Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 416173, 13 pages
doi:10.1155/2011/416173

Research Article

Results for Twin Singular Nonlinear Problems with Damping Term

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Received 6 December 2010; Accepted 7 March 2011

Academic Editor: Wolfgang Castell

By using the functional type cone expansion and compression fixed-point theorem in cones, some new and general results on the existence of positive solution for twin singular boundary value problems with damping term are obtained. An example is given to illustrate our results.

1. Introduction

The study of multipoint boundary value problem for linear second-order ordinary differential equation was initiated by I1'in and Moiseev [1] motivated by the work of Bitsadze and Samarskiĭ [2–4] on nonlocal linear elliptic boundary value problem, which is a new area of still fairly theoretical exploration in mathematics. Several authors have expounded on various aspects of this theory; see the survey paper by Gupta et al. [5–7] and the references cited therein. Thereamong, the study of singular boundary value problem for ordinary differential equation has led to several important applications in applied mathematics and physical science, such as the Thomas-Fermi problem

$$\begin{gathered} x''(t) - t^{-1/2}x^{3/2} = 0, \quad 0 < t < 1, \\ x(0) = 0 = x(1), \end{gathered} \tag{1.1}$$

which appears in determining the electrical potential in an atom. For other application results, we refer to [8–10]. With regards to this, increasing attention is paid to question of singular boundary value problem and has obtained many excellent results of the existence of the positive solution for two multiple points nonlinear singular boundary value problem [11–15]. The main techniques are the upper and lower solutions method [16], the Leray-Schauder continuation theory [17], and the fixed-point theory in cones [18] et al. We would like to mention some results of Ma and O'Regan [13] and Yao [15], which motivated us to consider the singular boundary value problems. In [13], the authors have studied a multipoint boundary value problem

$$x''(t) = f\big(t, x(t), x'(t)\big) + e(t), \quad 0 < t < 1,$$
$$x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{1.2}$$

where $\xi_i \in]0,1[$, $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a function satisfying Carathéodory's conditions and $(1-t)e(t) \in L^1]0,1[$; the Leray-Schauder continuation theorem leads to the existence of single $C^1[0,1[$ solution. The literature [15] has discussed a second-order boundary value problem

$$x''(t) + h(t)f(x(t)) = 0, \quad 0 < t < 1,$$
$$ax(0) - bx'(0) = 0, \qquad ax(1) + bx'(1) = 0, \tag{1.3}$$

where $h(t)$ is symmetric on $]0,1[$ and may be singular at both end points $t = 0$ and $t = 1$. The author has proved the existence of n symmetric positive solutions and established a corresponding iterative scheme, the main tool being the monotone iterative technique.

A powerful tool for proving existence of solution to boundary value problem is the fixed-point theory. In many cases, it is possible to find single, double, or multiple solutions for boundary value problem, and for the same problem, using the various methods, one can obtain different results under some appropriate conditions. To my best knowledge, very little work has been done on the existence of positive solution for boundary value problem by using the functional type cone expansion and compression fixed-point theorem. The aim of this paper is to establish some new and general results on the existence of positive solution to singular boundary value problems with damping term

$$u''(t) - \lambda u'(t) + h(t)f(t, u(t)) = 0, \quad a < t < b, \tag{1.4}$$

$$u'(a) - \lambda u(a) = 0, \qquad \gamma u(b) + \delta u'(b) = \sum_{i=1}^{n} a_i u(t_i), \tag{1.5}$$

$$\gamma u(a) + \delta u'(a) = \sum_{i=1}^{n} a_i u(t_i), \quad u'(b) - \lambda u(b) = 0, \tag{1.6}$$

where λ, γ, δ, a_i, t_i, $(i = 1, 2, \ldots, n)$, h, and f satisfy

(H_1) $-\infty < a < t_1 < \cdots < t_n < b < \infty$, $\lambda, \gamma, \delta \in]0,\infty[$, $a_i \in]0,\infty[$, $(i = 1, 2, \ldots, n)$,

(H_2) $d_1 = (\gamma+\lambda\delta)\exp(\lambda b) - \sum_{i=1}^{n} a_i \exp(\lambda t_i) > 0$, $d_2 = \sum_{i=1}^{n} a_i \exp(\lambda t_i) - (\gamma+\lambda\delta)\exp(\lambda a) > 0$,

(H_3) $h :]a,b[\to [0,\infty[$ is continuous function, $h(t)$ may be singular at $t = a$ and/or $t = b$, and $0 < \int_a^b h(s)ds < \infty$,

(H_4) $f : [a,b] \times [0,\infty[\to [0,\infty[$ satisfies the Carathéodory condition, that is, for each $x \in [0,\infty[$, the mapping $t \to f(t,x)$ is Lebesgue measurable on $[a,b]$ and for a.e. $t \in [a,b]$, the mapping $x \to f(t,x)$ is continuous on $[0,\infty[$, and for each $r > 0$, there exists $\phi_r \in L^1[a,b]$ such that $f(t,x) \le \phi_r(t)$ for all $u \in [0,r]$ and for a.e. $t \in [a,b]$.

