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Research Article

Comparison between Certain Equivalent Norms Regarding Some Familiar Properties Implying WFPP

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In a Banach space with a basis we define a similar norm to the norm shown by Lin to make l_1 into a space with FPP and make a comparative study of certain geometric properties such as the Opial property, WNS, and uniform nonsquareness of the original space and the space with the new norm.

1. Introduction

Dowling et al. in [1] defined a norm in l_1 which was used by Lin [2] to exhibit an equivalent norm which makes l_1 into a space with the fixed point property (FPP). A similar norm can be defined in every Banach space X with a basis. Since l_1 with its usual norm does not have FPP, we asked ourselves if this norm in these spaces would also improve properties that imply the weak fixed point property (WFPP). We found out that in some instances it does, in some cases the original norm has better properties, and in some cases you cannot compare them. We give several examples to illustrate our assertions.

2. The Γ Norm in a Banach Space

We start by giving the definition of the generalization of the norm used by Lin in a Banach space with a basis.

Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space with a basis $\{e_n\}$. Let $x = \sum_{i=1}^{\infty} x_i e_i \in X$ and $Q_n : X \to X$ be the projection $Q_n(\sum_{i=1}^{\infty} x_i e_i) = \sum_{i=n}^{\infty} x_i e_i$. The basis $\{e_n\}$ is called premonotone if $\|Q_n x\| \geq \|Q_{n+1} x\|$ and monotone if $\|P_n x\| \leq \|P_{n+1} x\|$ where $P_n x = (I - Q_{n+1})x$ for every $x \in X$ and for every $n \in \mathbb{N}$.

Definition 2.2. Let $(X, \|\cdot\|)$ be a Banach space with a basis $\{e_n\}$ and $\Gamma = \{\gamma_n\} \subset \mathbb{R}$ with $0 < \gamma_n < \gamma_{n+1}$ and $\lim_n \gamma_n = 1$. Let $x = \sum_{i=1}^{\infty} x_i e_i \in X$. Then if

$$|||x||| = \sup_n \gamma_n \|Q_n x\|, \tag{2.1}$$

$|||\cdot|||$ is a norm in X which we will call Γ-norm.

Clearly

$$\gamma_1 \|x\| \leq |||x||| \leq \left(\sup_n \|Q_n\|\right) \|x\|. \tag{2.2}$$

Observe that, if $\{e_n\}$ is a basis in $(X, \|\cdot\|)$, then it is always premonotone in $(X, |||\cdot|||)$ and, if $\{e_n\}$ is monotone in $(X, \|\cdot\|)$, then it is also monotone in $(X, |||\cdot|||)$. Also observe that since for every $x \in X$ we have that $\lim_n \gamma_n \|Q_n x\| = 0$, there exists n_0 such that $|||x||| = \gamma_{n_0} \|Q_{n_0} x\|$.

Next we define the properties related to wfpp we are going to analyze. The definition of GGLD is not the original one found in [3], but an equivalent one found in [4].

Definition 2.3. Let Y be a Banach space.

(1) Y has the Opial property if for every weakly null sequence $\{x_n\} \subset Y$ and for every $x \in Y$, $x \neq 0$,

$$\limsup_n \|x_n\| < \limsup_n \|x_n - x\|. \tag{2.3}$$

(2) If Y has a basis, it has the generalized Gossez-Lami Dozo property (GGLD) [4] if, for every weakly null normalized block basic sequence $\{y_n\}$, we have that

$$\lim_n \sup_{i,j \geq n} \left\| y_i - y_j \right\| > 1. \tag{2.4}$$

(3) A bounded sequence $\{y_n\} \subset Y$ is called diametral if

$$\lim_n d\left(y_{n+1}, \operatorname{conv}\{y_i\}_{i=1}^{n}\right) = \operatorname{diam}\{y_n\}_{n=1}^{\infty}. \tag{2.5}$$

Y has weak normal structure (WNS) if there is no weakly null diametral nonzero sequence in Y.

(4) The coefficient $J(Y)$, related to uniform nonsquareness, since $J(Y) < 2$ if and only if Y is uniformly nonsquare, is given by

$$J(Y) = \sup\left\{\min\left(\|x + y\|, \|x - y\|\right) : x, y \in B_Y\right\}. \tag{2.6}$$

(5) The coefficient $R(Y)$ [5] is defined by

$$R(Y) = \sup\left\{\liminf_n \|x_n + x\| : \{x\}, \{x_n\} \subset B_Y, x_n \underset{w}{\longrightarrow} 0\right\}. \tag{2.7}$$

(6) Coefficients $\mathrm{RW}(a, Y)$ and $\mathrm{MW}(Y)$ [6] are defined as follows: for each $a > 0$

$$\begin{gathered}\mathrm{RW}(a, Y) = \sup\left\{\min\left(\liminf_n \|x_n + x\|, \liminf_n \|x_n - x\|\right) : \|x\| \le a, \{x_n\} \subset B_Y, x_n \underset{w}{\longrightarrow} 0\right\}, \\ \mathrm{MW}(Y) = \sup\left\{\frac{1+a}{\mathrm{RW}(a, Y)} : a > 0\right\}.\end{gathered} \tag{2.8}$$

