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Research Article

Two New Types of Rings Constructed from Quasiprime Ideals

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Keigher showed that quasi-prime ideals in differential commutative rings are analogues of prime ideals in commutative rings. In that direction, he introduced and studied new types of differential rings using quasi-prime ideals of a differential ring. In the same sprit, we define and study two new types of differential rings which lead to the mirrors of the corresponding results on von Neumann regular rings and principally flat rings (PF-rings) in commutative rings, especially, for rings of positive characteristic.

1. Introduction

The derivatives of rings play important roles in ring theory. In particular, they are used to define various ring constructions, for example, see Sections 3.4 to 3.7 of the monograph [1].

Rings considered in this paper are all commutative with unity. Recall that a ring R is regular if for every element $a \in R$ there exists an element $b \in R$ such that $a = a^2b$. Also a ring R is called a PF-ring if every principal ideal aR of R is an R-flat module. These two types of rings were investigated extensively in the literature, see von Neumann [2], Endo [3], Matlis [4], Goodearl [5], and Abu-Osba et al. [6]. In this paper, we generalize these concepts to ordinary differential rings. Some well-known properties of regular rings and PF-rings are given in the following theorems. Before that, recall that an ideal I in a ring R is called a pure ideal if, for each $a \in I$, there exists $b \in I$ such that $ab = a$. These ideals classify certain important types of rings, see, for example, Borceux and Van Den Bosch [7] and AL-Ezeh [8, 9].

Theorem 1.1 (Goodearl, [5]). *Let R be a ring. Then the following are equivalent.*

(1) *R is von Neumann regular.*

(2) *R is reduced and every prime ideal is a maximal one.*

(3) *Every maximal ideal of R is pure.*

(4) *Every element of R can be written as a product of a unit and an idempotent element.*

(5) *Every localization R_M at each maximal ideal M is a field.*

Theorem 1.2 (Goodearl, [5]). (1) *If R is a von Neumann regular ring and S is a multiplicative subset of R, then ring of fractions $S^{-1}R$ is a von Neumann regular ring.*

(2) *A direct product of von Neumann regular rings is von Neumann regular.*

(3) *If R is von Neumann regular ring and I is an ideal of R, then R/I is von Neumann regular.*

Theorem 1.3. *Let R be a ring. Then the following are equivalent.*

(1) *R is PF-ring.*

(2) *R is reduced and each prime ideal contains a unique minimal prime ideal, see Matlis [4].*

(3) *For each $a \in R$, $\operatorname{ann}_R(a) = \{x \in R : xa = 0\}$ is pure ideal in R, see Al-Ezeh [8].*

(4) *Every localization R_M at each prime ideal M is an integral domain.*

Theorem 1.4. *If R is PF-ring and I is pure ideal of R, then R/I is PF-ring.*

Recall that by a *derivation* of a ring R we mean any additive map $\delta : R \to R$ satisfying $\delta(ab) = \delta(a)b + a\delta(b)$ for every $a, b \in R$. A differential ring R is a ring with a derivation δ. A subset S of R is said to be *differential* if $\delta(S) \subseteq S$. For any subset S of R, the set $S_\Delta = \{x \in S : \delta(x) \in S\}$ is called the *differential* of S. Many properties of S_Δ were studied by Keigher in [10, 11]. Let R be a differential ring. Then a differential ideal I is called a *quasiprime* ideal if there is a multiplicative subset of R such that I is maximal among differential ideals of R disjoint from S. Clearly, a quasiprime ideal of a differential ring is a generalization of a prime ideal of a ring R. Note that I is quasiprime ideal of R if there is a prime ideal P of R such that $I = P_\Delta$ and $r(I) = P$, where $r(I)$ is the radical ideal of I in R, see Keigher [10]. Every prime differential ideal is quasiprime, while the converse need not be true. Also every maximal differential ideal is quasiprime but it need not be prime differential ideal. Quasiprime ideals were studied extensively by Keigher in [10–12]. Also, in Keigher [10, 11], the differential rings constructed from quasiprime ideals via quotient rings and rings of fractions were studied too. Recall that a differential ideal I of a *differential ring* R is a *quasimaximal* ideal if $r(I)$ is maximal ideal. So, I is called a quasimaximal ideal if there exists a maximal ideal M such that $I = M_\Delta$ and $r(I) = M$. It is clear that every maximal differential ideal is a quasimaximal but the converse need not be true. A *differential ring* R is called *quasireduced* ring if the differential of the nilradical, $\operatorname{nil}(R)$, equals zero (i.e., $\Delta - \operatorname{nil}(R) = O_R$).