We will impose some advisable conditions on the nonlinearity f to ensure the existence of at least one positive solution for the above problems. In order to obtain our results, we construct special operator which is the base for further discussion and provide two crucial functionals on cones. Applying the functional type cone expansion and compression fixed-point theorem to the operator and functionals, we obtain some new and general results on the existence of at least one positive solution for the twin singular problems (1.4), (1.5) and (1.4), (1.6). Our results improve and generalize those in [15, 19].

Let α and β be nonnegative continuous functionals on a cone $\mathcal{P}$ in real Banach space $\mathcal{B}$. For positive numbers r and L, we define the sets

$$\begin{aligned} \mathcal{P}(\alpha, r) &= \{x \in \mathcal{P} : r < \alpha(x)\}, \\ \mathcal{P}(\beta, L) &= \{x \in \mathcal{P} : \beta(x) < L\}, \\ \mathcal{P}(\beta, \alpha, r, L) &= \{x \in \mathcal{P} : r < \alpha(x),\ \beta(x) < L\}. \end{aligned} \tag{1.7}$$

We state the functional type cone expansion and compression fixed-point theorem [20].

Lemma 1.1. *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$, and let α and β be nonnegative continuous functionals on $\mathcal{P}$. Let $\mathcal{P}(\beta, \alpha, r, L)$ be a nonempty bounded subset of $\mathcal{P}$,*

$$\mathcal{T} : \overline{\mathcal{P}(\beta, \alpha, r, L)} \longrightarrow \mathcal{P} \tag{1.8}$$

is a completely continuous operator with

$$\inf_{x \in \partial\mathcal{P}(\beta,\alpha,r,L)} \|\mathcal{T}x\| > 0, \qquad \overline{\mathcal{P}(\alpha, r)} \subseteq \mathcal{P}(\beta, L), \tag{1.9}$$

for $\mathcal{P}(\beta, L)$. If $\alpha(\mathcal{T}x) \geq r$ for all $x \in \partial\mathcal{P}(\alpha, r)$, $\beta(\mathcal{T}x) \leq L$ for each $x \in \partial\mathcal{P}(\beta, L)$, and for each $y \in \partial\mathcal{P}(\alpha, r)$, $z \in \partial\mathcal{P}(\beta, L)$, $\theta \in]0, 1]$, and $\mu \in [1, \infty[$, the functionals satisfy the properties

$$\alpha(\theta y) \leq \theta\alpha(y), \qquad \beta(\mu z) \geq \mu\beta(z), \qquad \beta(0) = 0, \tag{1.10}$$

then $\mathcal{T}$ has at least one positive fixed-point x such that

$$r \leq \alpha(x), \qquad \beta(x) \leq L. \tag{1.11}$$

2. Main Results

Let $\mathcal{B}$ be the Banach space $C^1[a, b]$ with the norm $\|u\| = \max\{\|u\|_0, \|u'\|_1\}$, where $\|u\|_0 = \sup_{t\in[a,b]}|u(t)|$ and $\|u'\|_1 = \sup_{t\in]a,b[}|u'(t)|$, and let a cone in $\mathcal{B}$

$$\mathcal{P}_1 = \{v \in \mathcal{B} : v \text{ is nondecreasing on } [a, b], v(a) = 0\}. \tag{2.1}$$

For $v \in \mathcal{P}_1$, define the operator $\mathcal{T}_1$ by

$$\mathcal{T}_1 v(t) = \int_a^t h(s) f\left(s, \exp(\lambda s)\left(c_1(v) + \int_s^b v(\omega)\exp(-\lambda\omega)d\omega\right)\right)ds, \quad t \in [a,b], \tag{2.2}$$

where $c_1(v) = (1/d_1)(\delta v(b) + \sum_{i=1}^n a_i \exp(\lambda t_i)\int_{t_i}^b v(\omega)\exp(-\lambda\omega)d\omega)$.