It is known that GGLD $\Rightarrow$ WNS $\Rightarrow$ wfpp and the Opial property implies wfpp. Also

$$R(Y) < 2 \Longrightarrow \mathrm{MW}(Y) > 1 \Longrightarrow \text{wfpp}, \tag{2.9}$$

$$J(Y) < 2 \Longrightarrow \mathrm{MW}(Y) > 1 \Longrightarrow \text{wfpp}. \tag{2.10}$$

First we will show that the Opial property is inherited from $(X, |||\cdot|||)$ to $(X, \|\cdot\|)$ and that $(X, \|\cdot\|)$ has GGLD if and only if $(X, |||\cdot|||)$ has GGLD. In order to achieve this, we need the following result shown in [7].

Lemma 2.4. *Let $(X, \|\cdot\|)$ be a Banach space with a premonotone basis $\{e_n\}$. Then*

(1) *if $\{x_n\}$ converges weakly to x, $\lim_n \|x_n - x\| = a$ if and only if $\lim_n |||x_n - x||| = a$,*

(2) *if $\{x_n\}$ converges weakly to 0 and $\lim_n \lim_r \|x_n - x_r\| = a$, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\lim_n \lim_r |||y_n - y_r||| = a$.*

Lemma 2.5. *Let $(X, \|\cdot\|)$ be a Banach space with a premonotone basis $\{e_n\}$. If $(X, |||\cdot|||)$ has the Opial property, then $(X, \|\cdot\|)$ also has the Opial property, but the converse is false.*

Proof. Let $\{x_n\}$ be weakly null in X and $x \in X, x \neq 0$. Then, by Lemma 2.4 and by (2.2),

$$\limsup_n \|x_n\| = \limsup_n |||x_n||| < \limsup_n |||x_n - x||| \le \limsup_n \|x_n - x\|. \tag{2.11}$$

It is known that, for $1 < p < \infty$, $(l_p, \|\cdot\|)$ has the Opial property. Consider any Γ-norm $|||\cdot|||$ in l_p with the canonical basis $\{e_n\}$, and let $\delta > 0$ be such that $\delta < ((\gamma_2^p - \gamma_1^p)/\gamma_1^p)^{1/p}$. Then, for $n \ge 2$,

$$\begin{gathered}|||\delta e_1 + e_n||| = \max\left[\gamma_1(\delta^p + 1)^{1/p}, \gamma_n\right] = \gamma_n, \\ \lim_n |||e_n||| = \lim_n \gamma_n = \lim_n |||\delta e_1 + e_n||| = 1.\end{gathered} \tag{2.12}$$

Thus, $(l_p, |||\cdot|||)$ does not have the Opial property. ∎

Lemma 2.6. *Let $(X, \|\cdot\|)$ be a Banach space with a premonotone basis $\{e_n\}$. Then, $(X, \|\cdot\|)$ has GGLD if and only if $(X, |||\cdot|||)$ has GGLD.*

Proof. Let $\{y_n\}$ be a weakly null normalized block basic sequence. By Lemma 2.4, $\lim_n\|y_n\|$ exists if and only if $\lim_n|||y_n|||$ exists and in this case $\lim_n\|y_n\| = \lim_n|||y_n|||$. Also

$$\lim_n \sup_{i,j\geq n}\|y_i - y_j\| = \lim_n \sup_{i,j\geq n}|||y_i - y_j|||. \tag{2.13}$$

The above equality follows immediately from the following inequality, for $i, j \geq n$:

$$|||y_i - y_j||| \leq \|y_i - y_j\| \leq \frac{1}{\gamma_n}|||y_i - y_j|||. \tag{2.14}$$

This proves the lemma. □

Now we will show that there exists a space with WNS such that with the Γ-norm it does not have WNS.

Lemma 2.7. *Let X be the space c_0 with the norm $\|x\| = \sup|b_i| + \sum_{i=1}^{\infty}\varepsilon_i|b_i|$, where $x = \sum_{i=1}^{\infty} b_i e_i$, $\varepsilon_i > 0$ and $\sum_{i=1}^{\infty}\varepsilon_i < \infty$. Then X has WNS.*

Proof. Let $\{x_n\} \subset X$ be a weakly null nonzero sequence. We may assume that $x_1 \neq 0$ and that there exists a block basic sequence $\{u_n\} \subset X$ with $\|u_n - x_n\| \underset{n\to\infty}{\rightarrow} 0$. Suppose that $u_n = \sum_{i=p_n}^{q_n} a_i e_i$ with $p_n \leq q_n < p_{n+1}$ for $n \in \mathbb{N}$ and that $x_1 = \sum_{i=1}^{\infty} b_i e_i$. Let k be such that $\sum_{i=1}^{k}\varepsilon_i|b_i| = \delta \neq 0$. Let $\varepsilon < \delta/2$, $s > k$ with $\|Q_s x_1\| < \varepsilon$ and $n > s$. Then,

$$\|x_1 - u_n\| + \varepsilon \geq \|P_s x_1 - u_n\| = \max(\|P_s x_1\|_\infty, \|u_n\|_\infty) + \sum_{i=1}^{s}\varepsilon_i|b_i| + \sum_{i=p_n}^{q_n}\varepsilon_i|a_i| \geq \|u_n\| + \delta. \tag{2.15}$$