Now, we define two new types of differential rings that can be constructed using quasiprime ideals.

Definition 1.5. A differential ring (R, δ) is said to be quasiregular if R is quasireduced and every quasiprime ideal is quasimaximal.

Definition 1.6. A differential ring (R, δ) is called a quasi-PF ring if R is quasireduced and every quasiprime ideal of it contains a unique minimal quasiprime ideal.

It is clear that the concept of quasiregular rings and quasi-PF rings are generalizations to the differential context of von Neumann regular rings and PF-rings in ordinary commutative rings. Our aim in this paper is to study the classes of quasiregular rings and quasi-PF rings and how their structures closely mirrors that of classes of von Neumann regular rings and PF-rings in commutative rings. Also, we investigate when the Hurwitz series ring is quasiregular or quasi-PF, in particular, for rings of positive characteristic.

2. Quasiregular Ring

In this section we study some basic properties of quasiregular rings. We will show that the structure of these classes of rings is very closely connected to the structure of the corresponding class in commutative rings, especially, for rings of positive characteristic. We start by stating an easy lemma that will be used frequently later on.

Lemma 2.1. *Let S be a multiplicative subset with nonzero divisors of R, then R is quasireduced if and only if $S^{-1}R$ is quasireduced.*

The following theorem was proved by Keigher in [10].

Theorem 2.2. *Let R be a differential ring, and let S be a multiplicative subset of R.*

(1) *If P is a prime ideal such that $P \cap S = \phi$, then in the differential ring $S^{-1}R$, we have $[S^{-1}P]_{\Delta} = S^{-1}P_{\Delta}$.*

(2) *There is a one to one correspondence between quasiprime ideals in $S^{-1}R$ and quasiprime ideals in R disjoint from S.*

Now, we can conclude the following.

Theorem 2.3. *If R is a quasiregular ring and S is a multiplicative subset, which does not contain any zero divisors of R, then $S^{-1}R$ is a quasiregular ring.*

Proof. By Lemma 2.1, if every quasiprime ideal of $S^{-1}R$ is quasimaximal, then $S^{-1}R$ is quasiregular ring. Let P be a prime ideal of $S^{-1}R$ such that $r(P_{\Delta}) = P$. Then there exists a prime ideal T of R disjoint from S such that $r(T_{\Delta}) = T$, $S^{-1}T_{\Delta} = P_{\Delta}$ and $S^{-1}T = P$. Since R is a quasiregular ring and T_{Δ} is a quasiprime ideal of R, we see that T is a maximal ideal of R and hence P is a maximal ideal of $S^{-1}R$. □

It is well known that if R is a reduced ring, and let I is an ideal of R, then the factor ring R/I is reduced if and only if $r(I) = I$. Also one can easily show that if I is a pure ideal of a reduced ring R, then $r(I) = I$ and hence R/I is reduced ring. We can generalize these results to differential rings as follows.

Theorem 2.4. *Let R be a quasireduced ring and I be a differential ideal of R.*

(1) *The factor ring R/I is quasireduced if and only if $[r(I)]_{\Delta} = I$.*

(2) *I is a pure ideal of R implies that $[r(I)]_{\Delta} = I$.*

(3) *I is pure ideal of R implies that the factor ring R/I is quasireduced.*

Proof. (1) Obvious.

(2) First note that every pure ideal of a differential ring is a differential ideal. Now, suppose that R is quasireduced ring and I is pure ideal of R. Let $x \in [r(I)]_\Delta$. Then there exist positive integers m, n such that $x^n = x^n t$ and $(\delta(x))^m = (\delta(x))^m r$ for some $t, r \in I$. Since $t \in I$ and I is a differential ideal, there exists $k \in I$ such that $\delta(t) = \delta(t)k$. Take $y = x(1-t)(1-k)(1-r)$. Then it is easy to verify that $y \in \Delta - \text{nil}(R)$. But R is a quasireduced ring, so $y = 0$. Therefore, $x = xl$, where $l = r + t - tr + k[1 - r - t + tr] \in I$. Thus $x \in I$.

(3) It follows easily from (1) and (2). □

Now, we can determine when a factor ring R/I of a quasiregular ring R is quasiregular.