Lemma 2.1. *If $v \in \mathcal{P}_1$ is a fixed-point $\mathcal{T}_1$, then*

$$u(t) := \exp(\lambda t)\left(c_1(v) + \int_t^b v(\omega)\exp(-\lambda\omega)d\omega\right) \tag{2.3}$$

is one solution of the problem (1.4), (1.5).

Proof. Suppose that $v \in \mathcal{P}_1$ is a fixed-point $\mathcal{T}_1$ and $u(t) = \exp(\lambda t)(c_1(v) + \int_t^b v(\omega)\exp(-\lambda\omega)d\omega)$, thus we have

$$u'(t) - \lambda u(t) = -v(t) = -\mathcal{T}_1 v(t) = -\int_a^t h(s) f\left(s, \exp(\lambda s)\left(c_1(v) + \int_s^b v(\omega)\exp(-\lambda\omega)d\omega\right)\right)ds. \tag{2.4}$$

Further,

$$u''(t) - \lambda u'(t) = -h(t) f\left(t, \exp(\lambda t)\left(c_1(v) + \int_t^{t_n} v(\omega)\exp(-\lambda\omega)d\omega\right)\right) = -h(t) f(t, u(t)). \tag{2.5}$$

The boundary condition (1.5) is satisfied due to the relation between u, v, and $c_1(v)$. □

For $v \in \mathcal{P}_1$, we define the nonnegative continuous functionals α and β on $\mathcal{P}_1$ by

$$\alpha(v) = \int_{t_1}^b v(\omega)\exp(-\lambda\omega)d\omega, \quad \beta(v) = v(t_1). \tag{2.6}$$

Lemma 2.2. *Let $r_1 > 0$. If $v \in \partial\mathcal{P}_1(\alpha, r_1)$, then*

$$\begin{gathered} v(t_1) \le \frac{\exp(\lambda b)}{b - t_1} r_1, \\ \int_a^b v(\omega)\exp(-\lambda\omega)d\omega \ge r_1. \end{gathered} \tag{2.7}$$

Proof. For $v \in \partial \mathcal{P}_1(\alpha, r_1)$, that is, $\alpha(v) = r_1$. Since $v(t)$ nondecreasing on $[a, t_1]$, we have

$$r_1 = \alpha(v) = \int_{t_1}^{b} v(\omega)\exp(-\lambda\omega)d\omega \le \int_{t_1}^{b} v(b)\exp(-\lambda\omega)d\omega \le v(b)\exp(-\lambda t_1)(b - t_1), \tag{2.8}$$

so we get (2.7). Furthermore,

$$\int_{a}^{b} v(\omega)\exp(-\lambda\omega)d\omega = \int_{a}^{t_1} v(\omega)\exp(-\lambda\omega)d\omega + \int_{t_1}^{b} v(\omega)\exp(-\lambda\omega)d\omega \ge r_1. \tag{2.9}$$

□

Lemma 2.3. *Let $L_1 > 0$. If $v \in \partial \mathcal{P}_1(\beta, L_1)$, then*

$$\int_{t_1}^{b} v(\omega)\exp(-\lambda\omega)d\omega \ge L_1 \exp(-\lambda b)(b - t_1). \tag{2.10}$$

Proof. For $v \in \partial \mathcal{P}_1(\beta, L_1)$, that is to say, $\beta(v) = L_1$. In view of $v(t)$ is nondecreasing on $[a, b]$, for $\omega \in [t_1, b]$, we have

$$v(\omega) \ge v(t_1) = \beta(v) = L_1. \tag{2.11}$$

Hence,

$$\int_{t_1}^{b} v(\omega)\exp(-\lambda\omega)d\omega \ge L_1 \int_{t_1}^{b} \exp(-\lambda\omega)d\omega \ge L_1 \exp(-\lambda b)(b - t_1). \tag{2.12}$$

□

Lemma 2.4. *Let (H_1)–(H_4) hold, then $\mathcal{T}_1 : \mathcal{P}_1 \to \mathcal{P}_1$ is completely continuous.*

Proof. By (H_1)–(H_4), we observe that $\mathcal{T}_1 v \in C^1[a, b]$, $(\mathcal{T}_1 v)(t)$ is nondecreasing on $[a, b]$, and $(\mathcal{T}_1 v)(a) = 0$, so $\mathcal{T}_1 : \mathcal{P}_1 \to \mathcal{P}_1$. Since $h(t)$ may be singular at $t = a$ and/or $t = b$, we take the arguments to show that $\mathcal{T}_1$ is completely continuous.