Hence, $\limsup_n\|x_1 - x_n\| \geq \limsup_n\|x_n\| + \delta/2$ and $\{x_n\}$ cannot be a diametral sequence, since for a diametral sequence $\{x_n\}$ it is true that $\lim_n\|x - x_n\| = \operatorname{diam}\{x_n\}$ for every $x \in \overline{\operatorname{conv}}\{x_n\}$. □

Lemma 2.8. *Let $\Gamma = \{\gamma_n\} \subset (0,1)$ be an increasing sequence with $\lim_n\gamma_n = 1$. Then there is a space $(X, \|\cdot\|)$ with WNS, such that X with the Γ-norm $|||\cdot|||$ does not have WNS.*

Proof. Let $\{\gamma_{n_j}\}_j$ be a subsequence of $\{\gamma_n\}$ such that, if $\varepsilon_{n_j} = (1/3)(1/\gamma_{n_j} - 1)$, then $\sum_{i=1}^{\infty}\varepsilon_{n_i} < \infty$. Observe that $\varepsilon_{n_j} < \varepsilon_{n_{j+1}}$. Let X be the space c_0 with the norm $\|x\| = \sup|a_i| + \sum_{i=1}^{\infty}\varepsilon_{n_i}|a_{n_i}| + \sum_{i=1, i\neq n_j}^{\infty}(|a_i|/2^i)$, where $x = \sum_{i=1}^{\infty} a_i e_i$. By Lemma 2.7, since $\sum_{i=1}^{\infty}(\varepsilon_{n_i} + 1/2^i) < \infty$, we know that X has WNS. Now let $|||\cdot|||$ be the Γ-norm in X with respect to $\{\gamma_i\}$. Let $u_j = e_{n_j}$; then $\gamma_{n_j} = 1/(1 + 3\varepsilon_{n_j})$, and thus

$$|||u_j||| = \gamma_{n_j}\left(1 + \varepsilon_{n_j}\right) < 1, \tag{2.16}$$

and, if $j < m$,

$$\begin{aligned} |||u_j - u_m||| &= \max\left(\gamma_{n_j}\left(1 + \varepsilon_{n_j} + \varepsilon_{n_m}\right), \gamma_{n_m}(1 + \varepsilon_{n_m})\right) \\ &\leq \max\left(\gamma_{n_j}\left(1 + 2\varepsilon_{n_j}\right), \gamma_{n_m}(1 + \varepsilon_{n_m})\right) < 1. \end{aligned} \tag{2.17}$$

Therefore, since $0 \in \overline{\mathrm{conv}}\{u_n\}$ and $\lim_n |||u_n||| = 1$, we have that $\mathrm{diam}_{|||\cdot|||}\{u_n\} = 1$. Also, if $0 \leq \lambda_i$, $\sum_{i=1}^{n-1} \lambda_i = 1$,

$$|||u_n||| \leq \left|\left|\left|\sum_{i=1}^{n-1} \lambda_i u_i - u_n\right|\right|\right| \leq 1. \tag{2.18}$$

Hence, $\{u_n\}$ is diametral in $(X, |||\cdot|||)$. □

The above example is another proof of the fact that for every $\varepsilon > 0$ there are Banach spaces X and Y with $d(X, Y) < 1 + \varepsilon$ so that X has WNS but Y does not.

With regard to the coefficient MW(X), we will see that, if X has a premonotone basis and $\mathrm{MW}(X, \|\cdot\|) > 1$, then $\mathrm{MW}(X, |||\cdot|||) > 1$ and we will show a sufficient condition for the reverse implication. For this we need the following lemma.

Lemma 2.9. *Let X be a Banach space with a basis. If*

$$\begin{aligned} RW_1(a, X) = \sup\Big\{ &\min\left(\liminf\|u_n + y\|, \liminf\|u_n - y\|\right) : u_n \underset{w}{\longrightarrow} 0, \\ &\{u_n\} \subset B_X \text{ is a block basic sequence}, \\ &\|y\| \leq a \text{ and support of } y \text{ is finite}\Big\}, \end{aligned} \tag{2.19}$$

then $RW(a, X) = RW_1(a, X)$.

Proof. It is clear that $\mathrm{RW}_1(a, X) \leq \mathrm{RW}(a, X)$.