Theorem 2.5. *Let R be a quasiregular ring and I be a differential ideal of R. Then the factor ring R/I is a quasiregular ring if and only if $[r(I)]_\Delta = I$.*

Proof. (⇒) Obvious.

(⇐) It is enough to show that every quasiprime ideal of R/I is quasimaximal. Let P be a prime ideal of R/I such that $[r(P_\Delta)] = P$. Then there exists a prime ideal T of R such that $[r(T_\Delta)] = T$ and $P = T/I$, because for any differential ideal I of R, there is one to one correspondence between quasiprime ideals in R/I and quasiprime ideals in R that contain I, see Proposition 1.12 in [10]. Since R is a quasiregular ring, T is a maximal ideal of R. Therefore, P is a maximal ideal of R/I. □

The following corollary follows directly from Theorems 2.4 and 2.5.

Corollary 2.6. *Let R be a quasiregular ring. For any pure ideal I of R, the ring R/I is quasiregular.*

Next, we will show that a finite direct product of quasiregular rings is quasiregular. The following observation is trivial but useful for our purpose.

Lemma 2.7. *Let R_1 and R_2 be two differential rings, and let $R = R_1 \times R_2$ and $\pi_i : R \to R_i$, $i = 1, 2$, be the two projections. If I is an ideal of R, then*

(1) $\pi_i(r(I)) = r(\pi_i(I))$, $i = 1, 2$,

(2) $\pi_i(I_\Delta) = [\pi_i(I)]_\Delta$, $i = 1, 2$, *and*

(3) $r(I_\Delta) = I$ *implies that* $\pi_i(I) = r([\pi_i(I)]_\Delta)$, $i = 1, 2$.

Proof. (1) $x \in \pi_1(r(I))$ if and only if $(x, 0) \in r(I)$ if and only if $(x^n, 0) \in I$ if and only if $x^n \in \pi_1(I)$ if and only if $x \in r(\pi_1(I))$.

(2) $x \in \pi_1(I_\Delta)$ if and only if $(x, 0) \in I$ and $(\delta(x), 0) \in I$ if and only if $x \in \pi_1(I)$ and $\delta(x) \in \pi_1(I)$ if and only if $x \in [\pi_1(I)]_\Delta$.

(3) It follows easily from (1) and (2). □

Theorem 2.8. *Let $R = \prod_{i=1}^{n} R_i$ where R_i is a quasiregular ring. Then R is a quasiregular ring.*

Proof. We give the proof for the product of two quasiregular rings R_1 and R_2. The general result follows by induction. Let $R = R_1 \times R_2$ where R_1 and R_2 are quasiregular rings. Since $\text{nil}(R) = \text{nil}(R_1) \times \text{nil}(R_2)$, we have $[\Delta - \text{nil}(R)] = [\Delta - \text{nil}(R_1)] \times [\Delta - \text{nil}(R_2)]$. But R_i, $i = 1, 2$, are quasiregular rings so, $\Delta - \text{nil}(R_i) = O_{R_i}$, $i = 1, 2$. Therefore, $\Delta - \text{nil}(R) = O_R$ and hence R

is quasireduced. We may assume that $P = P_1 \times R_2$ is a prime ideal of R such that $r(P_\Delta) = P$. From Lemma 2.7, we conclude that $r((P_1)_\Delta) = P_1$. So, $(P_1)_\Delta$ is a quasiprime ideal of R_1 and hence it is a quasimaximal ideal of R_1. Thus P is a maximal ideal of R. □

Keigher in [10] introduced the following definitions of differential rings.

(1) R is said to be a q-local ring if R is a local ring whose unique maximal ideal M satisfies $r(M_\Delta) = M$ (i.e., M_Δ is quasimaximal).

(2) R is a quasidomain ring if R is quasireduced and every zero divisor in R is nilpotent.

(3) R is called a quasifield if R is quasireduced and every nonunit of R is nilpotent.

It is clear that R is a quasidomain if and only if O_R is a quasiprime ideal. Also, R is a quasifield if and only if O_R is a quasimaximal ideal. So, every quasidomain is a quasifield, and every quasifield is q-local. For more details about these classes of rings see Keigher [10].

Next, we give a characterization of q-local quasiregular rings and a characterization of quasiregular rings through localization, when the ring is of positive characteristic. First, we state the following result which is quite helpful.