Assume that $v_n, v_0 \in \mathcal{P}_1$. In view of f satisfying the Carathéodory condition, it is easy to see that

$$\begin{aligned}
&\|v_n - v_0\|_0 \longrightarrow 0 \text{ implies that} \\
&\qquad \sup_{s \in \Omega} \left| f\left(s, \exp(\lambda s)\left(c_1(v_n) + \int_{s}^{b} v_n(\omega)\exp(-\lambda\omega)d\omega\right)\right)\right. \\
&\qquad\qquad \left. - f\left(s, \exp(\lambda s)\left(c_1(v_0) + \int_{s}^{b} v_0(\omega)\exp(-\lambda\omega)d\omega\right)\right)\right| ds \longrightarrow 0
\end{aligned} \tag{2.13}$$

as $n \longrightarrow \infty$, where $\Omega = [a,b]$ or $]a,b[$. Thus, we have

$$\begin{aligned}
\|\mathcal{T}_1 v_n - \mathcal{T}_1 v_0\|_0 &= \sup_{t\in[a,b]} |(\mathcal{T}_1 v_n)(t) - (\mathcal{T}_1 v_0)(t)| \\
&\le \int_a^b h(s)\left| f\left(s, \exp(\lambda s)\left(c_1(v_n) + \int_s^b v_n(\omega)\exp(-\lambda\omega)d\omega\right)\right)\right. \\
&\qquad \left. - f\left(s, \exp(\lambda s)\left(c_1(v_0) + \int_s^b v_0(\omega)\exp(-\lambda\omega)d\omega\right)\right)\right| ds, \\
\left\|(\mathcal{T}_1 v_n)' - (\mathcal{T}_1 v_0)'\right\|_1 &= \sup_{t\in]a,b[} \left|(\mathcal{T}_1 v_n)'(t) - (\mathcal{T}_1 v_0)'(t)\right| \\
&\le \sup_{t\in]a,b[} \left(h(t)\left| f\left(t, \exp(\lambda t)\left(c_1(v_n) + \int_s^b v_n(\omega)\exp(-\lambda\omega)d\omega\right)\right)\right.\right. \\
&\qquad \left.\left. - f\left(t, \exp(\lambda t)\left(c_1(v_0) + \int_s^b v_0(\omega)\exp(-\lambda\omega)d\omega\right)\right)\right|\right).
\end{aligned} \tag{2.14}$$

Therefore,

$$\|\mathcal{T}_1 v_n - \mathcal{T}_1 v_0\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.15}$$

This means that the operator $\mathcal{T}_1 : \mathcal{P}_1 \longrightarrow \mathcal{P}_1$ is continuous.

Choose two sequences $\{\varphi_n\}_{n=1}^{\infty}$, $\{\psi_n\}_{n=1}^{\infty} \subset]a,b[$ satisfying $\varphi_n \le \psi_n$ for any $n \ge 1$, such that $\varphi_n \longrightarrow a$ and $\psi_n \longrightarrow b$ as $n \longrightarrow \infty$, respectively. Define

$$h_n(t) = \begin{cases} \inf\limits_{a\le t\le \varphi_n} h(t), & \\ h(t), & \varphi_n < t < \psi_n \\ \inf\limits_{\psi_n \le t\le b} h(t), & \end{cases} \tag{2.16}$$

and an operator sequence $\{\mathcal{T}_{1n}\}_{n=1}^{\infty}$ by

$$\mathcal{T}_{1n} v = \int_a^t h_n(s) f\left(s, \exp(\lambda s)\left(c_1(v) + \int_s^b v(\omega)\exp(-\lambda\omega)d\omega\right)\right) ds. \tag{2.17}$$

Clearly, $h_n : [a,b] \longrightarrow [0,\infty)$ is a piecewise continuous function, and the operator $\mathcal{T}_{1\mathrm{n}} : \mathcal{P}_1 \longrightarrow \mathcal{P}_1$ is well defined. Further, we can see that $\mathcal{T}_{1n} : \mathcal{P}_1 \longrightarrow \mathcal{P}_1$ is completely continuous.

Let $R > 0$, $B_R := \{v \in \mathcal{P}_1 : \|v\|_0 \le R\}$, and $M_R = \sup\{f(t,u) : (t,u) \in [a,b] \times [0,\overline{R}]\}$, where $\overline{R} = (R\exp(\lambda b)/d_1)(\delta + (1/\lambda)\sum_{i=1}^{n} a_i(1-\exp(-\lambda(b-t_i)))) + (R/\lambda)(\exp\lambda(b-a)-1) > 0$.