Now let $\varepsilon > 0$, $x \in X$, $\|x\| \leq a$, and $\{x_n\} \subset B_X$, with $x_n \underset{w}{\rightarrow} 0$ such that $\min(\lim_n \|x_n + x\|, \lim_n \|x_n - x\|) > \mathrm{RW}(a, X) - \varepsilon$. By passing to a subsequence, we may assume that there exist a block basic sequence $\{u_n\} \subset B_X$ with $\|x_n - u_n\| < \varepsilon$ and $m \in \mathbb{N}$ such that $\|x - P_m x\| < \varepsilon$ and $\|P_m x\| < (1 + \varepsilon)\|x\| \leq a(1 + \varepsilon)$. Then,

$$\begin{aligned} \left\|\frac{P_m x}{1 + \varepsilon} + u_n\right\| &\geq \|x_n + x\| - \|x_n - u_n\| - \|x - P_m x\| - \|P_m x\| \frac{\varepsilon}{1 + \varepsilon}, \\ \left\|\frac{P_m x}{1 + \varepsilon} - u_n\right\| &\geq \|x_n - x\| - \|x_n - u_n\| - \|x - P_m x\| - \|P_m x\| \frac{\varepsilon}{1 + \varepsilon}. \end{aligned} \tag{2.20}$$

Let $y = P_m x / (1 + \varepsilon)$; then $\|y\| \leq a$, and we conclude that $\mathrm{RW}_1(a, X) \geq \mathrm{RW}(a, X) - (3 + a)\varepsilon$, thus proving the assertion. □

Similarly one can prove that, if X is a space with a basis,

$$R(Y) = \sup\Big\{ \liminf_n \|u_n + x\| : \{x\}, \{u_n\} \subset B_Y, u_n \xrightarrow[w]{} 0\ ,\\ \{u_n\} \text{ is a block basic sequence and support of } x \text{ is finite} \Big\}. \tag{2.21}$$

It is known (see [6, 8]) that, if X, Y are Banach spaces, then

$$\mathrm{MW}(X) \le \mathrm{MW}(Y) d(X,Y), \qquad J(X) \le J(Y) d(X,Y). \tag{2.22}$$

So, if X is a Banach space with a basis, $\Gamma = \{\gamma_n\}$ with $0 \le \gamma_n \le \gamma_{n+1} \le 1$ and $\gamma_1 > 1/\mathrm{MW}(X)$, and Y is X with the Γ-norm, then $\mathrm{MW}(Y) \ge \mathrm{MW}(X)/d(X,Y) \ge \gamma_1 \mathrm{MW}(X) > 1$, and if $\gamma_1 > 1/\mathrm{MW}(Y)$, $\mathrm{MW}(X) > 1$. Similarly, if $\gamma_1 > J(X)/2$, then $J(Y) < 2$ and, if $\gamma_1 > J(Y)/2$, then $J(X) < 2$. But the next proposition shows that in fact $\mathrm{MW}(X) > 1$ always implies $\mathrm{MW}(Y) > 1$. For the coefficient J, in general neither $J(X) < 2$ implies $J(Y) < 2$ nor the other way round, as we will see in Examples 2.16 and 2.17.

Proposition 2.10. *Suppose that X is a Banach space with a premonotone basis $\{e_n\}$. Then $MW = MW(X, \|\cdot\|) > 1$ implies that $MW_1 = MW(X, |||\cdot|||) > 1$.*

Proof. Let $\mathrm{RW}(a, (X, \|\cdot\|)) = R(a)$ and $\mathrm{RW}(a, (X, |||\cdot|||)) = R_1(a)$. Suppose that $\mathrm{MW}_1 = 1$. Then, $R_1(a) = 1 + a$ for every $a > 0$. Let $a > 0$ and $0 < \varepsilon < a$, $y \in X$ with finite support, $|||y||| \le a$, and let $\{u_n\}$ be a weakly null block basic sequence with $|||u_n||| \le 1$ such that $\lim_n |||u_n + y||| > 1 + a - \varepsilon$ and $\lim_n |||u_n - y||| > 1 + a - \varepsilon$. Then we may suppose that for every n, $|||u_n + y||| > 1 + a - \varepsilon$ and $|||u_n - y||| > 1 + a - \varepsilon$. Hence,

$$|||u_n||| \ge 1 - \varepsilon, \qquad |||y||| \ge a - \varepsilon. \tag{2.23}$$

We may also assume that the supports of y and u_n are disjoint. Let $u_n = \sum_{i=l_n}^{r_n} a_i e_i$ and $y = \sum_{i=1}^{r} b_i e_i$.

Suppose that $|||u_n + y||| = \gamma_{m_n} \| \sum_{i=m_n}^{r_n} (a_i + b_i) e_i \|$ for some $m_n \le r_n$. It is not possible that $m_n > r$, because this would mean that

$$1 + a - \varepsilon \le |||u_n + y||| = \gamma_{m_n} \left\| \sum_{m_n}^{r_n} a_i e_i \right\| \le |||u_n||| \le 1. \tag{2.24}$$

So, by passing to a subsequence if necessary, we may assume that for every n we have that $m_n = i_0 \le r$. Thus,

$$\begin{aligned} 1 + a - \varepsilon \le |||u_n + y||| &= \gamma_{i_0} \left\| \sum_{i=i_0}^{r_n} (a_i + b_i) e_i \right\| \\ &\le \gamma_{i_0} \left(\left\| \sum_{i=l_n}^{r_n} a_i e_i \right\| + \left\| \sum_{i=i_0}^{r} b_i e_i \right\| \right) \le \frac{\gamma_{i_0}}{\gamma_{l_n}} + a. \end{aligned} \tag{2.25}$$