Theorem 2.9 (Keigher, [10]). *Suppose that R has characteristic $l > 0$, and let P be a prime ideal in R. Then $r(P_\Delta) = P$.*

So, one can conclude the following.

Corollary 2.10. *Let R be a differential ring of positive characteristic. Then there is a one to one correspondence between quasiprime ideals in R and prime ideals in R.*

Now, we give the following result which is analogous to the corresponding one in commutative rings.

Theorem 2.11. *Let R be a differential ring of positive characteristic. Then a q-local ring R is quasiregular if and only if R is a quasifield.*

Proof. Since R is a q-local ring, $\mathrm{nil}(R)$ is a prime ideal. From Theorem 2.11, we get $r(\Delta - \mathrm{nil}(R)) = \mathrm{nil}(R)$. Since R is a quasiregular ring, $\Delta - \mathrm{nil}(R) = O_R$ and $\mathrm{nil}(R)$ is a maximal ideal of R. Thus R is a quasifield.

Conversely, it is clear that in any, R being a quasifield implies that R is quasiregular. □

For rings of positive characteristic, as in von Neumann regular rings we can characterize quasiregular ring by localizations.

Theorem 2.12. *Let R be a differential ring of positive characteristic. Then, R is quasiregular if R_M is a quasifield for each maximal ideal M of R.*

Proof. Let P be a prime ideal of R such that $r(P_\Delta) = P$. Then there exists a maximal ideal M of R such that $P \subseteq M$ and $r(M_\Delta) = M$. Therefore, PR_M is a prime ideal of R_M and $r(P_\Delta R_M) = PR_M$. Since R_M is a quasifield, we have $P_\Delta R_M = O_R R_M = M_\Delta R_M$. By Corollary 2.10, $P_\Delta = M_\Delta = O_R$. Hence $P = M$ and $\Delta - \mathrm{nil}(R) = O_R$. □

For a ring R, denote by $Z(R)$ and $J(R)$ the set of zero divisors and Jacobson radical, respectively.

As a simple consequence of Theorems 2.3, 2.11, and 2.12 we have the following.

Theorem 2.13. *Let R be a ring of positive characteristic with $Z(R) \subseteq J(R)$. Then, R is quasiregular if and only if every localization R_M at each maximal ideal M of R is quasifield.*

3. Quasi-PF Ring

Recall that a ring R is a PF-ring if and only if it is reduced and every prime ideal of R contains a unique minimal prime ideal. So, one can introduce quasi-PF rings. A ring R is called a quasi-PF ring if R is quasireduced and every quasiprime ideal of R contains a unique minimal quasiprime ideal. It is clear that every quasiregular ring is a quasi-PF ring and that every quasidomain is a quasi-PF ring. For a pure ideal I of a quasi-PF ring R, the factor ring R/I is a quasi-PF ring. This follows directly from Theorem 1.12 of [10] and Theorem 2.4. From Lemma 2.1 and Theorem 2.2, we can conclude that, if R is quasi-PF ring and S is a multiplicative subset with nonzero divisors of R, then $S^{-1}R$ is a quasi-PF ring. Furthermore, for rings with positive characteristic, a localization of quasi-PF ring is a quasidomain. This result is given in the following theorem.

Theorem 3.1. *Suppose that R is a differential ring with positive characteristic. A localization R_P of a quasi-PF ring R is a quasidomain for each prime ideal P of R.*

Proof. Let P be a prime ideal of R. Then $r(P_\Delta) = P$ and R_P has unique maximal ideal, PR_P. Since R is quasi-PF ring with positive characteristic, PR_P has unique minimal prime ideal TR_P where T is a unique minimal prime ideal of P in R such that $r(T_\Delta) = T$. Consequently, $\mathrm{nil}(R_P) = TR_P$. Furthermore, R_P is a quasireduced ring, hence O_{R_P} is a quasiprime ideal of R_P. □

Theorem 3.2. *A differential ring R is a quasi-PF ring if every localization R_P is a quasidomain for each prime ideal P of R with $r(P_\Delta) = P$.*

Proof. Let P be a prime ideal of R with $r(P_\Delta) = P$. Then R_P is quasidomain and hence $O_R R_P$ is a unique quasiminimal prime ideal of R_P. Let T and S be two minimal prime ideals of P with $r(T_\Delta) = T$ and $r(S_\Delta) = S$. Then $T_\Delta R_P = S_\Delta R_P = O_R R_P$. But, there is a one-to-one correspondence between quasiprime ideals of R_P and quasiprime ideals of R contained in P. So $T_\Delta = S_\Delta = O_R$. Consequentially, $T = S$ and $\Delta - \mathrm{nil}(R) = O_R$. □

Now, we prove an analogous result for localizations of maximal ideals.