We will prove that $\mathcal{T}_{1n}$ approach $\mathcal{T}_1$ uniformly on B_R. From the absolute continuity of integral, we obtain

$$\lim_{n\to\infty}\int_{l(n)} h(s)ds = 0, \tag{2.18}$$

where $l(n) = [a,\varphi_n] \cup [\psi_n, b]$. For each $v \in B_R$, $t \in [a,\varphi_n]$, we have

$$\begin{aligned}\|\mathcal{T}_{1n}v - \mathcal{T}_1 v\|_0 &= \sup_{t\in[a,\varphi_n]}\left|\int_a^t (h_n(s)-h(s))f\left(s,\exp(\lambda s)\left(c_1(v)+\int_s^b v(\omega)\exp(-\lambda\omega)d\omega\right)\right)ds\right.\\ &\le M_R\int_a^{\varphi_n}|h_n(s)-h(s)|ds \longrightarrow 0 \quad (n\longrightarrow\infty).\end{aligned} \tag{2.19}$$

For each $v \in B_R$, $t \in [\psi_n, b]$, we have

$$\begin{aligned}\|\mathcal{T}_{1n}v - \mathcal{T}_1 v\|_0 &= \sup_{t\in[\psi_n,b]}\left|\int_a^t (h_n(s)-h(s))f\left(s,\exp(\lambda s)\left(c_1(v)+\int_s^b v(\omega)\exp(-\lambda\omega)d\omega\right)\right)ds\right.\\ &\le M_R\int_a^{b}|h_n(s)-h(s)|ds \longrightarrow 0 \quad (n\longrightarrow\infty).\end{aligned} \tag{2.20}$$

It is easy to see that, for each $v \in B_R$ and $\varphi_n < t < \psi_n$, there is $\|\mathcal{T}_{1n}v - \mathcal{T}_1 v\|_0 \to 0$ as $n \to \infty$. Similarly, for any $v \in B_R$, and $t \in [a,\varphi_n]$, $]\varphi_n, \psi_n[$, $[\psi_n, b]$, respectively, we can obtain that

$$\begin{aligned}\left\|(\mathcal{T}_{1n}v)' - (\mathcal{T}_1 v)'\right\|_1 &= \sup_t\left|(h_n(t)-h(t))f\left(t,\exp(\lambda t)\left(c_1(v)+\int_t^b v(\omega)\exp(-\lambda\omega)d\omega\right)\right)\right|\\ &\le M_R|h_n(t)-h(t)| \longrightarrow 0 \quad (n\longrightarrow\infty).\end{aligned} \tag{2.21}$$

From the above argument, we obtain

$$\|\mathcal{T}_{1n}v - \mathcal{T}_1 v\| = \max\left\{\|\mathcal{T}_{1n}v - \mathcal{T}_1 v\|_0, \left\|(\mathcal{T}_{1n}v)' - (\mathcal{T}_1 v)'\right\|_1\right\} \longrightarrow 0 \quad (n\longrightarrow\infty). \tag{2.22}$$

That is to say, the sequence $\mathcal{T}_{1n}$ is uniformly approximate $\mathcal{T}_1$ on any bounded subset of $\mathcal{P}_1$. Therefore, $\mathcal{T}_1 : \mathcal{P}_1 \to \mathcal{P}_1$ is completely continuous. $\square$

For convenience, we set

$$m_1 = \int_{t_1}^{b} \exp(-\lambda t)\left(\int_a^t h(s)ds\right)dt, \qquad M_1 = \int_a^b h(s)ds,$$

$$u_1 = r_1 \exp(\lambda a)\left(\frac{\delta \exp(\lambda a)}{d_1(b-a)} + 1\right), \qquad u_2 = L_1 \exp(\lambda t_1)\frac{\delta + \Sigma_{i=1}^{n} a_i(b-t_i)\exp(\lambda(t_i - b))}{d_1}. \tag{2.23}$$

We are now ready to apply a functional type cone expansion and compression fixed-point theorem to the operator $\mathcal{T}_1$ to give the sufficient conditions for the existence of at least one positive solution to the problem (1.4), (1.5).