Since $\lim_n \gamma_{l_n} = 1$, by passing to the limit, we obtain that

$$\gamma_{i_0} \geq 1 - \varepsilon. \tag{2.26}$$

Similarly, there exists $\gamma_{i_1} \geq 1 - \varepsilon$ with

$$1 + a - \varepsilon \leq \left| \| u_n - y \| \right| \leq \frac{\gamma_{i_1}}{\gamma_{l_n}} + a. \tag{2.27}$$

Suppose that $i_1 \geq i_0$, and let $y_0 = \sum_{i=i_0}^{r} b_i e_i$ and $y_1 = \sum_{i=i_1}^{r} b_i e_i$. Then, since the basis is premonotone, $1 + a - \varepsilon \leq |||u_n + y||| = |||u_n + y_0||| \leq \|u_n + y_0\|$ and $1 + a - \varepsilon \leq |||u_n - y||| = |||u_n - y_1||| \leq \|u_n - y_1\| \leq \|u_n - y_0\|$; thus,

$$a - \varepsilon \leq \left| \| y_0 \| \right| \leq \left| \| y \| \right| \leq a. \tag{2.28}$$

Therefore,

$$a - \varepsilon \leq \left| \| y_0 \| \right| \leq \| y_0 \| \leq \frac{1}{\gamma_{i_0}} \left| \| y_0 \| \right| \leq \frac{a}{\gamma_{i_0}} \leq \frac{a}{1 - \varepsilon} \tag{2.29}$$

and $|\|y_0\| - a| \leq \max\{\varepsilon, a\varepsilon/(1-\varepsilon)\}$. Further, since $\gamma_{i_0} \leq \gamma_{l_n}$,

$$1 - \varepsilon \leq |||u_n||| \leq \|u_n\| \leq \frac{1}{\gamma_{l_n}} |||u_n||| \leq \frac{1}{\gamma_{l_n}} \leq \frac{1}{1 - \varepsilon} \tag{2.30}$$

and $|\|u_n\| - 1| \leq \varepsilon/(1-\varepsilon)$.

Hence,

$$\begin{aligned} \left\| \frac{u_n}{\|u_n\|} + \frac{y_0}{\|y_0\|} a \right\| &\geq \|u_n + y_0\| - |\|u_n\| - 1| - \left| \|y_0\| - a \right| \\ &\geq 1 + a - \varepsilon - \frac{\varepsilon}{1 - \varepsilon} - \max\left\{ \varepsilon, \frac{a\varepsilon}{1 - \varepsilon} \right\}. \end{aligned} \tag{2.31}$$

Similarly,

$$\left\| \frac{u_n}{\|u_n\|} - \frac{y_0}{\|y_0\|} a \right\| \geq 1 + a - \varepsilon - \frac{\varepsilon}{1 - \varepsilon} - \max\left\{ \varepsilon, \frac{a\varepsilon}{1 - \varepsilon} \right\}. \tag{2.32}$$

We deduce that $\min(\liminf_n \|(u_n/\|u_n\|) + (y_0/\|y_0\|)a\|, \liminf_n \|(u_n/\|u_n\|) - (y_0/\|y_0\|)a\|) \geq 1 + a - \varepsilon - \varepsilon/(1-\varepsilon) - \max\{\varepsilon, a\varepsilon/(1-\varepsilon)\}$, and letting ε tend to zero we obtain $R(a) = 1 + a$ and $\mathrm{MW} = 1$. □

Examples 2.14 and 2.17 exhibit spaces in which $\mathrm{MW}(X, \|\cdot\|) = 1$ and $\mathrm{MW}(X, |||\cdot|||) > 1$. There is however a special case in which $\mathrm{MW}(X, \|\cdot\|) \geq \mathrm{MW}(X, |||\cdot|||)$.

Recall that a basis $\{e_n\}$ of a Banach space X is 1-spreading if, whenever $x = \sum_{i=1}^{\infty} a_i e_i \in X$ and $\{e_{n_i}\}_i$ is a subsequence of $\{e_n\}$, $\sum_{i=1}^{\infty} a_i e_{n_i} \in X$ and $\| \sum_{i=1}^{\infty} a_i e_i \| = \| \sum_{i=1}^{\infty} a_i e_{n_i} \|$.

Proposition 2.11. *If X is a Banach space with a premonotone 1-spreading basis, then* $MW(X, \|\cdot\|) \geq MW(X, |\|\cdot\||)$.

Proof. Let $T_m : X \to X$ be the translation given by $T_m \sum_{i=1}^{\infty} a_i e_i = \sum_{i=1}^{\infty} a_i e_{i+m}$, and let $a > 0$,$y \in X$, $\|y\| \leq a$, $\{u_n\} \subset B_X$, where y has finite support and $\{u_n\}$ is a weakly null block basic sequence such that $\min(\|y + u_n\|, \|y - u_n\|) > RW(a, (X, \|\cdot\|)) - \varepsilon$ for every n. Let $m \in \mathbb{N}$; then there exists $N \in \mathbb{N}$ such that for $n > N$ the supports of $T_m y$ and u_n are disjoint, and thus, since the basis is 1-spreading, $\|y - u_n\| = \|T_m y - u_n\|$ and $\|y + u_n\| = \|T_m y + u_n\|$. Then, $|\|T_m y\|| \leq \|T_m y\| \leq a$, $|\|u_n\|| \leq \|u_n\| \leq 1$ and for $n > N$

$$\begin{aligned} |\|T_m y - u_n\|| &\geq \gamma_m \|T_m y - u_n\| = \gamma_m \|y - u_n\|, \\ |\|T_m y + u_n\|| &\geq \gamma_m \|T_m y + u_n\| = \gamma_m \|y + u_n\|. \end{aligned} \tag{2.33}$$