Theorem 3.3. *Let R be a differential ring with positive characteristic and $Z(R) \subseteq J(R)$. Then, R is a quasi-PF ring if and only if R_M is a quasidomain for every maximal ideal M of R.*

Proof. ($\Rightarrow$) Let M be a maximal ideal of R. Since R is a quasireduced ring with positive characteristic and $Z(R) \subseteq J(R)$, to prove that R_M is a quasidomain it is enough to show that $\mathrm{nil}(R_M)$ is a prime ideal of R_M. But R is a quasi-PF ring so, the maximal ideal M has a unique minimal prime ideal P of R. Hence PR_M is a unique minimal prime ideal of R_M. Thus, $\mathrm{nil}(R_M) = PR_M$.

($\Leftarrow$) It follows directly from Theorem 3.1. □

4. Hurwitz Series

The Hurwitz series ring over R is denoted by HR and is defined as follows. The elements of HR are functions $a : \mathbb{N} \to R$, where $\mathbb{N}$ is the set of natural numbers and a is a sequence of the form(a_n). The operation of addition in HR is componentwise and for each $a = (a_n)$, $b = (b_n) \in$ HR multiplication is defined by $(a_n)(b_n) = (c_n)$, where $c_n = \sum_{k=0}^{n} C_k^n a_k b_{n-k}$ for all $n \in \mathbb{N}$. It can be easily shown that HR is a ring with zero element $0 = (0, 0, 0, \ldots, 0, \ldots)$, the unity of this ring is the sequence with 0th term 1 and nth term 0 for all $n \geq 1$. The ring HR has been named *the ring of Hurwitz series* in honors to Hurwitz who was the first to consider the product of sequences using the binomial coefficients [13]. The product of sequences using the binomial coefficients was studied extensively, for example, see Bochner and Marttin [14], Fliess [15], and Taft [16]. The ring of Hurwitz series has been of interest and has had important applications in many areas. In the discussion of weak normalization [4]. In differential algebra, Keigher in [11] and Keigher and Pritchard in [17] demonstrated that the ring HR of Hurwitz series over a commutative ring R with unity is very important in differential algebra. Some properties, which are shared between R and HR have been studied by Keigher [11], Liu [18]. The structure of Hurwitz series of positive characteristic is very close to the structure of R. Accordingly, for ring of positive characteristic, we prove that R is regular (resp., quasi-PF) if and only if HR is quasiregular (resp., quasi-PF). But before that, recall from Keigher [11] that for any ring R there is a natural ring homomorphism ε_R : HR $\to R$ defined as follows: for any $(a_n) \in$ HR, $\varepsilon_R((a_n)) = (a_{n+1})$ is a derivative of HR, a shift operator, making (HR, δ_R) a differential ring. For any ideal I of R, Keigher in [10], defined a differential ideal HI of HR as follows: HI $= \{(a_n) \in$ HR $: a_n \in I,\ \forall n \in \mathbb{N}\}$ and he proved that HR/HI $\cong H(R/I)$.

Theorem 4.1 (Keigher, [11]). *Let R be a ring with positive characteristic l.*

(1) $\mathrm{ch}(HR) = l$.

(2) *For any $x = (x_n) \in HR$, $x^l = (x_0^l, 0, 0, 0, \ldots)$.*

(3) *If I is an ideal of R then $r(HI) = \varepsilon_R^{-1}(r(I))$.*

(4) *HR is quasireduced if and only if R is reduced.*

Now, we prove the following new theorem which is the key to our main results of this section.

Theorem 4.2. *Let* $\mathrm{ch}(R) = l > 0$.

(1) *If P is prime ideal of HR, then $\varepsilon_R(P)$ is a prime ideal of R.*

(2) *P is a prime ideal of HR if and only if $P = \varepsilon_R^{-1}(T)$ for some prime ideal T of R.*

(3) *There is a one-to-one correspondence between prime ideals in R and quasiprime ideals in HR.*

Proof. (1) Let P be a prime ideal of HR. Let $xy \in \varepsilon_R(P)$. Then P has an element t with 0th term xy. Therefore, $t^l = ((xy)^l, 0, 0, \ldots) = (x^l, 0, 0, 0, \ldots)(y^l, 0, 0, 0, \ldots) \in P$. Hence $(x, 0, 0, 0, \ldots)$ or $(y, 0, 0, 0, \ldots)$ belongs to P and thus $x \in \varepsilon_R(P)$ or $y \in \varepsilon_R(P)$.