Theorem 2.5. *Suppose that* (H_1)–(H_4) *hold. Assume that there exist positive numbers* k_1, r_1, *and* L_1 *with* $(\exp(\lambda b))/(b-t_1)r_1 < L_1$ *such that*

(A_1) $f(t,w) \geq k_1$, $(t,w) \in [a,b] \times [r_1, \infty[$,

(A_2) $f(t,w) \geq r_1/m_1$, $(t,w) \in [a,t_1] \times [u_1, \infty[$,

(A_3) $f(t,w) \leq L_1/M_1$, $(t,w) \in [t_1,b] \times [u_2, \infty[$,

then the operator $\mathcal{T}_1$ *has at least one fixed-point* v *such that* $r_1 \leq \alpha(v)$ *and* $\beta(v) \leq L_1$, *and the problem* (1.4), (1.5) *has at least one positive solution* u *such that*

$$u(t) = \exp(\lambda t)\left(c_1(v) + \int_t^b v(\omega)\exp(-\lambda\omega)d\omega\right). \tag{2.24}$$

Proof. The cone $\mathcal{P}_1$ and operator $\mathcal{T}_1$ are defined by (2.1) and (2.2), respectively. By the properties of operator $\mathcal{T}_1$, it suffices to show that the conditions of Lemma 1.1 hold with respect to $\mathcal{T}_1$. In view of Lemma 2.1, it is not difficult to prove that a fixed point of $\mathcal{T}_1$ is coincident with the solution of the boundary value problem (1.4), (1.5), so we concentrate on the existence of the fixed point of the operator $\mathcal{T}_1$. Set $\mathcal{P}_1(\beta,\alpha,r_1,L_1)$ is a nonempty bounded subset of $\mathcal{P}_1$. From Lemma 2.4, it can be shown that

$$\mathcal{T}_1 : \overline{\mathcal{P}_1(\beta,\alpha,r_1,L_1)} \longrightarrow \mathcal{P}_1 \tag{2.25}$$

is completely continuous by the Arzela-Ascoli lemma. For $v \in \partial\mathcal{P}_1(\beta,\alpha,r_1,L_1)$, the assumption ($A_1$) implies that

$$\begin{gathered}\|\mathcal{T}_1 v\|_0 = \int_a^b h(s) f\left(s, \exp(\lambda s)\left(c_1(v) + \int_s^b v(\omega)\exp(-\lambda\omega)d\omega\right)\right)ds \geq k_1 \int_a^b h(s)ds, \\ \left\|(\mathcal{T}_1 v)'\right\|_1 = \sup_{t\in]a,b[} h(t) f\left(t, \exp(\lambda t)\left(c_1(v) + \int_t^b v(\omega)\exp(-\lambda\omega)d\omega\right)\right) \geq k_1 \sup_{t\in]a,b[} h(t).\end{gathered} \tag{2.26}$$

the hypotheses of (H_3) lead to

$$\inf_{v\in\partial\mathcal{P}_1(\beta,\alpha,r,L)}\|\mathcal{T}_1 v\| \geq k_1 \min\left\{\sup_{t\in]a,b[} h(t), \int_a^b h(s)ds\right\} > 0. \tag{2.27}$$

If $v \in \overline{\mathcal{P}_1(\alpha,r_1)}$, then $\beta(v) = v(t_1) \leq (\exp(\lambda b))/(b-t_1)r_1 < L_1$, and so $v \in \overline{\mathcal{P}_1(\beta,L_1)}$, that is, $\overline{\mathcal{P}_1(\alpha,r_1)} \subseteq \mathcal{P}_1(\beta,L_1)$. It follows that the conditions of Lemma 1.1 hold with respect to $\mathcal{T}_1$. By the definition of functionals α and β, we can check that the functionals satisfy the properties $\alpha(\theta y) = \int_{t_1}^b \theta y(\omega)\exp(-\lambda\omega)d\omega = \theta\int_{t_1}^b y(\omega)\exp(-\lambda\omega)d\omega = \theta\alpha(y)$ for $y \in \partial\mathcal{P}_1(\alpha,r_1)$ and $\theta \in]0,1]$, $\beta(\mu z) = \mu z(t_1) = \mu\beta(z)$ for $z \in \partial\mathcal{P}_1(\beta,L_1)$ and $\mu \in [1,\infty[$, $\beta(0) = 0$.

