_r in $PS(q)$ is not a semigroup. To see this, write $X = A \dot\cup B \dot\cup C$ where $|A| = p$ and $|B| = r = |C|$. Then $\mathrm{id}_B,\ \mathrm{id}_C \in PS(q) \setminus J_r$ whereas $\mathrm{id}_B.\ \mathrm{id}_C = \emptyset \in J_r$. Hence, unlike what was done in Section 3, we cannot use Lemma 3.1 to find maximal subsemigroups of $PS(q)$ when $p = q$. In this section, we determine some maximal subsemigroups of $PS(q)$, for $p = q$, using a different approach. We first describe some properties of each maximal subsemigroup in this case.

Lemma 4.1. *Suppose that $p = q \geq \aleph_0$ and M is a maximal subsemigroup of $PS(q)$. Then the following statements hold:*

(a) *M contains all $\alpha \in PS(q)$ with $r(\alpha) < p$,*

(b) *if $R(q) \subseteq M$, then $M \cap BL(q) = \emptyset$.*

Proof. Suppose that there exists $\alpha \notin M$ with $r(\alpha) = k < p$. Then $g(\alpha) = p$, and we write in the usual way

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}. \tag{4.2}$$

Also, write $X \setminus \{a_i\} = P \dot{\cup} Q$ and $X \setminus \{x_i\} = R \dot{\cup} S$ where $|P| = |Q| = p = |R| = |S|$, and define $\beta,\ \gamma$ in $P(X)$ by

$$\beta = \begin{pmatrix} a_i & P \\ a_i & P \end{pmatrix}, \qquad \gamma = \begin{pmatrix} a_i & Q \\ x_i & R \end{pmatrix}, \tag{4.3}$$

where $\beta|P$ and $\gamma|Q$ are bijections. Then $\beta, \gamma \in PS(q)$. Also,

$$\alpha = \beta \cdot \alpha \cdot \mathrm{id}_{X\alpha} \in PS(q) \cdot \alpha \cdot PS(q), \tag{4.4}$$

thus $M \subsetneq M \cup (PS(q) \cdot \alpha \cdot PS(q))$. But $M \cup (PS(q) \cdot \alpha \cdot PS(q))$ is a subsemigroup of $PS(q)$ and this means that $M \cup (PS(q) \cdot \alpha \cdot PS(q)) = PS(q)$ by the maximality of M. Since all mappings in $PS(q) \cdot \alpha \cdot PS(q)$ have rank at most k, it follows that M contains all mappings with rank greater than k. Therefore $\beta, \gamma \in M$ and thus $\alpha = \beta\gamma \in M$, a contradiction.

To show that (b) holds, suppose that $R(q) \subseteq M$. If there exists $\alpha \in M \cap BL(q)$, then [3, Theorem 5] implies that $PS(q) = \alpha \cdot R(q) \subseteq M$ (note that $S_q = PS(q)$ when $p = q$), so $M = PS(q)$, contrary to the maximality of M. Thus $M \cap BL(q) = \emptyset$. □

Remark 4.2. If $p > q$, then every $\alpha \in PS(q)$ has rank p. This contrasts with Lemma 4.1(a). Also, by Corollary 3.13, if $p > q$ and $q < k \leq p$, N_k is a maximal subsemigroup of $PS(q)$ containing $R(q) \cup BL(q)$, this contrasts with Lemma 4.1(b).

As in Section 3, for any cardinal k, we let

$$N_k = \{\alpha \in PS(q) : g(\alpha) \neq k\}. \tag{4.5}$$

By Lemma 3.10 and Remark 3.11, if $p = q$, then N_k is a subsemigroup of $PS(q)$ exactly when $k = 0$ or $k = p$. From Corollary 3.13(b), when $p > q$, N_p is a maximal subsemigroup of $PS(q)$. But when $p = q$, Lemma 4.1(a) implies that N_p is not maximal since $\emptyset \notin N_p$. Moreover, Lemma 4.1(a) implies that every maximal subsemigroup of $PS(q)$ must contain the largest proper ideal

$$J_p = \{\alpha \in PS(q) : r(\alpha) < p\}. \tag{4.6}$$

Note that J_p itself is a subsemigroup of $PS(q)$, but it is not maximal since $J_p \subsetneq R(q)$ (in case $p = q$, $r(\alpha) < p$ implies $g(\alpha) = p$).

Theorem 4.3. *Suppose that $p = q \geq \aleph_0$, and let A be a nonempty subset of X such that $|X \setminus A| \geq q$. The following are maximal subsemigroups of $PS(q)$:*

(a) $\overline{M}_A$;

(b) N_0;

(c) $N_p \cup J_p$.

Proof. If $p = q$, then $S_q = PS(q)$, and so (a) holds by Theorem 3.9. Also, by taking $r = p$ in Theorem 3.12(a), we see that (b) holds. To show that (c) holds, take $r = p = k$ in Lemma 3.10, we have $N_p = S_p \setminus G_p$ is a subsemigroup of $PS(q)$. Moreover, $N_p \cup J_p$ is also a subsemigroup of $PS(q)$ since J_p is an ideal. To show the maximality of $N_p \cup J_p$, let $\alpha, \beta \in PS(q) \setminus (N_p \cup J_p)$. Then $g(\alpha) = g(\beta) = p = r(\alpha) = r(\beta)$. Write in the usual way

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \qquad \beta = \begin{pmatrix} b_i \\ y_i \end{pmatrix}, \tag{4.7}$$

where $|I| = p$, and let

$$X \setminus \{a_i\} = A \,\dot{\cup}\, B, \quad X \setminus \{y_i\} = C \,\dot{\cup}\, D, \tag{4.8}$$

where $|A| = |B| = |C| = |D| = p$. Then define $\gamma, \mu \in P(X)$ by

$$\gamma = \begin{pmatrix} b_i & X \setminus \{b_i\} \\ a_i & A \end{pmatrix}, \qquad \mu = \begin{pmatrix} x_i & X \setminus \{x_i\} \\ y_i & C \end{pmatrix}, \tag{4.9}$$

where $\gamma|(X \setminus \{b_i\})$ and $\mu|(X \setminus \{x_i\})$ are bijections. Thus $\gamma, \mu \in PS(q)$ since $d(\gamma) = |B| = p = |D| = d(\mu)$. Moreover $\gamma, \mu \in N_p \cup J_p$ since $g(\gamma) = g(\mu) = 0 < p$. It is clear that $\beta = \gamma\alpha\mu$ and therefore $N_p \cup J_p$ is maximal in $PS(q)$. □

Remark 4.4. When $p = q$, if M is a maximal subsemigroup containing $R(q)$, then

$$M \subseteq (PS(q) \setminus BL(q)) = N_0 \tag{4.10}$$

by Lemma 4.1(b). Thus, $M = N_0$ by the maximality of M. So we conclude that N_0 is the only maximal subsemigroup of $PS(q)$ containing $R(q)$.

Remark 4.5. As we showed in Section 3, to see all maximal subsemigroups of $PS(q)$ when $p > q$, it is necessary to describe all maximal subsemigroups of S_r where $q \leq r < p$. So we leave this as a direction for future research.

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