(2) Suppose that P is a prime ideal of HR. Then $\varepsilon_R(P)$ is a prime ideal of R. Let $r \in \varepsilon_R^{-1}(\varepsilon_R(P))$ and r_0 be the 0th term of r. Then $r_0 \in \varepsilon_R(P)$ and hence $r^l = (r_0^l, 0, 0, 0, \ldots) \in P$. Since P is a prime ideal, $r \in P$. Now, let $T = \varepsilon_R(P)$. Then $P = \varepsilon_R^{-1}(T)$ where T is a prime ideal of R.

Conversely, note that $\varepsilon_R : \mathrm{HR} \to R$ is an epimorphism. So for any prime ideal T of R, $\varepsilon_R^{-1}(T)$ is a prime ideal of HR.

(3) From (1) and (2) the result holds. □

From the above theorem we get the following result, which was proved differently in Keigher [11].

Corollary 4.3. *Let* $\mathrm{ch}(R) = l > 0$.

(1) *I is a prime ideal of R if and only if HI is a quasiprime ideal of HR.*

(2) *I is a maximal ideal of R if and only if HI is a quasimaximal ideal of HR.*

Proof. (1) (⇒) Obvious.

(⇐) Suppose that $r(\mathrm{HI})$ is a prime ideal of HR. Then $r(\mathrm{HI}) = \varepsilon_R^{-1}(T)$ for some prime ideal T of R. But $r(\mathrm{HI}) = \varepsilon_R^{-1}(I)$ and ε is an epimorphism. So, $T = I$. □

Now, we prove the following theorem that characterizes when the differential ring HR is quasiregular.

Theorem 4.4. *Let R be a ring with* $\mathrm{ch}(R) = l > 0$. *Then R is a regular ring if and only if HR is a quasiregular ring.*

Proof. By using Corollary 2.10 and Theorem 4.1, it is enough to prove that every prime ideal of R is a maximal ideal if and only if every prime ideal of HR is maximal.

Let T be a prime ideal of HR. Then $T = \varepsilon_R^{-1}(P)$ for some prime ideal P of R. Hence P is a maximal ideal of R. Thus T is a maximal ideal of HR.

Conversely, let P be a prime ideal of R. Then HP is a quasiprime ideal of HR. Therefore, HP is a quasimaximal ideal of HR and thus P is a maximal ideal of R. □

Now, we give a similar result for when HR is a quasi-PF ring.

Theorem 4.5. *Let R be a ring with* $\mathrm{ch}(R) = l > 0$. *Then R is a PF-ring if and only if HR is a quasi-PF ring.*

Proof. Note that T is prime ideal of HR if and only if $T = \varepsilon_R^{-1}(P)$, P is prime ideal of R. Moreover, T_0 is a unique minimal prime ideal of T if and only if $T_0 = \varepsilon_R^{-1}(P_0)$, P_0 is a unique minimal prime ideal contains in P. □

From Theorems 4.4 and 4.5 we can obtain the following.

Theorem 4.6. *Let R be a ring with* $\mathrm{ch}(R) = l > 0$.

(1) *If R is a regular ring and I is an ideal of R, then HR/HI is a quasiregular ring.*

(2) *If R is a PF-ring and I is a pure ideal of R, then HR/HI is a a quasi-PF ring.*

Proof. Note that, $\mathrm{HR}/\mathrm{HI} \cong H(R/I)$. □

Remark 4.7. Every quasiregular ring is a quasi-PF ring and every quasidomain is a quasi-PF ring. But the converse is not true. For example, $H\mathbb{Z}_2[x]$ is quasi-PF ring but not quasiregular ring since $\mathbb{Z}_2[x]$ is PF-ring but not regular ring. $H\mathbb{Z}_6$ is a quasi-PF ring but not a quasidomain because $\mathbb{Z}_6$ is a PF-ring but not an integral domain, see Theorem 4.3 of Keigher [11].

Open Questions. (1) Give alternative characterizations of quasiregular rings and quasi-PF rings.

(2) Is it true that R is a quasiregular rings if and only if every differential ideal I is pure?

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