We now prove that $\alpha(\mathcal{T}_1 v) \geq r_1$, in Lemma 1.1, holds. In fact, if $v \in \partial\mathcal{P}_1(\alpha,r_1)$, by the properties of $c_1(v)$ and Lemma 2.2, for each $t \in [a,t_1]$,

$$\begin{aligned} u(t) &= \exp(\lambda t)\left(c_1(v) + \int_t^b v(\omega)\exp(-\lambda\omega)d\omega\right) \\ &\geq \exp(\lambda a)(c_1(v) + r_1) \\ &\geq r_1\exp(\lambda a)\left(\frac{\delta\exp(\lambda a)}{d_1(b-a)} + 1\right). \end{aligned} \tag{2.28}$$

Hence, by the assumption (A_2) and (2.28), there is

$$\begin{aligned} \alpha(\mathcal{T}_1 v) &= \int_{t_1}^b (\mathcal{T}_1 v)(t)\exp(-\lambda t)dt \\ &= \int_{t_1}^b \exp(-\lambda t)\left(\int_a^t h(s) f\left(s, \exp(\lambda s)\left(c_1(v) + \int_s^b v(\omega)\exp(-\lambda\omega)d\omega\right)\right)ds\right)dt \\ &\geq \frac{r_1}{m_1}\int_{t_1}^b \exp(-\lambda t)\left(\int_a^t h(\mathrm{s})ds\right)dt = r_1. \end{aligned} \tag{2.29}$$

Finally, we assert that $\beta(\mathcal{T}_1 v) \leq L_1$, in Lemma 1.1, also holds. If $v \in \partial\mathcal{P}_1(\beta,L_1)$, by Lemma 2.3, for $t \in [t_1,b]$,

$$\begin{aligned} u(t) &= \exp(\lambda t)\left(c_1(v) + \int_t^b v(\omega)\exp(-\lambda\omega)d\omega\right) \\ &\geq v(t_1)\exp(\lambda t_1)\frac{\delta + \Sigma_{i=1}^n a_i(b-t_i)\exp(\lambda(t_i-b))}{d_1} \\ &= L_1\exp(\lambda t_1)\frac{\delta + \Sigma_{i=1}^n a_i(b-t_i)\exp(\lambda(t_i-b))}{d_1}. \end{aligned} \tag{2.30}$$

The assumption (A_3) and (2.30) imply that

$$\begin{aligned}\beta(\mathcal{T}_1 v) &= (\mathcal{T}_1 v)(b)\\ &= \int_a^b h(s) f\left(s, \exp(\lambda s)\left(c_1(v) + \int_s^b v(\omega)\exp(-\lambda\omega)d\omega\right)\right)ds \\ &\le \frac{L_1}{M_1}\int_a^b h(s)ds = L_1.\end{aligned} \tag{2.31}$$

To sum up, the hypotheses of Lemma 1.1 are satisfied. Hence, the operator $\mathcal{T}_1$ has at least one fixed point, that is, the problem (1.4), (1.5) has at least one positive solution. □

Let the cone

$$\mathcal{P}_2 = \{v \in \mathcal{B} : v \text{ is nonincreasing on } [a,b],\ v(b) = 0\}. \tag{2.32}$$

Evidently, $\mathcal{P}_2 \subset \mathcal{B}$. For $v \in \mathcal{P}_2$, define the operator $\mathcal{T}_2$ by

$$\mathcal{T}_2 v(t) = \int_t^b h(s) f\left(s, \exp(\lambda s)\left(c_2(v) + \int_a^s v(\omega)\exp(-\lambda\omega)d\omega\right)\right)ds, \quad t \in [a,b], \tag{2.33}$$

where $c_2(v) = (1/d_2)(\delta v(a) - \sum_{i=1}^n a_i \exp(\lambda t_i)\int_a^{t_i} v(\omega)\exp(-\lambda\omega)d\omega)$.

We only give the preliminary lemmas and result of the problem (1.4), (1.6), the proofs are similar to the above argument.

Lemma 2.6. *If $v \in \mathcal{P}_2$ is a fixed-point $\mathcal{T}_2$, then*

$$u(t) = \exp(\lambda t)\left(c_2(v) + \int_a^t v(\omega)\exp(-\lambda\omega)d\omega\right) \tag{2.34}$$

is one solution of the problem (1.4), (1.6).

For $v \in \mathcal{P}_2$, the nonnegative continuous functionals α and β on $\mathcal{P}_2$ are defined by

$$\alpha(v) = \int_a^{t_n} v(\omega)\exp(-\lambda\omega)d\omega, \qquad \beta(v) = v(t_n). \tag{2.35}$$

Lemma 2.7. *Let $r_2 > 0$. If $v \in \partial\mathcal{P}_2(\alpha, r_2)$, then*

$$v(t_n) \le \frac{\exp(\lambda t_n)}{t_n - a} r_2, \qquad \int_a^b v(\omega)\exp(-\lambda\omega)d\omega \ge r_2. \tag{2.36}$$