Hence, $\mathrm{RW}(a, (X, |\|\cdot\||)) \geq \gamma_m \mathrm{RW}(a, (X, \|\cdot\|))$, and, by passing to the limit as m tends to infinity, we get $\mathrm{RW}(a, (X, |\|\cdot\||)) \geq \mathrm{RW}(a, (X, \|\cdot\|))$ and thus the desired result. □

Similarly to Propositions 2.10 and 2.11 we can prove the following.

Proposition 2.12. *Suppose that X is a Banach space with a premonotone basis* $\{e_n\}$. *Then* $R = R(X, \|\cdot\|) < 2$ *implies that* $R_1 = R(X, |\|\cdot\||) < 2$ *and, if the basis is premonotone and 1-spreading, then* $R = R(X, \|\cdot\|) \leq R(X, |\|\cdot\||) = R_1$.

Corollary 2.13. *If X is a Banach space with a premonotone 1-spreading basis, then* $MW_1 > 1$ *if and only if* $MW > 1$*; also* $R_1 < 2$ *if and only if* $R < 2$.

Next we will show an example of a space without a 1-spreading basis, such that $R = 2$, $\mathrm{MW} = 1$ but $R_1 < 2$ and thus $\mathrm{MW}_1 > 1$.

Example 2.14. Let X be c_0 with the following norm:

$$\|\{a_n\}\| = |a_1| + \max_{i \geq 2} |a_i|. \tag{2.34}$$

Let $\{e_n\}$ denote the canonical basis of c_0. Then for every $a > 0$, $\|ae_1 + e_n\| = \|ae_1 - e_n\| = 1 + a$, and thus $R = 2$ and $\mathrm{MW} = 1$. On the other hand let $x \in X$ with finite support and $|\|x\|| \leq 1$ and suppose that $\{u_n\}$ is a block basic sequence with $|\|u_n\|| \leq 1$ for $n \in N$, $u_n = \sum_{i=m_n}^{r_n} a_i e_i$ with $m_n \leq r_n < m_{n+1}$ and the support of x does not intersect the support of u_n. Then, for every $m \geq 2$ and $n \geq m$, $\gamma_m \|Q_m x + u_n\| = \max(\gamma_m \|Q_m x\|_{c_0}, \gamma_m \|u_n\|_{c_0}) \leq 1$,

$$\gamma_1 \|x + u_n\| \leq 1 + \gamma_1 \|u_n\|_{c_0} \leq 1 + \frac{\gamma_1}{\gamma_{m_n}}. \tag{2.35}$$

Thus, $\liminf |\|x + u_n\|| \leq 1 + \gamma_1$. Hence $R_1 \leq 1 + \gamma_1 < 2$ and, by (2.9), $\mathrm{MW}_1 > 1$.

With regard to the coefficient J we have the following which is proved similarly to Proposition 2.11.

Proposition 2.15. *Suppose that X is a Banach space with a premonotone 1-spreading basis* $\{e_n\}$. *Then* $J(X, \|\cdot\|) \le J(X, |||\cdot|||)$.

In general neither $J(X, \|\cdot\|) < 2$ implies $J(X, |||\cdot|||) < 2$ nor the other way round, as the following examples show.

Example 2.16. Let $1 > \mu \ge 1/\sqrt{2}$ and $X = \mathbb{R}^2_\mu \bigoplus_2 l_2$, where $\mathbb{R}^2_\mu = (\mathbb{R}^2, \|\cdot\|_{(\mu)})$ and $\|(x_1, x_2)\|_{(\mu)} = \max(|x_1|, |x_2|, \mu(|x_1| + |x_2|))$. Then, if $\mu \le \gamma_2/(\gamma_1 + \gamma_2)$, $J(X) < 2$ but for every Γ, $J(X, |||\cdot|||) = 2$.

Since $\mu \ge 1/\sqrt{2}$, it is easy to see that for $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\frac{\mu}{\sqrt{1 - 2\mu + 2\mu^2}} \|x\|_2 \le \|x\|_{(\mu)} \le \mu\sqrt{2}\|x\|_2. \tag{2.36}$$

Thus, $d(l_2, X) \le d(\mathbb{R}^2, \mathbb{R}^2_\mu) = \sqrt{2}\sqrt{1 - 2\mu + 2\mu^2} < \sqrt{2}$, and by (2.22), since $J(l_2) = \sqrt{2}$, we obtain that $J(X) < 2$.

Now let $\Gamma = \{\gamma_n\}$, $\mu \le \gamma_2/(\gamma_1 + \gamma_2)$, $x = (1/\gamma_1, 1/\gamma_2, 0, 0, \ldots)$, and $y = (-1/\gamma_1, 1/\gamma_2, 0, 0, \ldots)$. Then $|||x||| = |||y||| = 1$ but $|||x + y||| = |||x - y||| = 2$.