Lemma 2.8. *Let $L_2 > 0$. If $v \in \partial\mathcal{P}_2(\beta, L_2)$, then*

$$\int_a^{t_n} v(\omega)\exp(-\lambda\omega)d\omega \geq L_2\exp(-\lambda t_n)(t_n - a). \tag{2.37}$$

Lemma 2.9. *Let (H_1)–(H_4) hold, then $\mathcal{T}_2 : \mathcal{P}_2 \to \mathcal{P}_2$ is completely continuous.*

For convenience, we set

$$m_2 = \int_a^{t_n} \exp(-\lambda t)\left(\int_t^b h(s)ds\right)dt > 0, \qquad M_2 = \int_a^b h(s)ds > 0,$$
$$u_3 = r_2\exp(\lambda b), \qquad u_4 = L_2\exp(\lambda a)\frac{\delta - \Sigma_{i=1}^n a_i(t_i - a)\exp(\lambda(t_i - a))}{d_2}. \tag{2.38}$$

Theorem 2.10. *Suppose that (H_1)–(H_4) hold. Assume that $\delta - \Sigma_{i=1}^n a_i\exp(\lambda(t_i - a))(t_i - a) > 0$, then there exist positive numbers k_2, r_2, and L_2 with $((\exp(\lambda t_n))/(t_n - a))r_2 < L_2$ such that*

(B_1) $f(t,w) \geq k_2$, $(t,w) \in [a,b] \times [r_2, \infty[$,

(B_2) $f(t,w) \geq (r_2/m_2)$, $(t,w) \in [t_n,b] \times [u_3, \infty[$,

(B_3) $f(t,w) \leq (L_2/M_2)$, $(t,w) \in [a,t_n] \times [u_4, \infty[$.

Then the operator $\mathcal{T}_2$ has at least one fixed-point v such that $r_2 \leq \alpha(v)$ and $\beta(v) \leq L_2$, and the problem (1.4), (1.6) *has at least one positive solution u such that*

$$u(t) = \exp(\lambda t)\left(c_2(v) + \int_a^t v(\omega)\exp(-\lambda\omega)d\omega\right). \tag{2.39}$$

3. Examples

Consider the problems

$$u''(t) - u'(t) + h(t)f(t,u(t)) = 0, \quad 0 < t < 1, \tag{3.1}$$

$$u'(0) - u(0) = 0, \qquad u(1) + 2u'(1) = 2u\left(\frac{1}{2}\right), \tag{3.2}$$

$$u(0) + 2u'(0) = 2u\left(\frac{1}{2}\right), \qquad u'(1) - u(1) = 0. \tag{3.3}$$

Let

$$h(t) = \frac{1}{\sqrt{t(1-t)}}, \qquad f(t,w) = \begin{cases} 10^{-3}t + \dfrac{2}{\sqrt{w}+2}, & w < 2, \\ 10^{-3}t + 2 - \sqrt{2}, & w \geq 2. \end{cases} \tag{3.4}$$

It is easy to check that hypotheses (H_1)–(H_4) hold. For the problem (3.1), (3.2), by some calculations, we have $d_1 \approx 4.858$, $m_1 \approx 0.348$, and $M_1 \approx 3.142$. Taking $k_1 = 0.001$, $r_1 = 0.2$, and $L_1 = 2$, satisfying the following conditions: $f(t,w) \geq 0.001$, $(t,w) \in [0,1] \times [0.2,\infty[$, $f(t,w) \geq r_1/m_1 \approx 0.575$, $(t,w) \in [0,1/2] \times [0.141,\infty[$, $f(t,w) \leq L_1/M_1 \approx 0.637$, and $(t,w) \in [1/2,1] \times [2.242,\infty[$. Thus, the hypotheses of Lemma 1.1 are fulfilled, and so the operator $\mathcal{T}_1$ has at least one fixed point, that is to say, the problem (3.1), (3.2) has at least one positive solution. For the problem (3.1), (3.3), by some calculations, we obtain $d_2 \approx 0.297$, $m_2 \approx 0.348$, and $M_2 \approx 3.142$. Taking $k_2 = 0.001$, $r_2 = 0.1$, and $L_2 = 2$, combining with the following conditions: $f(t,w) \geq 0.001$, $(t,w) \in [0,1] \times [0.1,\infty[$, $f(t,w) \geq r_2/m_2 \approx 0.287$, $(t,w) \in [1/2,1] \times [0.272,\infty[$, $f(t,w) \leq L_2/M_2 \approx 0.637$, and $(t,w) \in [0,1/2] \times [2.364,\infty[$. So the problem (3.1), (3.3) has at least one positive solution.

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