This last example is another proof of the known fact that for every $\varepsilon > 0$ there are Banach spaces X and Y with $d(X, Y) < 1 + \varepsilon$, $J(X) < 2$ but $J(Y) = 2$. In the following example we exhibit a space X with $J(X) = 2$ such that $J(X, |||\cdot|||) < 2$.

Example 2.17. Let $X = (l_2, \|\cdot\|)$, where, for $x = (a_n) \in l_2$, $\|x\| = |a_1| + \left(\sum_{i=2}^{\infty} a_i^2\right)^{1/2}$. Then $J(X) = 2, \mathrm{MW}(X) = 1$, and, if Γ is such that $\gamma_2 > 1/\sqrt{2}$ and $\gamma_1/\gamma_2 < 1/\sqrt{2}$, $J(X, |||\cdot|||) < 2$ and thus $\mathrm{MW}(X, |||\cdot|||) > 1$.

Obviously if $\{e_n\}$ is the canonical basis of l_2, and $a > 0$, then $\|ae_1 + e_n\| = \|ae_1 - e_n\| = 1 + a$ for $n > 1$ and thus $J(X) = 2$ and $\mathrm{MW}(X) = 1$.

Suppose now that $J(Y) = J(X, |||\cdot|||) = 2$. Then there exist sequences $\{x_n\}, \{y_n\} \subset B_Y$ such that $\lim_n |||x_n + y_n||| = \lim_n |||x_n - y_n||| = 2$ and $\{m_n\}, \{l_n\} \subset \mathbb{N}$ so that

$$\begin{aligned} \big|\big\|x_n + y_n\big\|\big| &= \gamma_{m_n} \big\|Q_{m_n}(x_n + y_n)\big\|, \\ \big|\big\|x_n - y_n\big\|\big| &= \gamma_{l_n} \big\|Q_{l_n}(x_n - y_n)\big\|. \end{aligned} \tag{2.37}$$

By passing to a subsequence, we may assume that, for every $n \in \mathbb{N}$, $e_1^*(x_n) \ge 0$, $e_1^*(y_n) \ge 0$ and $e_1^*(x_n) \ge e_1^*(y_n)$, and that the subsequence satisfies one of the next three cases:

(1) $m_n > 1$ and $l_n > 1$ for every n,

(2) $m_n = l_n = 1$ for every n,

(3) $m_n = 1$ and $l_n > 1$ for every n.

Observe that

$$\gamma_m \big\|Q_m(x + y)\big\| \ge 2 - \varepsilon \quad \text{implies } \gamma_m \|Q_m(x)\| \ge 1 - \varepsilon \text{ and } \gamma_m \big\|Q_m(y)\big\| \ge 1 - \varepsilon. \tag{2.38}$$

Case 1. Let $Z = (l_2, |||\cdot|||_{\Gamma_1})$, where $|||\cdot|||_{\Gamma_1}$ is the $\Gamma_1 = \{\gamma_i\}_{i=2}^{\infty}$-norm. Then

$$\begin{aligned} |||Q_2 x_n|||_{\Gamma_1} &\le 1, & \left|\left|\left|Q_2 y_n\right|\right|\right|_{\Gamma_1} &\le 1, \\ \left|\left|\left|Q_2 (x_n + y_n)\right|\right|\right|_{\Gamma_1} &\ge 2 - \varepsilon, & \left|\left|\left|Q_2 (x_n - y_n)\right|\right|\right|_{\Gamma_1} &\ge 2 - \varepsilon. \end{aligned} \tag{2.39}$$

Since $1/\gamma_2 < \sqrt{2}$, then, by (2.2), $d(Z, l_2) < \sqrt{2}$ and, by (2.22), since $J(l_2) = \sqrt{2}, J(Z) < 2$ and this is a contradiction.

Case 2. Let $\varepsilon > 0$. Suppose that $x = \sum_{i=1}^{\infty} a_i e_i$, $y = \sum_{i=1}^{\infty} b_i e_i \in B_Y$, $a_1 > 0$, $b_1 > 0$, $a_1 > b_1$ and

$$\begin{aligned} \gamma_1 \left(\sum_{i=2}^{\infty} (a_i + b_i)^2 \right)^{1/2} &\ge 2 - \varepsilon - \gamma_1 (a_1 + b_1), \\ \gamma_1 \left(\sum_{i=2}^{\infty} (a_i - b_i)^2 \right)^{1/2} &\ge 2 - \varepsilon - \gamma_1 (a_1 - b_1). \end{aligned} \tag{2.40}$$

Squaring both inequalities and adding them, since $\gamma_2 (\sum_{i=2}^{\infty} a_i^2)^{1/2} \le |||x|||$, we get

$$\frac{2\gamma_1^2}{\gamma_2^2} \ge \gamma_1^2 \sum_{i=2}^{\infty} \left(a_i^2 + b_i^2 \right) \ge (2 - \varepsilon - \gamma_1 a_1)^2 + \gamma_1^2 b_1^2 \ge (2 - \varepsilon - \gamma_1 a_1)^2. \tag{2.41}$$

By passing to the limit as $\varepsilon \to 0$, since $\gamma_1 a_1 \le 1$, we obtain that $\sqrt{2}(\gamma_1/\gamma_2) \ge 2 - \gamma_1 a_1 \ge 1$, and this contradicts $\gamma_1/\gamma_2 < 1/\sqrt{2}$.

Case 3. Let $2 > \varepsilon > 0$. Suppose that $x = \sum_{i=1}^{\infty} a_i e_i$, $y = \sum_{i=1}^{\infty} b_i e_i \in B_Y$, $a_1 > 0$, $b_1 \ge 0$, $a_1 > b_1$ and $|||x + y||| = \gamma_1 (|a_1 + b_1| + (\sum_{i=2}^{\infty} (a_i + b_i)^2)^{1/2}) \ge 2 - \varepsilon$ and $|||x - y||| = \gamma_m (\sum_{i=m}^{\infty} (a_i - b_i)^2)^{1/2} \ge 2 - \varepsilon$. Then, by (2.38),

$$\gamma_m \left(\sum_{i=m}^{\infty} a_i^2 \right)^{1/2} \ge 1 - \varepsilon. \tag{2.42}$$

Since in l_2 if $u, v \in B_{l_2}$, one has that $\|u - v\| \ge \delta$ implies $\|u + v\| \le 2\sqrt{1 - (\delta/2)^2}$, then $\gamma_m \left(\sum_{i=m}^{\infty} (a_i + b_i)^2\right)^{1/2} \le \sqrt{4\varepsilon - \varepsilon^2}$. Also

$$\begin{aligned}
2 - \varepsilon &\le \gamma_1 \left(|a_1 + b_1| + \left(\sum_{i=2}^{m-1} (a_i + b_i)^2 \right)^{1/2} + \left(\sum_{i=m}^{\infty} (a_i + b_i)^2 \right)^{1/2} \right) \\
&\le \gamma_1 |a_1 + b_1| + \gamma_1 \left(\sum_{i=2}^{m-1} (a_i + b_i)^2 \right)^{1/2} + \frac{\gamma_1}{\gamma_m} \sqrt{4\varepsilon - \varepsilon^2} \\
&\le \gamma_1 |a_1 + b_1| + \gamma_1 \left(\sum_{i=2}^{m-1} (a_i + b_i)^2 \right)^{1/2} + \sqrt{4\varepsilon - \varepsilon^2}.
\end{aligned} \tag{2.43}$$

Let $\phi = \varepsilon + \sqrt{4\varepsilon - \varepsilon^2}$; then

$$2 - \phi \le \gamma_1 |a_1 + b_1| + \gamma_1 \left(\sum_{i=2}^{m-1} (a_i + b_i)^2 \right)^{1/2}. \tag{2.44}$$

By (2.38), $\gamma_1 |a_1| + \gamma_1 \left(\sum_{i=2}^{m-1} a_i^2\right)^{1/2} \ge 1 - \phi$, and since, for $A, B > 0$,

$$(A + B)^{1/2} - A^{1/2} = \frac{B}{(A + B)^{1/2} + A^{1/2}}, \tag{2.45}$$

we have, using (2.42), that

$$\begin{aligned}
1 &\ge \gamma_1 |a_1| + \gamma_1 \left(\sum_{i=2}^{\infty} a_i^2 \right)^{1/2} \\
&\ge \gamma_1 |a_1| + \gamma_1 \left(\sum_{i=2}^{m-1} a_i^2 + \frac{(1 - \varepsilon)^2}{\gamma_m^2} \right)^{1/2} \\
&= \gamma_1 |a_1| + \gamma_1 \left(\sum_{i=2}^{m-1} a_i^2 \right)^{1/2} + \frac{\gamma_1 (1 - \varepsilon)^2}{\gamma_m^2} \frac{1}{(A + B)^{1/2} + A^{1/2}} \\
&\ge 1 - \phi + \frac{\gamma_1 (1 - \varepsilon)^2}{\gamma_m^2} \frac{1}{(A + B)^{1/2} + A^{1/2}},
\end{aligned} \tag{2.46}$$

where $A = \left(\sum_{i=2}^{m-1} a_i^2\right)^{1/2}$ and $B = (1 - \varepsilon)^2 / \gamma_m^2$. But

$$(A + B)^{1/2} + A^{1/2} \le \left(\frac{1}{\gamma_2^2} + \frac{(1 - \varepsilon)^2}{\gamma_m^2} \right)^{1/2} + \frac{1}{\gamma_2} \le \frac{3}{\gamma_2}; \tag{2.47}$$

therefore,

$$\gamma_1 < \frac{\gamma_1}{\gamma_m^2} \le \frac{\phi}{(1-\varepsilon)^2}\frac{3}{\gamma_2}, \tag{2.48}$$

and, taking the limit as $\varepsilon \to 0$, we get $\gamma_1 = 0$ which is a contradiction.

Hence, $J(X, ||| \cdot |||) < 2$, and, by (2.10), we have that $\mathrm{MW}(X, ||| \cdot |||) > 1$.

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